Anderson localization in one-dimensional randomly disordered optical systems that are periodic on average

A. R. McGurn
Department of Physics, Western Michigan University, Kalamazoo, Michigan 49008

K. T. Christensen and F. M. Mueller
Center for Materials Science, Los Alamos National Laboratory, Los Alamos, New Mexico 87544

A. A. Maradudin
Department of Physics, University of California, Irvine, California 92717
(Received 29 October 1992)

We compute the frequency dependence of the localization length in a one-dimensional randomly disordered optical system, which on average is periodic, by studying the dependence of the transmissivity on the length of a finite random sample. Specifically, we consider a layered system of dielectric slabs with electromagnetic waves propagating perpendicular to the interfaces and compute the localization length for frequencies of these waves in and around the neighborhood of the band gaps in the photonic band structure of the average periodic system. The localization length is found to be very small in the gaps and much larger in the bands. We also compute the dependence of the localization length in the presence of dissipation (complex dielectric constant) and obtain a simple relationship, for frequencies of the electromagnetic waves in the allowed bands, between the localization length in the nondissipative system, the decay length in the nonrandom periodic system with dissipative terms, and the localization length in the presence of dissipation. For frequencies in the gaps the localization length appears to be insensitive to the presence of dissipation.

I. INTRODUCTION

There has recently been great interest in the Anderson localization of classical waves in randomly disordered systems that are periodic on average rather than homogeneous. This interest has been prompted by the recent work of John[1-3] in which it is argued that the Ioffe-Regel criterion for localization, $kl_s < 1$, where k is the wave number of the wave and l_s is its scattering mean free path, should be much more easily satisfied when the frequency of the wave is in the neighborhood of a band edge at the Brillouin-zone boundary in the band structure for the waves in the average periodic system. This is because in this case the wave number k is replaced by the "crystal momentum" k_{cryst} of the wave in the average periodic system, and this goes to zero at the band edge at the zone boundary. This result is also consistent with the commonly held belief that the localization length of the wave in the random system is shortest in the frequency ranges where the density of states of the classical waves is lowest, viz. within the gaps in the band structure of the waves in the average periodic system. A short localization length would facilitate the observation of many optical and acoustical phenomena related to Anderson localization, and this motivates efforts to formulate criteria for the establishment of short localization lengths.

The argument given by John seems very reasonable, as in systems that are weakly randomly disordered away from an originally periodic system we would expect to find large regions that are essentially periodic. The propagation of excitations in the band gaps of such periodic regions should certainly exhibit exponential attenuation. Nevertheless, little work has been done to verify this argument experimentally or by numerical simulation. In this paper we study by numerical simulation a one-dimensional system that illustrates the Anderson localization properties proposed by the above conjecture. In addition, we shall also discuss the effects of dissipative losses (complex dielectric constants) in media which form our random systems, on the Anderson localization of excitations in such systems.

We consider two types of random systems. Both of these systems are formed by introducing small random perturbations into a regular array of $2N-1$ dielectric slabs of thickness a, and alternating dielectric constants ϵ and 1. (For $N \rightarrow \infty$ the unperturbed array is periodic.) In our first type of disordered system we consider the addition of a small random dielectric disorder to the slabs of dielectric constant ϵ, i.e., for a given slab, ϵ becomes $\epsilon + \delta$ where the random addition δ is different for each dielectric slab of the system. In our second type of disordered system we consider fixing the dielectric constants of the system to be ϵ and 1 but adding small random increments to the widths of the slabs so that each slab has a different width.

