Ballistic conductance of interacting electrons in the quantum Hall regime

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We propose a quantitative electrostatic theory for a gate-confined narrow channel of the two-dimensional electron gas in the integer and fractional quantum Hall regimes. Our theory is based on the zero-magnetic-field electrostatic solution, which yields a domelike profile of electron density. This solution is valid when the width of the channel is larger than the Bohr radius in the semiconductor. In a strong magnetic field H, alternating strips of compressible and incompressible liquids are formed in the channel. When the central strip in the channel is incompressible, the conductance G is quantized in units of $e^2/2\pi\hbar$, i.e., there are plateaus in G as a function of the magnetic field H. However, we have found that in a much wider range of magnetic fields there is a compressible strip in the center of the channel. We also argue, based on the exact solution in a simple case, that conductance, in units of $e^2/2\pi\hbar$, of a short and "clean" channel is given by the filling factor in the center of the channel, allowing us to calculate conductance as a function of magnetic field and gate voltage, including both the positions of the plateaus and the rises between them. We apply our theory to a quantum point contact, which is an experimental implementation of a narrow channel.

I. INTRODUCTION

Magnetotransport of the high-mobility two-dimensional electron gas (2DEG) in narrow channels has attracted significant theoretical and experimental attention in recent years.[1] The quantization of conductance has been observed as a function of the magnetic field and channel width,[2] and can be explained by employing the concept of edge states that are formed along the lines of constant potential in a high-mobility 2DEG.[3,4] According to the Landauer-Büttiker transmission approach,[5,6] if one ignores backscattering, conductance is given by the number of edge states which pass through a narrow channel.

The one-electron picture of a channel is based on the assumption that a smooth parabolic potential bends the Landau levels; the position of the edge states is given by the intersection of the Landau levels with the constant Fermi level [see Figs. 1(a)–1(c)]. According to this picture, as the magnetic field is lowered, narrow edge channels appear in pairs in the middle of the channel. At any given magnetic field there is an even number of edge channels, with half of them going in one direction and the other half in the opposite direction; conductance is strictly quantized in units of $e^2/2\pi\hbar$. Thus, the two-terminal conductance G as a function of magnetic field should vary in a steplike manner, with the plateaus connected by steep rises. This prediction of the one-electron picture does not agree with experiment very well even for short and "clean" channels:[7] rises can have the same extent as the plateaus or be even wider. This disagreement casts doubts on the applicability of the one-electron picture of edge states.

The effect of screening in the presence of a magnetic field was included in a qualitative picture of edge states by Beenakker[8] and Chang.[9] They divided the electron gas, confined by a slowly varying external potential, into alternating strips of incompressible and compressible liquids, the former originating from the discontinuities of the chemical potential dependence on the filling factor $\mu(\nu)$. [For the integer quantum Hall effect (IQHE), incompressible and compressible strips correspond to integer and noninteger numbers of filled Landau levels, respectively.] Screening is almost perfect within the compressible strips, which behave like metal strips at constant potential. They are separated by insulatorlike incompressible strips where all the potential drops occur. The works of Beenakker[8] and Chang[9] offer only a qualitative picture of edge channels, but leave open the question of the widths of compressible and incompressible strips. The quantitative approach was developed recently by Chklovskii, Shklovskii, and Glazman,[10] who showed that the width of a strip of incompressible liquid is much smaller than the width of an adjacent strip of compressible liquid.

Chklovskii, Shklovskii, and Glazman solved analytically the electrostatics problem for the gate-induced edge of the 2DEG, exploiting the smallness of the screening length in the 2DEG in comparison with the width $2l$ of depletion layer between the gate and the 2DEG. In the absence of a magnetic field, l is the only relevant scale for the electron-density distribution. The application of a magnetic field does not change this distribution on a rough scale. The only exceptions are narrow strips near the lines where an integer number of Landau levels is fully occupied. A small portion of charge is redistributed forming incompressible *dipolar strips* in the vicinity of those lines. The dipolar strip produces a steep drop in

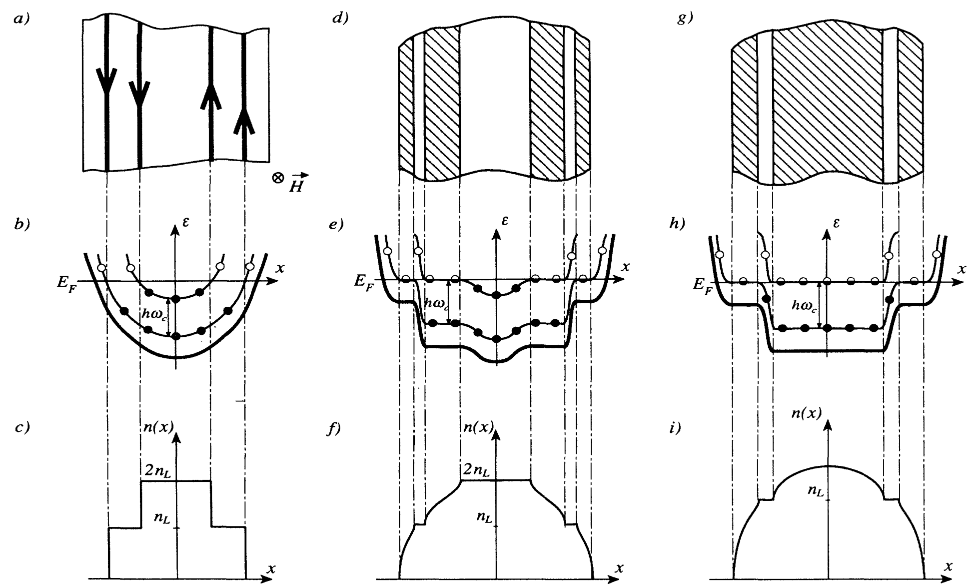

FIG. 1. Structure of a narrow 2DEG channel in the IQHE regime. (a)–(c), One-electron picture of the edge states. (a) Top view of the narrow 2DEG channel. Arrows designate electron flow direction in the edge states. (b) Adiabatic bending of Landau levels by a smooth external potential. Energy is measured from the Fermi level. Circles represent local filling of the Landau levels: ●, occupied and ○, empty. (c) Electron-density distribution in the channel. (d)–(i) Self-consistent electrostatic picture. (d)–(f) Narrow channel of the 2DEG in the I state. (d) Top view of the 2DEG channel. Shaded strips represent areas with a noninteger filling factor (compressible strips). Unshaded strips represent integer filling factor regions (incompressible liquid). (e) Bending of the electrostatic potential energy and the Landau levels. Circles represent local filling of the Landau levels: ●, occupied; ◒, partially occupied; and ○, empty. (f) Electron-density distribution in the channel. (g)–(i) Narrow channel of the 2DEG in the C state. (g) Top view of the 2DEG channel. (h) Bending of the electrostatic potential energy and the Landau levels. (i) Electron-density distribution in the channel.

the electrostatic potential which brings the next Landau level to the Fermi level. Chklovskii, Shklovskii, and Glazman obtained a complete analytical description of the dipolar strip, which agreed with the calculations performed by Kane[11] for a slightly different geometry. Similar results have been obtained by Efros[12] in the theory of screening of a random long-range potential.

In this paper we present a quantitative electrostatic treatment of the narrow channel formed by the gate-induced depletion. In this case the electron density has a domelike shape with characteristic width b which is still much larger than the screening length r_D (equal to the effective Bohr radius a_B in the semiconductor). At the periphery of the channel our results do not differ qualitatively from the description of the edge of the 2DEG occupying a half plane. New phenomena appear in the center of the channel near the maximum in electron density. Depending on the situation in the center, the channel can be in two different states. In the first state, there is a strip of incompressible liquid in the center of the channel and the total number of compressible strips is even [see Figs. 1(d)–1(f)]. We refer to this situation as an I state. In the second state the center is occupied by compressible liquid and there is an odd number of compressible strips, Figs. 1(g)–1(i). We call this a C state.