The easiest way to study the random layered systems described above is to begin by solving for the propagation of electromagnetic waves in the single slab system shown in Fig. 1. The results of this solution can be used inductively to obtain a general expression for the transmissivity of the random layered systems of the types described above. The system shown in Fig. 1 consists of a single

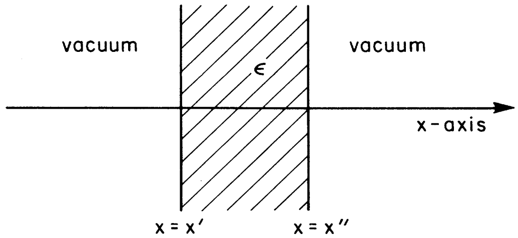

FIG. 1. The geometry of a single dielectric slab of dielectric constant ϵ surround by a vacuum. Left face at $x=x'$ and right face at $x=x''$.

slab of dielectric constant ϵ surrounded by a vacuum. The left edge of the slab is located at x' and the right edge is located at x''. The electric field in the region $x<x'$ is of the form

$$\mathbf{E}=[E_{1R}e^{i\omega(x/c-t)}+E_{1L}e^{-i\omega(x/c+t)}]\hat{j} \tag{1a}$$

and in the region $x>x''$ it is of the form

$$\mathbf{E}=[E_{2R}e^{i\omega(x/c-t)}+E_{2L}e^{-i\omega(x/c+t)}]\hat{j} \ . \tag{1b}$$

By satisfying the electromagnetic boundary conditions at x' and x'' we find the matrix equation relating the amplitudes $E_{2R,L}$ and $E_{1R,L}$,

$$\mathbf{E}_2=\underline{T}(x'',x';\epsilon)\mathbf{E}_1 \ , \tag{2}$$

where

$$\mathbf{E}_1=\begin{bmatrix} E_{1R} \\ E_{1L} \end{bmatrix} , \tag{3a}$$

$$\mathbf{E}_2=\begin{bmatrix} E_{2R} \\ E_{2L} \end{bmatrix} , \tag{3b}$$

and

$$\underline{T}(x'',x';\epsilon)=\begin{bmatrix} T_1(x'';x',\epsilon) & T_2(x'';x',\epsilon) \\ T_3(x'';x',\epsilon) & T_4(x'';x',\epsilon) \end{bmatrix} , \tag{4}$$

with

$$\begin{aligned} T_1(x'';x',\epsilon)&=\left\{\frac{i}{2}\left[\sqrt{\epsilon}+\frac{1}{\sqrt{\epsilon}}\right]\sin\left[\sqrt{\epsilon}\frac{\omega}{c}(x''-x')\right]+\cos\left[\sqrt{\epsilon}\frac{\omega}{c}(x''-x')\right]\right\}e^{-i\omega(x''-x')/c} \\ &=T_4^*(x'';x',\epsilon) \ , \end{aligned} \tag{5a}$$

$$\begin{aligned} T_2(x'';x',\epsilon)&=\frac{i}{2}\left[\sqrt{\epsilon}-\frac{1}{\sqrt{\epsilon}}\right]\sin\left[\sqrt{\epsilon}\frac{\omega}{c}(x''-x')\right]e^{-i\omega(x''+x')/c} \\ &=T_3^*(x'';x',\epsilon) \ . \end{aligned} \tag{5b}$$

II. THE AVERAGE PERIODIC STRUCTURE

The average periodic structure that underlies the calculations to be presented in the remainder of this paper consists of an infinite, alternating array of dielectric slabs of dielectric constants ϵ and 1, each of thickness a. The dispersion relation for electromagnetic waves incident normally on this structure is[4]

$$\cos 2ka=\cos\sqrt{\epsilon}\frac{\omega a}{c}\cos\frac{\omega a}{c}-\frac{1}{2}\left[\sqrt{\epsilon}+\frac{1}{\sqrt{\epsilon}}\right]\times\sin\sqrt{\epsilon}\frac{\omega a}{c}\sin\frac{\omega a}{c} \ , \tag{6}$$

where k is the wave number of the electromagnetic wave and ω is its frequency. In what follows we will assume that $\epsilon=9$, in which case Eq. (6) simplifies to

$$\cos 2ka=\tfrac{4}{3}\cos\frac{4\omega a}{c}-\tfrac{1}{3}\cos\frac{2\omega a}{c} \ , \tag{7}$$

from which it follows that

$$\frac{\omega a}{c}=\tfrac{1}{2}\cos^{-1}\{\tfrac{1}{16}[1\pm(129+96\cos 2ka)^{1/2}]\} \ . \tag{8}$$