Let us start from the C state at a strong magnetic field and consider a transition to the I state with decreasing magnetic field. When the magnetic field is lowered, the topmost Landau level becomes completely filled in the middle, which signals the appearance of the new incompressible strip in the center (C-I transition). Electrons that would be in the middle in the absence of a magnetic field are now pushed aside due to the gap in the electron spectrum. Charge redistribution creates what we call a *quadrupolar strip:* an additional charge density is positive in the center and negative on the sides. The potential from the quadrupolar strip lowers the first empty Landau level, and with decreasing magnetic field eventually brings it to the Fermi level. This induces the appearance of the new compressible strip in the center (I-C transition) which splits the quadrupolar strip into two dipolar strips of opposite polarity. In this work we present an analytic solution for the quadrupolar strip based on the existence of the small parameter a_B/b, and calculate the values of magnetic field at which all the described C-I and I-C transitions occur. The range of magnetic field at which an I state exists turns out to be narrower than the range of the adjacent C state.

The ultimate goal of this paper is to formulate a theory for the magnetoconductance of a narrow channel. The two-probe conductance G in the I state was considered by Beenakker[8] for the fractional quantum Hall regime. An extension of his approach to the case of interacting electrons in the integer quantum Hall regime gives

$$G=\frac{e^2}{2\pi\hbar}k \ , \tag{1}$$

where $k=0,1,2,3\ldots$ is the number of Landau levels occupied in the central incompressible strip. This result coincides with the prediction of the one-electron picture of edge states,[3,6] but is valid only for the range of magnetic fields corresponding to the I state. In the C state a new question of the conductance of the central compressible strip arises. The contribution of the partially filled Landau level to the two-terminal conductance measurement depends crucially on the presence of disorder. In a long channel with a sufficient degree of disorder, the conductance of the central strip is much smaller than $e^2/2\pi\hbar$ and can be neglected. Here we deal with the opposite case of the short channel and therefore neglect the influence of disorder. In this case the two-terminal conductance is quantized only in the I state. We calculate the widths of the plateaus and the shape of the rises using a general expression for conductance,

$$G=\frac{e^2}{2\pi\hbar}\nu_H(0) \ . \tag{2}$$

In this equation the occupation number $\nu_H(0)=n_H(0)/n_L$, where $n_H(0)$ is the electron concentration in the center of the channel as a function of magnetic field, and n_L is the electron density of one completely filled Landau level. Equation (2) is proven below for one simple case, and we make the hypothesis that it is true, in general. In the I state Eq. (2) is reduced to Eq. (1).

One can view Eq. (2) as a simple generalization of the Landauer-Büttiker transmission approach to a C state. One can imagine that the central compressible strip is symmetrically divided into an even number of "substrips" or "subchannels" running along the lines of constant density. Then, as in the conventional transmission approach, subchannels on the right and left sides of the compressible liquid acquire the electrochemical potential of the two opposite terminals. The electrochemical potential drop occurs in the center of the strip, where the whole nonequilibrium current is concentrated. This explains why the two-terminal conductance is proportional to the concentration in the center of the channel.

It is natural to present the dependence of G on H by using, instead of a magnetic field, an occupation number $\nu(0)=n(0)/n_L$, where $n(0)$ is the density in the center of the channel at $H=0$. The corresponding plot (see Sec. V) shows plateaus of constant $\nu_H(0)$ when the channel is in the I state, as well as the deviation of $\nu_H(0)$ from $\nu(0)$ on the rises corresponding to the C state. Note that the plateaus are substantially narrower than the rises. This is the main result presented in this paper.

This seems to differ from the conventional transmission approach which predicts almost vertical rises for a one-dimensional channel. We would like to explain the origin of this discrepancy. Both theories give steep rises as a function of the Fermi level. The difference is that if one takes a parabolic self-consistent potential (this was usually done in the one-electron picture) then steep rises as a function of the Fermi level translate into steep rises as a function of the external parameters such as magnetic field H or gate voltage V_g. In our theory the self-consistent potential (or better to say electron energy) is very peculiar: due to the metallic screening it is constant within a compressible strip. This means that steep rises as a function of the Fermi level translate into smooth rises as a function of external parameters H and V_g.

We begin (Sec. II) with the model for the gate-induced 2DEG channel and the charge distribution at zero magnetic field. In Sec. III we study the influence of high magnetic field on the distribution of electron density. We consider in detail the redistribution of charge near the center of the channel forming the quadrupolar strip. In Sec. IV we discuss magnetotransport in a narrow channel under the conditions of a two-terminal measurement. Section V contains the derivation and discussion of Eq. (2). In Sec. VI we apply our theory to a quantum point contact which is a practical realization of a narrow channel. Section VII contains our major conclusions.

II. ELECTRON-DENSITY DISTRIBUTION AT ZERO MAGNETIC FIELD

We adopt a simplified model of the split-gate device on the GaAs/Al_xGa_{1-x}As heterostructure, proposed by Glazman and Larkin,[13] and Larkin and Shikin.[14] In this model (see Fig. 2), ionized donors are represented by a uniform positive background of constant two-dimensional charge density en_0. Far from the gates the 2DEG compensates for the positive background, so the electron concentration in the bulk is equal to n_0. The split gate is represented by two semi-infinite metal planes separated by the gap centered at $x=0$. The width of this gap is $2d$. Negative voltage V_g is applied to both halves of the gate, depleting the 2DEG underneath them, and confining the electrons to a narrow channel. The whole system is translationally invariant along the y axis. In the model considered the positive background, the split gate, and the 2DEG are all in the same plane $z=0$ (Fig. 2), a simplification we will justify later. The half-space $z<0$ is occupied by a semiconductor with a high dielectric constant $\epsilon \gg 1$. Since a_B is much less than the characteristic length scale of the density distribution, the screening ra-

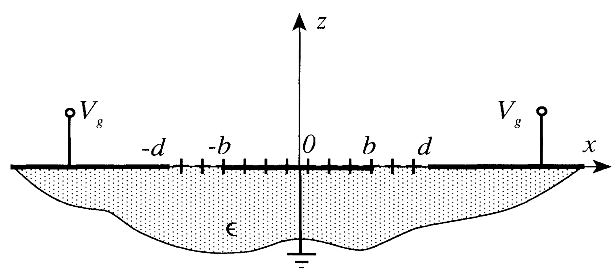

FIG. 2. Electrostatic system formed in a narrow 2DEG channel. Thick lines represent conductors: split gate at potential V_g with the grounded 2DEG in the middle. Pluses represent a uniform positive background due to ionized donors. Dotted area is occupied by a semiconductor with high dielectric constant ϵ, while the half-space $z>0$ is vacuum.

dius of the 2DEG is taken to be zero. Then the 2DEG in the channel behaves much like a metal strip of width $2b$, differing only in that the edges of the 2DEG can move. Thus we have to include a condition for them to be in mechanical equilibrium: the x component of electric field should be zero both to the left and to the right of each edge.

The problem is reduced to the solution of the Laplace equation $\Delta\phi=0$ in the half-space $z<0$, with mixed boundary conditions,

$$\phi(x,z=0)=\begin{cases}0\ , & |x|<b\\ V_g\ , & |x|>d\ ,\end{cases} \tag{3}$$

$$\left.\frac{d\phi(x,z)}{dz}\right|_{z\to-0}=-E_z(x,z)|_{z\to-0}=\frac{4\pi e n_0}{\epsilon}\ ,\quad b<|x|<d\ . \tag{4}$$

Positioning of all the charges at the interface of two media with dielectric constants ϵ and 1 leads to a factor $4\pi/(\epsilon+1)\approx 4\pi/\epsilon$ in Eq. (4). In order to ensure mechanical equilibrium of the 2DEG edges at $x=\pm b$, we set

$$E_x(x,z=0)|_{x\to-b-0}=E_x(x,z=0)|_{x\to b+0}=0\ . \tag{5}$$

The solution of the Laplace equation satisfying conditions (3), (4), and (5) was given by Larkin and Shikin.[14] They found an electron-density distribution of

$$n(x)=n_0\left[\frac{b^2-x^2}{d^2-x^2}\right]^{1/2} \tag{6}$$

in the 2DEG. The half-width of the 2DEG strip b can be found by solving the equation

$$V_g=-\frac{4\pi e n_0 d}{\epsilon}\left[E(\sqrt{1-b^2/d^2})-\frac{b^2}{d^2}K(\sqrt{1-b^2/d^2})\right]\ , \tag{7}$$

where $E(x)$, $K(x)$ are complete elliptic integrals. It was also pointed out in Ref. 14 that near the pinch off, when $b\ll d$, the electron-density distribution is close to that formed in a parabolic confining potential in the perfect screening approximation,

$$n(x)\approx n_0\frac{(b^2-x^2)^{1/2}}{d}\ . \tag{8}$$

In the opposite limit, $2l\equiv d-b\ll d$, the two edges can be treated independently, and each of them is described by the formulas of Ref. 10.