The three lowest-frequency branches of the dispersion curve for the propagation of electromagnetic waves through this structure are depicted in Fig. 2, for k in the first Brillouin zone, $0\le k\le\pi/(2a)$. It follows from Eq. (8) that the lower edge of the gap between the first and second band occurs at the frequency given by $(\omega a/c)=\frac{1}{2}\cos^{-1}[(1+\sqrt{33})/16]=0.5678$; the upper edge of this gap occurs at the frequency given by $(\omega a/c)=\frac{1}{2}\cos^{-1}[(1-\sqrt{33})/16]=0.9359$. It is the local-

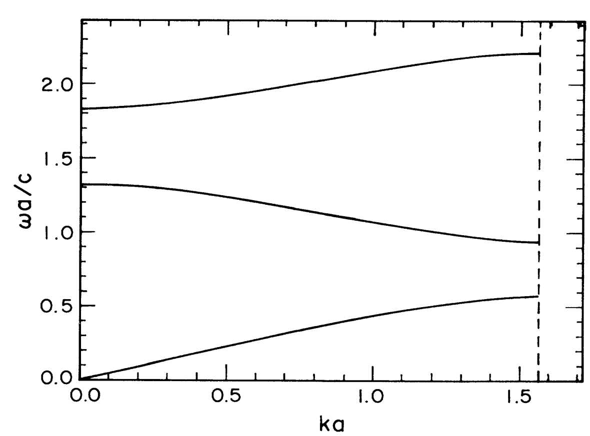

FIG. 2. The three lowest-frequency branches of the dispersion curve for electromagnetic waves propagating through the average periodic dielectric structure studied in this paper.

ization length of electromagnetic waves whose frequency is in the vicinity of this band gap that will be calculated when the periodic structure is disordered in the ways described in the Introduction.

III. RANDOM DIELECTRIC CONSTANTS

The first case we consider is a system of $2N-1$ slabs, each of thickness a, in which the dielectric constant ϵ_i of the ith slab satisfies

$$\epsilon_{2i-1}=\epsilon+\delta_{2i-1} , \quad (9a)$$

$$\epsilon_{2i}=1 , \quad (9b)$$

where δ_{2i-1} is a random number uniformly distributed over the interval $[-\delta,\delta]$ with $\delta \ll \epsilon$. The slab with dielectric constant ϵ_{2i-1} occupies the region $(2i-2)a<x_1<(2i-1)a$; the slab with dielectric constant ϵ_{2i} occupies the region $(2i-1)a<x_1<2ia$. We assume that the electromagnetic wave is incident on the system from the right at $x=(2N-1)a$ and that the transmitted light emerges from the array to the left at $x=0$.

For $x>(2N-1)a$ we take the electric field to be of the form

$$\mathbf{E}=[e^{-i\omega(x/c+t)}+re^{i\omega(x/c-t)}]\hat{j} , \quad (10)$$

and for $x<0$ it has the form

$$\mathbf{E}=te^{-i\omega(x/c+t)}\hat{j} . \quad (11)$$

By using Eqs. (1)–(5) we find that

$$\begin{bmatrix} r \\ 1 \end{bmatrix} = \prod_{j=1}^{N} \{\underline{T}[(2j-1)a,(2j-2)a;\epsilon_{2j-1}]\} \begin{bmatrix} 0 \\ t \end{bmatrix} , \quad (12)$$

where, for a given set of ϵ_i, the product of 2×2 matrices on the right-hand side of Eq. (12) can be easily evaluated by a computer. Disordered systems in one dimension always exhibit Anderson localization so we expect the transmissivity $T=|t|^2$ of our system to decrease rapidly with increasing N. In fact, it can be shown that the localization length l in a one-dimensional disordered medium whose disorder is described by a statistical distribution function is given by[5]

$$l=-\frac{L}{\langle \ln T \rangle} , \quad (13)$$

where $L(2N-1)a$ is the length of the one-dimensional random array, T is the transmissivity, and $\langle\ \rangle$ represents an average over the statistical distribution function of the disorder in the system.