Bringing all the charges into the same plane is justified if $d-b$ and b are much larger than the spacer thickness and the distance between the gate and the 2DEG plane. Let us check this condition for the channel of lithographic width $2d=5000$ Å: for $V_g=-1$ V, $n_0=4\times10^{11}$ cm^{-2}, and $\epsilon=12.5$, we find $2b=2600$ Å. This length, as well as $d-b$, is much larger than the spacer layer thickness and the distance from the 2DEG to the gate. These numbers also confirm the validity of the perfect screening approximation since in GaAs $a_B=100$ Å $\ll b$.

Despite the fact that this electron-density distribution has been found for the gate-confined 2DEG, we believe that the result can also be applied to etched structures. In that case the half-width of the forbidden gap would take the place of the gate voltage in Eqs. (3) and (7) due to the pinning of the Fermi level by the surface states.

III. NARROW CHANNEL IN A STRONG MAGNETIC FIELD: FORMATION OF THE QUADRUPOLAR STRIP

Let us consider the effect of a strong magnetic field H on the 2DEG in a narrow channel, while neglecting electron spin. Due to the smallness of the parameter $\hbar\omega_c/eV$ ($\omega_c=eH/m_{\rm eff}c$ is a cyclotron frequency) at any reasonable magnetic field, we expect that the electron-density distribution (6) obtained from electrostatics will not be altered significantly. This is because of the huge amount of work which must be performed against electrostatic forces in order to produce any variation.

The only effect of the magnetic field on the electron-density distribution is due to the periodic dependence of screening properties of the 2DEG on the filling factor ν, caused by the oscillations in the density of states. The density of states is given by a set of δ functions centered at $(k-\frac{1}{2})\hbar\omega_c$. The screening length r_D as a function of the filling factor takes the following form:

$$r_D=\begin{cases}\infty\ , & \nu=k\\ 0\ , & \nu\neq k\ ,\end{cases} \tag{9}$$

i.e., screening at integer filling factors is absent while at noninteger ν screening is very strong. This leads to the formation of the alternating strips of compressible and incompressible liquid.[8,9] The electrostatic potential remains constant throughout any one compressible strip, whereas it changes by $\hbar\omega_c$ between the inner and outer edges of an incompressible strip. As was shown in Ref. 10, incompressible strips are narrower than compressible ones. Their locations can be found by solving the equation $n(x)=kn_L$ using $n(x)$ from Eq. (6).

First, we would like to discuss qualitatively what happens as the magnetic field is lowered slowly, starting from a value high enough for all electrons in the channel to be on the first Landau level. Our results will be presented in terms of the quantity

$$\nu(0)=\frac{n(0)}{n_L}=\frac{b}{d}\frac{n_0}{n_L}\ , \tag{10}$$

where $n(0)=n(x)|_{x=0}$ stands for the electron concentration in the absence of magnetic field. Initially, when $\nu(0)<1$ the electron-density distribution is well described by Eq. (6) and is illustrated in Fig. 3(a). [Here we ignore the fractional quantum Hall effect (FQHE).] At the moment, when $\nu(0)=\nu_1=1$ a flat region, $n(x)=n_L$, starts to develop in density distribution [Fig. 3(b)], thus indicating the first C-I transition. The new density distribution $n_H(x)$ can be thought of as the solution $n(x)$ obtained without magnetic field [Eq. (6)], plus some redistributed density $\Delta n(x)$ [see Fig. 4(a)]. The distribution $n(x)$ can be approximated near its maximum as

$$n(x)=n(0)+\tfrac{1}{2}n''x^2\ , \tag{11}$$

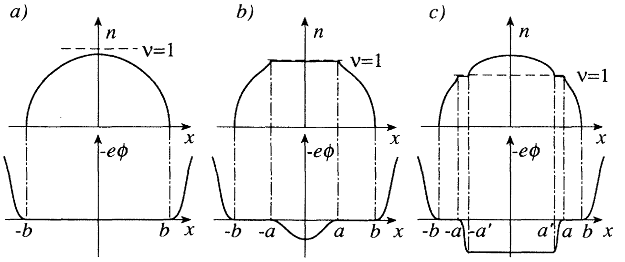

FIG. 3. Evolution of the potential and 2DEG density distribution with decreasing magnetic field.

where

$$n''=\left.\frac{d^2n}{dx^2}\right|_{x=0} .$$

The redistributed electron density has the form

$$\Delta n(x)=n_L-n(x)=n_L-n(0)-\tfrac{1}{2}n''x^2 . \tag{12}$$

One can see that $\Delta n(0)<0$ when $n_L<n(0)$, and that $\Delta n(x)$ changes sign at some x [$n''<0$, see Fig. 4(b)]. Equation (12) is valid only within the incompressible strip. One can see from Eq. (12) that the magnetic-field-induced redistribution of charge has a quadrupolar character: redistributed charge $-e\Delta n(x)$ is positive in the middle and negative on the sides. If one considers the dipolar strip of Refs. 10–12 as reminiscent of a p-n junction, then the quadrupolar strip can be said to resemble a p-n-p structure.

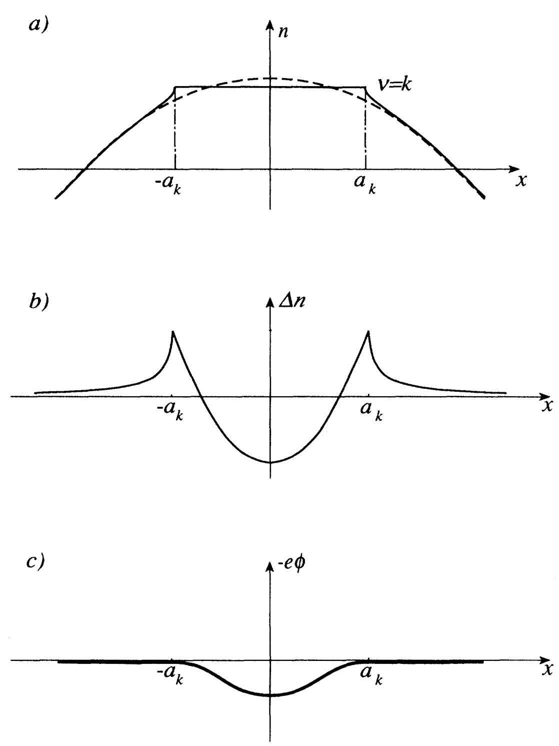

FIG. 4. The quadrupolar strip. (a) Magnetic-field-induced electron-density redistribution near the center of the channel. (b) Magnetic-field-induced additional electron density in the quadrupolar strip. (c) Electron potential energy as a function of position.

The quadrupolar strip creates a minimum of potential energy in its center [see Figs. 4(a) and 3(b)]. As $\nu(0)$ increases further and the depth of this minimum reaches the value of $\hbar\omega_c$, electrons start to fill up the second Landau levels [$\nu(0)=\nu_1'$]. This leads to the formation of a new compressible strip in the middle of the incompressible one, indicating the first I-C transition. Simultaneously, the quadrupolar strip breaks into two dipolar strips of opposite polarity.

Let us estimate the width of the incompressible strip at the I-C transition. The depth of the potential well $\Delta\phi$ in the center of the quadrupolar strip is of the order of the characteristic electric field $E\approx e|n''|a^2/\epsilon$ times ea, and the new compressible strip appears at $x=0$ when $\Delta\phi=\hbar\omega_c$ or

$$\frac{e^2|n''|a^3}{\epsilon}=\hbar\omega_c . \tag{13}$$

Equation (13) gives

$$a\approx\left[\frac{\epsilon\hbar\omega_c}{e^2|n''|}\right]^{1/3}\approx\frac{(a_Bb^2)^{1/3}}{[1-(b/d)^2]^{1/3}} , \tag{14}$$

where $a_B=\epsilon\hbar^2/m_{\text{eff}}e^2$ is the Bohr radius in the semiconductor and we used the expression

$$|n''|=n_0\frac{1}{bd}\left[1-\frac{b^2}{d^2}\right]=n(0)\frac{1}{b^2}\left[1-\frac{b^2}{d^2}\right] \tag{15}$$

that follows from Eq. (6).

Using Eq. (14) we obtain for the interval $\Delta\nu_1$ of the filling factor $\nu(0)$ corresponding to the first I state

$$\Delta\nu_1=\nu_1'-\nu_1\approx\frac{|n''|a^2}{n_L}\approx\left[\frac{a_B}{b}\right]^{2/3}\left[1-\frac{b^2}{d^2}\right]^{1/3} , \tag{16}$$

up to a numerical factor which cannot be obtained in this estimate. One can see that in the limit of wide channel $\Delta\nu_1$ goes to zero.

We now move from the order-of-magnitude estimates to the exact analytical theory of the kth quadrupolar strip, formed when k Landau levels are filled in the center. This theory yields numerical factors omitted in Eqs. (14) and (16). The theory is based on the fact that the half-width of the incompressible strip a is much smaller than the widths of the adjacent compressible strips. [One can see this from Eq. (14), keeping in mind that a_B is the smallest length in the problem.] The charge distribution can then be thought of as the one described by Eq. (6) plus some additional charge which is localized in the vicinity of $x=0$ due to the magnetic field.

We can now find the additional charge distribution by solving an electrostatics problem, where compressible strips are represented by the grounded plates of a two-dimensional capacitor. These plates are taken to be semi-infinite because they are much wider than the incompressible strip in between. The gap between two plates (of the width $2a$) is filled by a charged insulator. We must solve the Laplace equation in the half-space $z<0$ with mixed boundary conditions,

$$\phi(x,z=0)=0\ ,\quad |x|>a\ , \tag{17}$$

$$E_z(x,z)|_{z\to -0}=-\frac{2\pi e}{\epsilon}\left[\frac{n''}{2}x^2+[\nu(0)-k]n_L\right]\ ,\quad |x|<a\ . \tag{18}$$

Here we use a boundary condition (18) which differs from Eq. (4) by a factor of 2. This is because we anticipate the width of the quadrupolar strip to be much smaller than the distance from the 2DEG plane to the surface, which implies that the electric field is concentrated within the semiconductor. The approximation of semi-infinite plates can only be justified if we find a solution in which electric field E_z decays at large distances from the gap. Thus we set a condition

$$\lim_{x\to+\infty}E_z(x,0)=\lim_{x\to-\infty}E_z(x,0)=0\ . \tag{19}$$

We also have the condition of mechanical equilibrium at the edges which is similar to Eq. (5),

$$E_x(x,z=0)|_{x\to a-0}=E_x(x,z=0)|_{x\to -a+0}=0\ . \tag{20}$$

This problem can be solved by employing the methods of complex analysis. (As an alternative, the less general but very simple method involving Chebyshev polynomials is presented in the Appendix.) Because ϕ is the solution of the Laplace equation, it can be viewed as the imaginary part of an analytic function $F(\zeta)$, where $\zeta=x+iz$. The electric field is then given by

$$E_x=-\mathrm{Im}\left[\frac{dF}{d\zeta}\right]\ , \tag{21}$$

$$E_z=-\mathrm{Re}\left[\frac{dF}{d\zeta}\right]\ . \tag{22}$$

The solution satisfying conditions (17) and (18) and having the correct behavior at infinity (19) is given by

$$\frac{dF}{d\zeta}=\frac{2\pi e}{\epsilon}\left\{[\nu(0)-k]n_L\left[1-\frac{\zeta}{\sqrt{\zeta^2-a^2}}\right]+\frac{n''}{2}\left[\zeta^2-\frac{\zeta^3-\zeta a^2/2}{\sqrt{\zeta^2-a^2}}\right]\right\}\ . \tag{23}$$

In general, this solution has singularities in the electric field at $x=\pm a$. Therefore it satisfies the equilibrium condition (20) only if

$$[\nu(0)-k]n_L+n''a^2/4=0\ . \tag{24}$$

Then

$$\frac{dF}{d\zeta}=-\frac{\pi e n''}{\epsilon}\left\{\frac{a^2}{2}-\zeta^2+\zeta\sqrt{\zeta^2-a^2}\right\}\ . \tag{25}$$

By making use of Eq. (22), we find the redistributed electron density in the quadrupolar strip [see Fig. 4(b)]

$$\Delta n(x)=\frac{n''}{2}\begin{cases}\dfrac{a^2}{2}-x^2\ , & |x|<a\\[2mm] \dfrac{a^2}{2}-x^2+x\sqrt{x^2-a^2}\ , & |x|>a\ .\end{cases} \tag{26}$$

It follows from Eq. (25) that the electrostatic potential in the strip is given by

$$\phi(x,z=0)=-\frac{\pi e n'' a^3}{3\epsilon}[1-(x/a)^2]^{3/2}\ , \tag{27}$$

which has a maximum at $x=0$. Electrons start to fill up the $k+1$th Landau level $[\nu(0)=\nu_k']$ when

$$\phi(x=0,z=0)=\frac{\pi e|n''|a_k^3}{3\epsilon}=\hbar\omega_c/e\ . \tag{28}$$

Thus,

$$a_k=\left[\frac{3\epsilon\hbar\omega_c}{\pi e^2|n''|}\right]^{1/3}\ , \tag{29}$$

$$\Delta\nu_k=\nu_k'-k=\frac{|n''|a_k^2}{4n_L}=\left[\frac{9|n''|}{16n_L}\left[\frac{\epsilon\hbar\omega_c}{2\pi e^2 n_L}\right]^2\right]^{1/3}\ . \tag{30}$$

Finally, by noticing that $\epsilon\hbar\omega_c/2\pi e^2 n_L=a_B$, recalling that a_B/b is a small parameter, and using Eq. (15) we find

$$a_k=\left[\frac{6}{k(1-b^2/d^2)}\right]^{1/3}(a_B b^2)^{1/3}\ , \tag{31}$$

$$\Delta\nu_k=(\nu_k'-k)=[\tfrac{9}{16}k(1-b^2/d^2)]^{1/3}\left[\frac{a_B}{b}\right]^{2/3}\ . \tag{32}$$

Equations (31) and (32) give the numerical factors omitted in the qualitative derivation, as well as the dependence on the total number of filled Landau levels in the channel k. Let us estimate $\Delta\nu_k$ and a_k using $b=1300$ Å $\ll d$ and $a_B=100$ Å. In this case we obtain $a_k=1000/k^{1/3}$ Å and $\Delta\nu_k=0.15k^{1/3}$. One can see that the inequality $a_B\ll a_k$ holds when k is not very large, while at the same time the inequality $a_k\ll b$ is not valid for small k. It is possible to describe the problem with an exact system of equations, valid also when $a_k\approx b$. The numerical solution of this system of equations for $k=1$ yields a value of $\Delta\nu_1$ which is only 10% greater than the one given by Eq. (32).

Let us now verify the validity of an important assumption in our theory, namely that the compressible strips on both sides of the central one screen well on the scale of a_k, i.e., behave like a good metal. We see two conditions of such behavior.