We have computed the localization length given by Eq. (13) for a random system of slabs with $\epsilon=9$, $\delta=1$, and $N=640$ and 1280. The frequency dependence of the localization length was obtained for frequencies about the upper and lower edges of the lowest-frequency band gap in the photonic band structure of the average periodic system. In the results presented in Fig. 3 we see that the localization length decreases rapidly as the frequency of the electromagnetic wave enters the gap region, and that in the gap region, away from the immediate neighborhood of the band edges, the localization length remains fairly independent of frequency. It is important to note that outside the gap region the localization length may extend to over 100 slab widths, whereas in the gap the localization length is only of the order of a few dielectric slab widths. Hence the gap region greatly assists the disorder in creating wave functions that are localized over small regions of space in the disordered system.

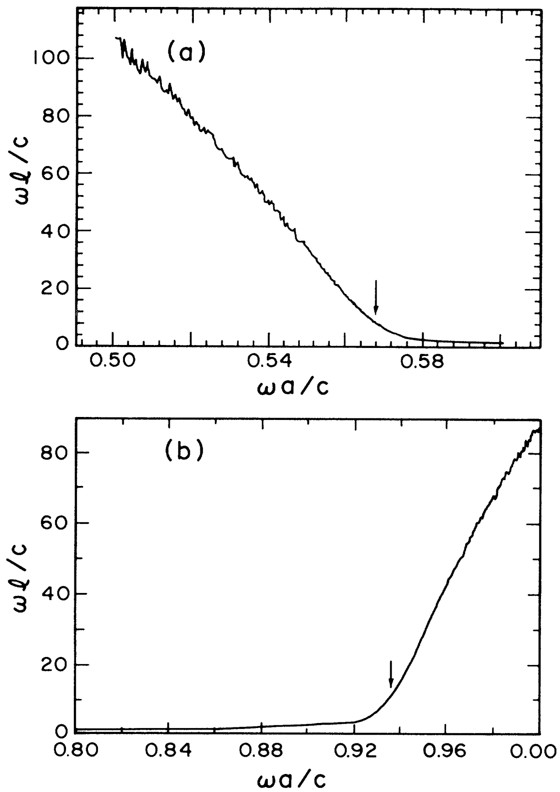

FIG. 3. Plot of (ω/c) (localization length) vs $\omega a/c$ for a system with random dielectric constants. The results are shown for (a) ω near the lower band-gap edge and (b) near the upper band-gap edge. The edges of the band gaps are indicated by arrows. The curves in the figures were computed for an average over 500 random samples.

IV. RANDOM SLAB WIDTHS

In our second case we consider a system of $2N-1$ slabs of fixed dielectric constants ϵ and 1 such that the dielectric constant ϵ_i of the ith slab satisfies

$$\epsilon_{2i-1}=\epsilon , \quad (14a)$$

$$\epsilon_{2i}=1 . \quad (14b)$$

The widths of the ϵ and 1 dielectric slabs are weakly randomly disordered, but both sets of dielectric slabs have the same average thickness a. The x coordinates of the left-hand sides of the slabs of dielectric constant ϵ are given by

$$x_{L,2i-1}=(2i-2)a+d_{2i-1} , \quad (15)$$

where $d_1=0$ and d_{2i-1}/a for $i>1$ is uniformly randomly distributed over the interval $[-\gamma,\gamma]$. The x coordinates of the right-hand sides of the slabs of dielectric constant ϵ are given by

$$x_{R,2i-1}=(2i-1)a+e_{2i-1} , \tag{16}$$

where e_{2i-1}/a is uniformly randomly distributed over the interval $[-\gamma,\gamma]$.