The first condition is related to the discreteness of the electron gas. The 2DEG cannot screen well on the distances less than an average distance between electrons. A similar statement can be made about the screening by holes of the almost filled kth Landau level. Therefore the hole concentration on the kth Landau level at a distance a_k from the central incompressible strip is larger than a_k^{-2}, i.e.,

$$n''a_k^4 \gg 1 \ . \quad (33)$$

The second condition is related to a large characteristic length (size) of the wave function for electrons on high Landau levels (large k). In the quasiclassical approach valid for $k \gg 1$ this length is given by the cyclotron radius $\lambda\sqrt{k}$, where $\lambda = \sqrt{\hbar c/eH}$ is the magnetic length. Local relationships between the filling factor $\nu(x)$ and the charge density, as well as between the electrostatic potential and electron energy, used in our theory are valid only at length scales larger than $\lambda\sqrt{k}$. This is why for the applicability of our theory the inequality

$$\lambda\sqrt{k} \ll a_k \quad (34)$$

should hold.

One can show that at $n(0)a_B^2 > 1$, condition (34) is violated earlier than condition (33) when the magnetic field is lowered. Using Eq. (31) we can rewrite (34) in the form

$$k < k_{c1} \equiv \left[\frac{b}{a_B} \right]^{1/2} [n(0)a_B^2]^{3/8} \ . \quad (35)$$

This means that for the parameters used above, Eqs. (31) and (32) are valid for the first several plateaus.

So far we have considered the case of spinless electrons. Our theory was based on the presence of a discontinuity in the chemical potential $\Delta\mu_k$ equal to $\hbar\omega_c$ at any integer occupation number k. In a real situation, however, $\Delta\mu_k$ is not equal to $\hbar\omega_c$, and the existence of an electron spin makes $\Delta\mu_k$ explicitly dependent on k. In particular, $\Delta\mu_k$ at odd k is determined by spin splitting; therefore we expect it to be smaller than $\Delta\mu_k$ at even k. In order to find a_k and $\Delta\nu_k$ for a given discontinuity in the chemical potential, we substitute $\Delta\mu_k$ in place of $\hbar\omega_c$ in Eqs. (29) and (30). This gives us the correct values of a_k and $\Delta\nu$,

$$a_k = \left[\frac{3\epsilon\Delta\mu_k}{\pi e^2 |n''|} \right]^{1/3} , \quad (36)$$

$$\Delta\nu_k = \nu_k' - k = \left[\frac{9|n''|}{16 n_L} \left[\frac{\epsilon\Delta\mu_k}{2\pi e^2 n_L} \right]^2 \right]^{1/3} . \quad (37)$$

At low enough temperatures, formation of the incompressible liquid becomes possible at fractional filling factors $f = p/q$, where q is an odd number and p is an integer number. This is due to the discontinuity in the chemical potential related to the FQHE. The number of the incompressible strips formed is determined by the temperature and the level of disorder. The incompressible liquid is characterized by the energy gap necessary to create a pair of quasiparticles with the charges $\pm e/q$,

$$\Delta_f = c_f \frac{e^2}{\epsilon\lambda} , \quad (38)$$

where c_f is a small numerical factor. Thus the discontinuity in the chemical potential for the electrons $\Delta\mu_f$ at $\nu = f$ is

$$\Delta\mu_f = q\Delta_f = qc_f \frac{e^2}{\epsilon\lambda} \ . \quad (39)$$

Substituting Eq. (39) in Eqs. (36) and (37), we find

$$a_f = \left[\frac{6qc_f}{f(1-b^2/d^2)} \right]^{1/3} (\lambda b^2)^{1/3} , \quad (40)$$

$$\Delta\nu_f = [\tfrac{9}{16} f (qc_f)^2 (1-b^2/d^2)]^{1/3} \left[\frac{\lambda}{b} \right]^{2/3} . \quad (41)$$

IV. TWO-TERMINAL MAGNETOCONDUCTANCE

We now consider magnetoconductance of a short channel where disorder-caused scattering can be totally neglected. Conductance of such a channel in the I state was derived by Beenakker for the fractional quantum Hall regimes.[8] Applying similar ideas to the integer quantum Hall regime gives quantization of conductance in units of $e^2/2\pi\hbar$, which is the same result as in the one-electron picture of edge states.[3,4] As shown above, intervals of $\nu(0)$ in which the I state exists $[\nu_k < \nu(0) < \nu_k']$ are quite narrow [Eq. (32)]. At other values of $\nu(0)$ the channel is in the C state, meaning that there is a compressible liquid strip in the center of the channel. We could not find in the literature any discussion of the conductance of such a strip. We calculate its conductance in the next section, accounting for Coulomb interaction in the mean-field framework only, and arrive at a very simple expression [Eq. (2)] for G. It gives a very natural and continuous transitions between plateaus. At this point, we are not able to prove Eq. (2) rigorously for an electron liquid in which correlations are allowed, but we believe that Eq. (2) is generally true. In this section, we will calculate $\nu_H(0)$ and then $G(H)$, using Eq. (2) as a plausible hypothesis.

If $\nu(0) < 1$, all the electrons are on the first Landau level and have a very short screening radius (we neglect the FQHE here). This means that the charge distribution is given by Eq. (6), the same as at zero magnetic field. Therefore $\nu_H(0) = \nu(0)$, and from Eq. (2) we find

$$G = \frac{e^2}{2\pi\hbar} \nu(0) \text{ at } \nu(0) < 1 \ . \quad (42)$$

At $\nu(0) > 1$, incompressible strips are present and the calculation of $\nu_H(0)$ in the C state becomes more complicated. We will perform it in the case when the central compressible strip is narrower than the two adjacent incompressible strips. Then, following the approach of Sec. III we consider an electrostatics problem of two conducting semiplanes representing compressible regions and separated by a gap of width $2a$. However, we now have a conducting strip of the width $2a'$ in the gap between them. Additional charge in the incompressible region $a' < |x| < a$ is described by Eq. (12), so we have now to solve the Laplace equation with the following boundary conditions:

$$\phi(x, z=0) = \begin{cases} 0 , & |x| > a \\ \dfrac{\hbar\omega_c}{e} , & |x| < a' , \end{cases} \quad (43)$$

$$E_z(x,z)|_{z \to -0} = -\frac{2\pi e}{\epsilon} \left[\frac{n''}{2} x^2 + [\nu(0) - k] n_L \right] , \quad a' < |x| < a \ . \quad (44)$$

To ensure the mechanical equilibrium of the 2DEG edges at $|x|=a',a$, we set

$$E_x(x,0)|_{x\to -a+0}=E_x(x,0)|_{x\to a-0} =E_x(x,0)|_{x\to -a'-0} =E_x(x,0)|_{x\to a'+0}=0 \ . \qquad (45)$$

The condition of the proper behavior at infinity is given by

$$\lim_{x\to +\infty} E_z(x,0)=\lim_{x\to -\infty} E_z(x,0)=0 \ . \qquad (46)$$

Once again we use complex variables and find the solution in terms of $dF/d\zeta$, which is related to the electric field as described by Eqs. (21) and (22),

$$\frac{dF}{d\zeta}=\frac{2\pi e}{\epsilon}\left\{[\zeta^2-\sqrt{(a^2-\zeta^2)(a'^2-\zeta^2)}]\frac{n''}{2} +[\nu(0)-k]n_L\right\} , \qquad (47)$$

where a and a' should also satisfy

$$[\nu(0)-k]n_L+n''\frac{a^2+a'^2}{4}=0 \qquad (48)$$

in order to fulfill (46). Also,

$$\frac{\pi e|n''|}{\epsilon}\int_{a'}^{a}dx\sqrt{(a^2-x^2)(x^2-a'^2)}=\frac{\hbar\omega_c}{e} \ . \qquad (49)$$

The last equation can be rewritten in terms of elliptic integrals[18]

$$\frac{\pi e|n''|a}{3\epsilon}\{(a^2+a'^2)E[\sqrt{1-(a'/a)^2}] -2(a')^2K[\sqrt{1-(a'/a)^2}]\}=\frac{\hbar\omega_c}{e} \ . \qquad (50)$$

From Eq. (47) the difference between the electron density at $x=0$ with and without the magnetic field is

$$\nu(0)-\nu_H(0)=\frac{|n''|(a-a')^2}{4n_L}=\frac{(a-a')^2}{a^2+a'^2}[\nu(0)-k] \ . \qquad (51)$$

Equations (48), (50), and (51) represent a complete system from which $\nu_H(0)$ can be found. In the two limiting cases, it yields equations consistent with the earlier results. Setting $a'=0$ in Eq. (50), one arrives at Eq. (28). When $a-a'\ll a$, Eq. (50) is reduced to

$$\frac{\pi^2 e|n''|}{4\epsilon}(a-a')^2a=\frac{\hbar\omega_c}{e} \ . \qquad (52)$$

By making use of Eq. (51) we find

$$\nu(0)-\nu_H(0)=\frac{\epsilon\hbar\omega_c}{\pi^2e^2an_L}=\frac{2}{\pi}\frac{a_B}{a} =\frac{\sqrt{2}}{\pi}\frac{a_B}{b}\left\{\frac{\nu(0)[1-(b^2/d^2)]}{[\nu(0)-k]}\right\}^{1/2} \ . \qquad (53)$$

In this case, we deal with the two narrow dipolar strips which appeared as a result of the splitting of the quadrupolar strip. Indeed, by substituting $n'=n''a$ one can verify that Eq. (52) yields a dipolar strip width $a-a'$ in agreement with Eq. (20) of Ref. 10. (We remind the reader that in Ref. 10 $\epsilon/2$ was used instead of ϵ.) The change of concentration produced by the dipolar strip in the neighboring compressible strips was also found in Ref. 10. At distance a which is much larger than the dipolar strip width, the change in concentration is

$$\Delta n=\frac{\epsilon\hbar\omega_c}{2\pi^2e^2}\frac{1}{a} \ . \qquad (54)$$

In our case, *two* dipolar strips of opposite polarities give equal decrements of concentration in the center of the channel. Indeed Eq. (53) gives Δn twice that of Eq. (54). As $\nu(0)$ grows, two dipolar strips move farther away from the center and their effect on $\nu_H(0)$ decreases.

In Fig. 5 we plot $\nu_H(0)$ as a function of $\nu(0)$, as obtained from the numerical solution of the system of Eqs. (48), (50), and (51) separately for each interval $k<\nu(0)<k+1$. We also calculate $\nu_H(0)$ as a function of gate voltage at fixed values of the magnetic field (Fig. 6).

One may wonder how good the approximation of the nearest dipolar strips is at low magnetic fields (large k). Indeed, distant dipolar strips also contribute to $n_H(0)$, but because of their large number and relatively slow decay of Δn with distance [see Eq. (54)], the absolute value of their contribution may be comparable to, or even larger than, that of the nearest strips. However the contribution of distant strips is monotonic in the magnetic field. We will discuss this briefly although it is more difficult to observe than the oscillatory contribution of the nearest ones.

The monotonic contribution of the distant dipolar

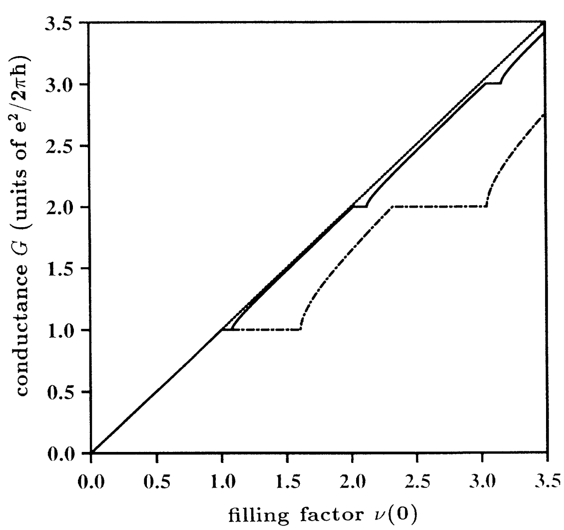

FIG. 5. Two-terminal conductance of a narrow channel as a function of the filling factor in the center of the channel $\nu(0)=n(0)/n_L$. Dotted line is $\nu_H(0)=\nu(0)$. Solid line corresponds to $a_B/b=0.05$; dashed-dotted line corresponds to $a_B/b=0.5$.

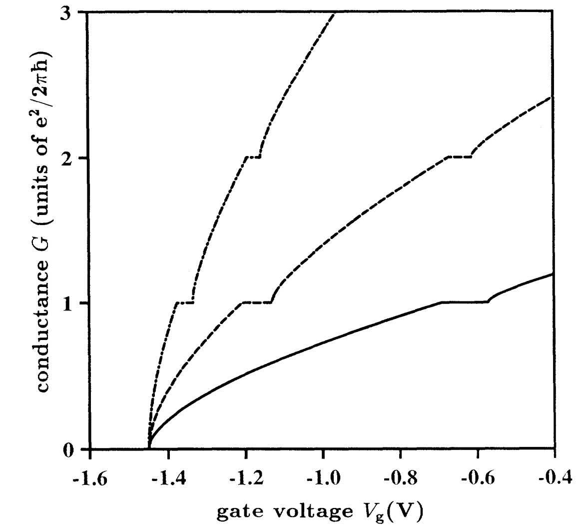

FIG. 6. Two-terminal conductance of a narrow channel $2d = 5000$ Å as a function of gate voltage V_g at different bulk filling factors ν_0 (determined by applied magnetic field). Solid line, $\nu_0 = 1.5$; dashed line, $\nu_0 = 3$; dashed-dotted line, $\nu_0 = 6$.

strips is intimately related to the small difference in the electron distribution between the extreme quantum limit $[\nu(0) < 1]$ and the zero magnetic field $[\nu(0) \to \infty]$. Up to now, we have not distinguished between these two regimes, describing both in the perfect screening approximation in which the charge distribution is given by Eq. (6). Actually, in both cases the screening radius is finite and screening is not perfect. Because of this, the channel is somewhat broader and $n(0)$ is slightly smaller than in the distribution of Eq. (6). Corrections to $n(0)$ are of the order of r_D/b. At $H=0$, when $r_D = a_B$, we found that the relative correction to the concentration in the center of the channel is $2a_B/\pi b$. For this purpose, we used the Thomas-Fermi approximation, which is valid when $na_B^2 \gg 1$. In the extreme quantum limit $[\nu(0) < 1]$, screening is related to electron-electron correlations and the screening radius is of the order of $n^{-1/2}$. If $na_B^2 \gg 1$ then r_D is smaller than a_B. In any case, the charge distribution in the extreme quantum limit is closer to the electrostatic solution given in Eq. (6). This means that as the magnetic field is lowered, $n_H(0)$ experiences a small monotonic decrease. This decrease can be understood as a result of the collective action of remote dipolar strips slightly depleting the center of the channel. Practically, this means that plateaus are centered at $\nu = k$ at $k \gg 1$ while at $\nu(0) < 1$ one gets $\nu_H(0) = (1+\delta_1)\nu(0)$, where δ_1 is of the order of a_B/b.

We remind the reader that our theory works only for $k < k_{c1}$ [see Eq. (35)]. At $k > k_{c1}$ a finite size of the wave function should be included in the theory. It can be shown[15] that $\Delta\nu_k$ grows linearly with k in the range $k_{c1} < k < k_{c2}$ where $k_{c2} = (b/a_B)(na_B^2)^{1/4}$. At $k \gg k_{c2}$ plateau widths are $\Delta\nu_k = 1$ and the conventional one-electron theory of ballistic transport[16] is valid.

V. RELATIONSHIP BETWEEN CONDUCTANCE AND THE FILLING FACTOR IN THE CENTER OF THE CHANNEL

In the previous section, we discussed the conductance of a short channel as a function of magnetic field. We used relation (2) between the two-terminal conductance and the electron density $n_H(0)$ in the center of the channel. To our knowledge, Eq. (2) has not yet been discussed for the most interesting case of the compressible liquid in the center of the channel. In this section, we substantiate hypothesis (2) for the channel in the C state.