For x to the right of our dielectric array we take $\mathbf{E}$ to be given by Eq. (10), and for $x<0$ we take $\mathbf{E}$ to be given by Eq. (11). By using Eqs. (1)–(5), we then find

$$\begin{bmatrix} r \\ 1 \end{bmatrix} = \prod_{i=1}^{N} [\underline{T}(x_{R,2i-1},x_{L,2i-1};\epsilon)] \begin{bmatrix} 0 \\ t \end{bmatrix} , \tag{17}$$

and define the localization length l of our system by

$$l=-\frac{L}{\langle \ln T\rangle} , \tag{18}$$

where $L=(2N-1)a$ is the average length of our system, T is the transmissivity, and $\langle\ \rangle$ represents an average over the statistical distribution function of the disorder in our system.

We have computed localization length given by Eq. (18) for a random system of slabs with $\epsilon=9$, $\gamma=0.2$, and $N=480$ and 960. The frequency dependence of the localization length, shown in Fig. 4, was obtained for frequencies about the upper and lower edges of the lowest-frequency band gap in the photonic band structure of the average, periodic system. The results are very similar to those for the case, considered in Sec. III in which the slab widths were fixed and the dielectric constants were given small random additions. The localization lengths dramatically decrease as the frequency of the electromagnetic radiation enters the region of the band gap for the periodic system. Within the gap, away from the immediate neighborhood of the band edges, the localization length again displays little dependence on frequency.

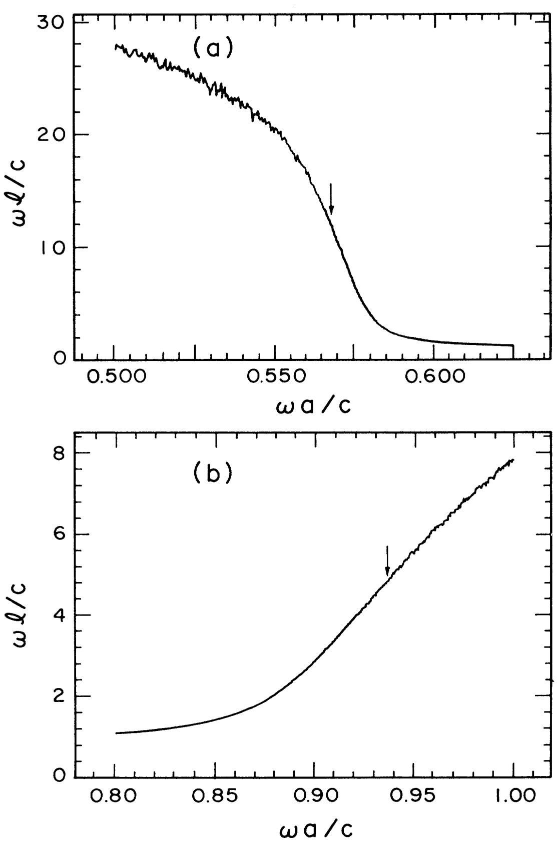

FIG. 4. Plof of (ω/c) (localization length) vs $\omega a/c$ for a system with random slab widths. The results are shown for (a) ω near the lower band-gap edge and (b) ω near the upper band-gap edge. The curves in the figures were computed for an average over 500 random samples.