Unfortunately, we are not able to prove Eq. (2) for the case of low temperatures $k_B T \ll n^{1/2}e^2/\epsilon$ where taking proper account of electronic correlations is necessary. We have therefore restricted ourselves to the case of high temperatures $k_B T \gg n^{1/2}e^2/\epsilon$. Note that we can still use the electrostatic solution (6), because even at $k_B T \gg n^{1/2}e^2/\epsilon$ the screening radius $r_D \sim k_B T\epsilon/ne^2$ may be much less than the widths of compressible and incompressible strips. For the sake of simplicity, we consider the two-terminal conductance at the extreme quantum limit, when the occupation number in the center of the channel $\nu_H(0) = n_H(0)/n_L < 1$. All the electrons occupy the lowest Landau level but their energy ε depends on the coordinate x. One can find $\varepsilon(x)$ from the condition

$$\left[\exp\left[\frac{\varepsilon(x)-\xi}{k_B T}\right]+1\right]^{-1} = \frac{n(x)}{n_L}, \tag{55}$$

where ξ is the electrochemical potential and $n(x)$ is determined by Eq. (6). Equation (55) means that the Fermi occupation numbers produce the density of electrons $n(x)$ which coincides with the result of the electrostatic problem, Eqs. (3)–(5). The dependence $\varepsilon(x)$ is shown schematically in Fig. 7(a). (This dependence may be viewed as the result of a correction to the constant electrostatic potential inside the compressible strip due to the nonvanishing screening radius $r_D \approx k_B T\epsilon/ne^2$.) The dependence of electron energy on the coordinate means that there is an electric field directed across the channel and consequently a drift of electrons along the channel with velocity

$$v(x) = \frac{\lambda^2}{\hbar}\frac{d\varepsilon}{dx}. \tag{56}$$

Apparently, electrons to the left and to the right of the channel center move in opposite directions, and the net current in the equilibrium is zero. Let us consider a nonequilibrium state with a small voltage $V \ll k_B T/e$ applied to the two terminals at the ends of the channel. The dependence $\varepsilon(x)$ is slightly modified as shown in Fig. 7(b). As electrons to the left and to the right of the center move in opposite directions, they are in equilibrium with different terminals. Thus, if we neglect all scattering processes, the electrochemical potential ξ has a step in the center of the channel,

$$\xi(x) = \begin{cases} \xi_L, & x < 0 \\ \xi_R = \xi_L + eV, & x > 0. \end{cases} \tag{57}$$

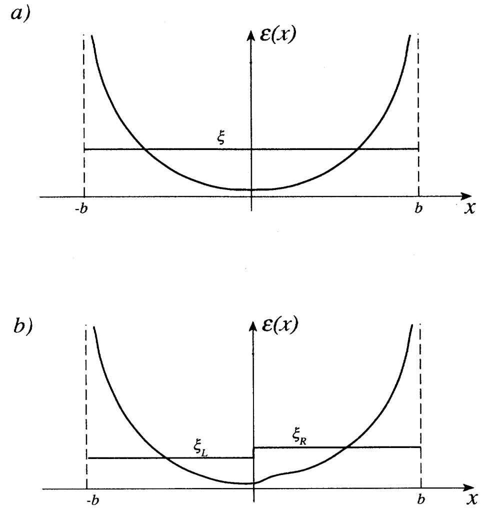

FIG. 7. Electron energy and electrochemical potential as the functions of the electron position in a high-temperature model. (a) Equilibrium state, (b) current-carrying state.

The current density has the form

$$j(x)=en(x)v(x)$$
$$=en_L\nu(x)\frac{\lambda^2}{\hbar}\frac{d\varepsilon}{dx}=\frac{e}{2\pi\hbar}f[\varepsilon(x)-\xi(x)]\frac{d\varepsilon}{dx}\ , \qquad (58)$$

where $\nu(x)=f[\varepsilon(x)-\xi(x)]$ is the occupation number for an electron state at point x, $f(\varepsilon)=(e^{\varepsilon/k_BT}+1)^{-1}$. We can now calculate the total current in the channel using Eqs. (57) and (58),

$$I=\int j(x)dx$$
$$=\frac{e}{2\pi\hbar}\left[\int_{\infty}^{\varepsilon(0)}f(\varepsilon-\xi_L)d\varepsilon+\int_{\varepsilon(0)}^{\infty}f(\varepsilon-\xi_R)d\varepsilon\right]$$
$$=\frac{e}{2\pi\hbar}f[\varepsilon(0)-\xi](\xi_R-\xi_L)=\frac{e^2}{2\pi\hbar}\nu(0)V\ . \qquad (59)$$

Thus, in the absence of disorder and electron-electron scattering, the linear conductance is determined by Eq. (2). It is interesting to note that in this approximation the conductivity does not depend on temperature. So far, we have considered the case of the high magnetic field when the channel contains only one compressible strip. However, the result (2) may be easily generalized for the case of arbitrary $\nu(0)$ by considering several Landau levels.

From the above consideration, it is clear that the conductance is always determined by the occupation number at point where the drop of electrochemical potential occurs. In other words, a nonequilibrium current is usually concentrated near the line of maximum $n(x)$. For example, if the voltages on the confining gates are different and the distribution of electron density across the channel $n(x)$ is asymmetric, $\nu_H(0)$ in Eq. (2) must be substituted by the maximum occupation number $(\nu_H)_{\max}=\max\{n(x)/n_L\}$.

VI. QUANTUM POINT CONTACTS

So far, we have considered a narrow channel which is translationally invariant in the y direction. In reality, experiments on ballistic transport are carried out with quantum point contacts (see, e.g., Ref. 7), so an electron channel has a finite length and is not necessarily translationally invariant. Usually it has a shape similar to the one shown in Fig. 8, and our theory can be generalized for this case as well. The electron density $n(x,y)$ at zero magnetic field has a saddle point at $x=y=0$ and can be described in its vicinity by the expression

$$n(x,y)=n(0,0)-\frac{n_x''}{2}x^2+\frac{n_y''}{2}y^2\ . \qquad (60)$$

Here, $n_x''=|d^2n/dx^2|$ and $n_y''=|d^2n/dy^2|$. Let us first consider the case of a very strong magnetic field, so that all electrons belong to the first Landau level. Thus, the entire channel is occupied by the compressible liquid. The density saddle point is also a saddle point of the energy $\varepsilon(x,y)$. The electrons move along the lines of constant energy, coinciding with the lines of constant density as shown in Fig. 8. We can now apply formulas (57)–(59) to the cross section $y=0$, and get the result (2) with the occupation number taken at the saddle point $x=y=0$.

In order to understand this result in the framework of the conventional transmission approach one can divide all the compressible liquid into many narrow "subchannels" along the lines of constant density. Then a nonequilibrium current will flow along these channels. It is obvious now that all the channels with $n<n(0,0)$ will pass through the quantum point contact and all the others will turn back. This explains why $n(0,0)$ plays such an important role.

To see that the current in any other cross section $(y=y_0)$ has the same value, one should note that according to the directions of arrows in Fig. 8, the drop of electrochemical potential occurs at the equipotential line A-A' which passes through the saddle point. It follows from Eq. (59) that the electron density at this line [being

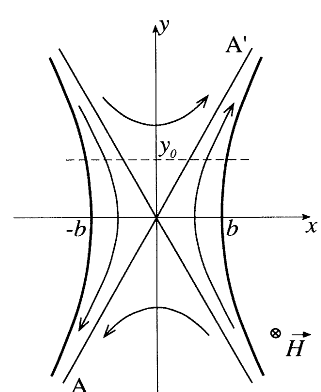

FIG. 8. Ballistic transport in the quantum point contact. Arrows show the direction of electron drift. Line A-A' is the equipotential on which the drop of electrochemical potential occurs.

equal to $n(0,0)$] determines the net current.