V. DISORDER AND DISSIPATION

We also wish to study the competition between Anderson localization and internal dissipation effects described by complex dielectric constants. Anderson localization causes the transmissivity of a random sample of length L to decrease with increasing L in such a way that

$$\langle \ln T\rangle \underset{L\to\infty}{\longrightarrow} -\frac{L}{l} , \tag{19}$$

where l is the localization length. Similarly, internal dissipation causes the transmissivity of a periodic sample of length L, in the absence of disorder, to decrease in such a way that

$$\ln T \underset{L\to\infty}{\longrightarrow} -\frac{L}{l_\epsilon} , \tag{20}$$

where

$$l_\epsilon=\frac{a}{\ln\left|\frac{x}{2}-\left[\left[\frac{x}{2}\right]^2-1\right]^{1/2}\right|} , \tag{21}$$

$$x=-\left[\sqrt{\epsilon}+\left[\frac{1}{\epsilon}\right]^{1/2}\right]\sin(\sqrt{\epsilon}\omega a/c)\sin(\omega a/c) + 2\cos(\sqrt{\epsilon}\omega a/c)\cos(\omega a/c) . \tag{22}$$

Equations (20)–(22) are obtained by expressing the transmissivity of the periodic structure in terms of the eigenvalues of the matrix $\underline{T}(x'';x',\epsilon)$ defined by Eqs. (4) and (5).

In the presence of both random disorder and dissipative losses we might expect, roughly, to have the exponential decay of both processes become multiplicative so that

$$\langle \ln T\rangle \underset{L\to\infty}{\longrightarrow} -L\left[\frac{1}{l}+\frac{1}{l_\epsilon}\right]\equiv-\frac{L}{l_T} , \tag{23}$$

where l_T is the decay length of the transmissivity in the presence of both random disorder and dielectric losses. We have investigated this expectation in Figs. 5 and 6.

In Fig. 5 we plot the localization length l as a function of the frequency for the case of the random slab widths treated in Sec. IV, when $\epsilon=\epsilon_1+i\epsilon_2$ with $\epsilon_1=9$ and

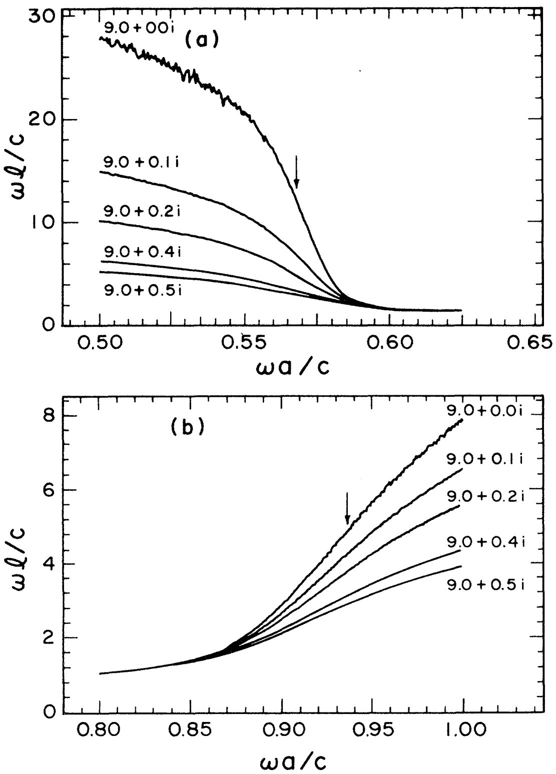

FIG. 5. Same as in Figs. 4(a) and 4(b) but now computed for $\epsilon=9.0$, $9.0+0.1$, $9.0+i0.2$, $9.0+i0.4$, and $9.0i0.5$.

$\epsilon_2=0.1n$ for integers $n=1-5$. For comparison, in Fig. 6 we have plotted l_T, defined by Eq. (23), as a function of frequency in the vicinity of the lower band edge. For the localization length l we have used the result plotted in Fig. 4(a). It is seen that below the gap, i.e., for $(\omega a/c)<0.5678$, the expression for l_T given in Eq. (23) represents the results plotted in Fig. 5(a) quite well. Inside the gap, i.e., for $(\omega a/c)>0.5678$, the relation (23) is not valid, however. From Fig. 5 the behavior of the localization length for frequencies within the gap appears to be determined primarily by the real part of the dielectric constant, and depends negligibly on the imaginary part. For frequencies within the gap l, l_ϵ, and l_T all have essentially the same value. This indicates that the spatial decay of the electromagnetic field amplitude in this region is dominated by the phase-coherent Bragg reflection properties of the average periodic medium rather than by the time-reversal phase coherence associated with Anderson localization.

VI. CONCLUSION

We see that one-dimensional structures of disordered systems which on average are periodic can be effectively used to create highly localized excitations. Specifically, excitations whose frequencies lie in the gaps of the photonic band structure of the underlying periodic system are seen to exhibit extremely short localization lengths. This should also be the case in two- and three-dimensional systems, where the existence of such highly localized states would facilitate the observation of many phenomena related to Anderson localization.

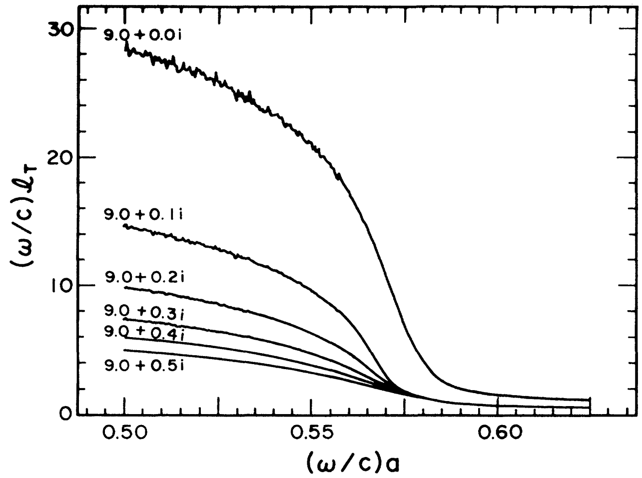

FIG. 6. Plot of $(\omega/c)l_T$ vs $\omega a/c$ obtained from Eqs. (21)–(23) and the data in Fig. 4(a).

We have also studied the exponential decay of localized states in the presence of dielectric losses. The decay length of the transmissivity of such states is found, for states outside the gap, to be simply related to the localization length in the absence of dielectric losses and to the decay length due to dielectric losses in the periodic system. For frequencies in the gap, however, the localization length is found to be relatively insensitive to the presence of dielectric losses.

After this work was completed we received a thesis in which the problem studied in Sec. III of this paper is also investigated, by analytic and numerical methods. The emphasis in Ref. 6 is on the Fabry-Pérot resonances in the localization length that occur at frequencies in the allowed bands of the average periodic system when the average index of refraction of the dielectric layer is a rational number. However, the decrease in the localization length as the frequency of the light enters the region of the gap in the photonic band structure of the average periodic system found in the present work is observed in the results of Ref. 6 as well.

ACKNOWLEDGMENTS

We are grateful to Dr. B. A. van Tiggelen for sending us a copy of his thesis. The work of A.A.M. was supported in part by NSF Grant No. DMR 89-18184. Part of this work was performed under the auspices of the U.S. Department of Energy. This work was also supported by the University of California, Irvine, through an allocation of computer time.

[1]S. John, Phys. Rev. Lett. **58**, 2486 (1987).
[2]S. John, in *Scattering and Localization of Classical Waves in Random Media,* edited by Ping Sheng (World Scientific, Singapore, 1990), p. 4.
[3]S. John, Phys. Today **44**, 32 (1991).
[4]See, for example, Pochi Yeh, *Optical Waves in Layered Media* (Wiley, New York, 1988), pp. 124–125.
[5]S. A. Gredeskul and Yu. S. Kivshar, Phys. Rep. **216**, 1 (1992).
[6]B. A. van Tiggelen, Ph.D. thesis, University of Amsterdam (1992).