To find the widths of the plateaus and the shape of the rises in the function $G(\nu)$, one has to find $n(0,0)$ in a strong magnetic field. Because of the absence of the translational invariance we could not solve this problem analytically as was done for a narrow channel in Secs. III and IV. However, one of us (D.C.) has solved this problem by a direct numerical minimization of electrostatic energy for different saddle parameters. A detailed account of this work will be published elsewhere.[17]

We would like to give here a brief qualitative consideration in the limiting cases. In the limit $n_x'' \gg n_y''$ one reproduces the results derived for a narrow channel provided $n''=n_x''$. In the opposite limit $n_y'' \gg n_x''$ the widths of the plateaus are obtained by substituting n_y'' in Eq. (31). However plateaus are shifted in the direction of small ν so that they end on the line $G=\nu e^2/2\pi\hbar$. Also, the square-root singularity occurs at the low-ν end of the plateaus rather than at the high-ν side. In the case $n_x''=n_y''=n''$ plateaus are centered on the line $G=\nu e^2/2\pi\hbar$. As shown in Ref. 17 plateaus are slightly wider than that given by Eq. (31).

The experimental data show that in some cases plateaus are wider than would follow from our calculation. We attribute this discrepancy to the presence of disorder. The impact of disorder on the formation of compressible strips was discussed in Ref. 10. Disorder may localize a compressible liquid of small enough density. When a new Landau level starts to fill up with the rising Fermi level the concentration of electrons as well as the width of the new compressible strip grows from zero. Under such conditions there is a range of $\nu(0)$ where all the electrons of the new Landau level are localized. In this range of $\nu(0)$ only totally occupied Landau levels contribute to the conductance, meaning that at zero temperature plateaus should be wider than calculated in this paper. Finite temperature may delocalize a part of electrons on the partially filled Landau level. It means that the plateaus should narrow with temperature. For very weak disorder a range of temperatures should exist where the localization is destroyed but the temperature is still lower than $\hbar\omega_c$. In this regime our results should be valid quantitatively.

VII. CONCLUSION

In this paper, we have studied the distribution of electrons in a narrow channel defined by a split gate. We started with the solution in the absence of a magnetic field, which yields a domelike distribution across the channel. The electron channel width is determined by the gate voltage and is assumed to be larger than the Bohr radius in the semiconductor. Application of a strong magnetic field breaks the channel into alternating strips of compressible and incompressible liquid, thus altering the electron-density distribution. We applied knowledge of the charge distribution in the strong magnetic field to study ballistic conductance of the quantum point contact using the following conjecture: *Ballistic conductance of the quantum point contact in the strong magnetic field is given by the filling factor at the saddle point of the electron-density distribution multiplied by* $e^2/2\pi\hbar$. Hence, we identify plateaus in conductance with the situation when there is an incompressible strip in the center of the channel (*I* state). This situation is similar to the one-electron picture of edge states.

We presented a complete electrostatic description of the central incompressible strip, which we call a quadrupolar strip, finding that it can exist only in narrow ranges of magnetic field or gate voltage. In wider ranges, there is a compressible strip in the center of the channel (*C* state), and conductance is not quantized. This situation has no analogy in the one-electron picture of edge states. We solved the electrostatics problem for the density distribution and, using our conjecture, calculated the total conductance curve as a function of magnetic field and gate voltage. Experimental results do not always show narrow plateaus with wide rises. We attribute this discrepancy to the presence of disorder in the channel. For a sufficiently "clean" channel, our theory gives the dependence of conductance on a long list of parameters such as magnetic field, gate voltage, channel width, concentration of ionized donors, and the discontinuities in chemical potential. This allows for a detailed experimental verification of the theory.

APPENDIX

We present here a general method for solving a certain kind of electrostatics problem in two dimensions which involves Chebyshev polynomials.[19] Consider two metal semiplanes lying in the xy plane and separated by the insulating strip of width $2a$ and centered at $x=0$. It carries some charge, characterized by a two-dimensional charge density $\rho(x)$ invariant in the y direction. All the charges are confined to the $z=0$ plane. There is also some voltage difference applied to the metal semiplanes. We therefore come to a two-dimensional problem in the xz plane with the boundary conditions specified at $z=0$. In principle, this problem can be resolved by solving the Laplace equation in each semiplane. However, this method is complicated. It was pointed out previously[20] that in this kind of problem, one can utilize the properties of the Chebyshev polynomials.[18]

The Coulomb law in the two-dimensional system, when all the charges are confined to $z=0$, yields an electric field $E(x)=E_x(x,z=0)$,

$$E(x)=\int_{-\infty}^{+\infty} dx' \frac{2\rho(x')}{x-x'} \ . \qquad \text{(A1)}$$

This integral should be understood in terms of the principal value. Because E_x and E_z can be understood as imaginary and real parts of an analytic function one can invert Eq. (A1) using the same line of argument as in the derivation of the Kramers-Kronig relations. This leads to the following relationship:

$$\rho(x)=-\frac{1}{2\pi^2}\int_{-\infty}^{+\infty} dx' \frac{E(x')}{x-x'} \ . \qquad \text{(A2)}$$

In our problem, the electric field is zero in the metal semiplanes (for $|x|>a$), so we rewrite Eq. (A1) as

$$\rho(x)=-\frac{1}{2\pi^2}\int_{-a}^{a}dx'\frac{E(x')}{x-x'} . \quad \text{(A3)}$$

We now expand the electric field and charge density in orthogonal Chebyshev polynomials,

$$\rho(x)=\Sigma C_i U_i(x/a) , \quad \text{(A4)}$$

$$E(x)=\frac{2\pi}{\sqrt{1-(x/a)^2}}\Sigma D_i T_i(x/a) , \quad \text{(A5)}$$

and use the following relationship between T_i and U_{i-1}:[18]

$$\int_{-a}^{a}dx'\frac{T_i(x'/a)}{(x'-x)\sqrt{1-(x'/a)^2}}=\pi U_{i-1}(x/a) . \quad \text{(A6)}$$

Combining Eqs. (A3)–(A6), we find

$$D_i=C_{i-1} . \quad \text{(A7)}$$

This equation provides the following algorithm for solving the given class of electrostatics problems. One should expand the charge density on the strip $\rho(x)$ in the Chebyshev polynomials U_i, and the expansion coefficients of the electric field in Eq. (A5) are then given by Eq. (A7). The coefficient D_0 should be taken to satisfy the condition on the voltage drop between the plates. Indeed, $T_0/\sqrt{a^2-x'^2}$ gives the electrostatic solution for $\rho(x)=0$ and a finite voltage drop.

We apply this algorithm to solve the problem which appeared for the quadrupolar strip. The charge density is given by

$$\begin{aligned}\rho(x)&=e[\nu(0)-k]n_L+en''\frac{x^2}{2}\\&=e[\nu(0)-k]n_L U_0(x/a)\\&\quad+en''\frac{a^2}{2}[\tfrac{1}{4}U_2(x/a)+\tfrac{1}{4}U_0(x/a)] .\end{aligned} \quad \text{(A8)}$$

This yields an electric field (taking into account the dielectric constant of the media ϵ)

$$\begin{aligned}E_x(x)&=\frac{2\pi e}{\epsilon}\left\{[\nu(0)-k]n_L\frac{T_1(x/a)}{\sqrt{1-(x/a)^2}}+\frac{n''a^2}{2}\frac{T_3(x/a)+T_1(x/a)}{4\sqrt{1-(x/a)^2}}\right\}\\&=\frac{2\pi}{\epsilon}\left\{[\nu(0)-k]n_L\frac{x}{\sqrt{a^2-x^2}}+\frac{n''}{2}\frac{x^3-xa^2/2}{\sqrt{a^2-x^2}}\right\}\end{aligned}$$

and, finally, the electrostatic potential

$$\phi(x)=\frac{2\pi e}{\epsilon}\left\{\left[\nu(0)-k]n_L\frac{n''a^2}{4}\right](a^2-x^2)^{1/2}-\frac{n''}{6}(a^2-x^2)^{3/2}\right\} . \quad \text{(A9)}$$

Note added in proof. After the completion of this work, we learned that Ruzin,[21] by using a different method, has concluded that the nonequilibrium current flows in the center of the channel. Recently, Cooper and Chalker[22] have also confirmed this result.

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[1]See, e.g., a review by C. W. J. Beenakker and H. van Houten, in *Solid State Physics,* edited by H. Ehrenreich and D. Turnbull (Academic, New York, 1991), Vol. 44.

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