High-frequency linear response of anisotropic type-II superconductors in the mixed state

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(Received 30 November 1992)

Effective-mass anisotropy is incorporated into a self-consistent phenomenological theory of the high-frequency electrodynamic response of type-II superconductors in the mixed state. The theory accounts for two-fluid effects, including a possible anisotropic normal-fluid contribution, in addition to nonlocal vortex interaction. The approach, applicable to the modeling of a wide range of complex response functions, is illustrated in the calculation of the complex penetration depths and surface impedance for uniaxially anisotropic type-II superconductors.

The influence of moving vortices upon the electrodynamic response functions describing the behavior of isotropic type-II superconductors in the mixed state has been self-consistently described.[1-5] In the context of such a phenomenological theory the isotropy refers to that of the Meissner response. That is, for an externally applied magnetic field below the lower critical field, the superconductor expels it from the sample interior independent of the field orientation. Alternatively, shielding supercurrents flow equally well in all directions in the bulk of the sample. Because important classes of type-II superconductors are anisotropic, an extension of this previous theory is required. Prominent among such superconductors are the high-T_c cuprates, which are well approximated as being uniaxially anisotropic owing to their layered structure. In this description, conduction of supercurrents is much easier in the *ab* plane, corresponding to the copper-oxide sheets, perpendicular to the *c* axis. In this paper we include in the model of high-frequency coupled electrodynamics a general Meissner response anisotropy by way of an effective-mass tensor or its equivalent.

Anisotropic superconductors exhibit striking electromagnetic properties. Among the static properties is that a vortex in an anisotropic superconductor has a transverse magnetic field when it is not aligned along a principal axis.[6] In addition, it is possible for vortices tilted away from a principal axis to experience a mutual attractive force in such a material when the intervortex spacing is of the order of the London penetration depth.[7] Further quantities that have been investigated in anisotropic superconductors include the critical fields, the torque between two vortices, and the elastic constants of the flux-line lattice.[6,8,9]

Effects of anisotropy have been suspected in a number of rf experiments.[10] The theory presented here can provide a tool in the analysis of such experiments since we include (a) not only an anisotropic Meissner response, but also the complicating angular dependence resulting from an obliquely oriented static field, and (b) a description of the coupling to an anisotropic normal fluid response which is essential at high temperatures.

After presenting a description of our governing equations we concentrate on the modeling of superconductors with uniaxial anisotropy due to the practical usefulness of this case. Thus, these results depend upon two general vectors: one describing the orientation of the vortex lattice and the other the anisotropy axis. Sources of anisotropy in the vortex dynamics are mentioned but not studied in detail.

In the presence of anisotropy the London equation modified to include vortices may be written in terms of an effective mass tensor[6] $\hat{M}$. The components of the mass tensor come from certain averages over the Fermi surface involving the gap parameter.[11] Using matrix-vector notation we have[12]

$$\hat{M}\,\mathbf{j}_s = -\frac{1}{\mu_0\lambda^2}\left[\mathbf{a}-\frac{\phi_0}{2\pi}\nabla\varphi\right] . \tag{1}$$

Here $\mathbf{j}_s$ is the local supercurrent density, $\mathbf{a}$ the local vector potential, φ the phase of the macroscopic order parameter, λ the geometric mean penetration depth, and ϕ_0 the flux quantum. We develop a two-fluid model wherein $\mathbf{j}_s=\mathbf{j}-\mathbf{j}_n$, the total current density $\mathbf{j}$ is given by Ampére's law (ignoring the displacement term[5,13]), the normal current density $\mathbf{j}_n=\hat{\sigma}_{\rm nf}\mathbf{e}$, and $\mathbf{e}$ is the local electric field, where we have introduced a normal-fluid conductivity tensor. We may then perform the curl operation on Eq. (1) and a local averaging over several intervortex spacings to obtain the equation

$$\frac{1}{\mu_0}\nabla\times(\hat{M}\nabla\times\mathbf{B})-\nabla\times(\hat{M}\hat{\sigma}_{\rm nf}\mathbf{E})=-\frac{1}{\mu_0\lambda^2}(\mathbf{B}-n\phi_0\hat{B}_0) , \tag{2}$$

where the unit vector $\hat{B}_0$ gives the local vortex direction.[5] In Eq. (2) the continuum areal density n has resulted from the average of a sum over δ-function contributions representing the vortex lattice. Equation (2) is equivalent to a generalized diffusion-London equation derived in Ref. 5 in the isotropic case. The superconductor electrodynamics are completed with Faraday's law to describe the coupling to the macroscopic electric field and a vortex equation of motion to describe the coupling to the density n.

Generally the vortex lattice response itself is nonlinear, nonlocal, and anisotropic.[14] In this study we will treat only the nonlocality, taking for the vortex displacement $\mathbf{u}=i(\tilde{\mu}_v/\omega)\mathbf{f}$, where $\tilde{\mu}_v$ is a scalar dynamic mobility[2,3,5] and $\mathbf{f}$ is the Lorentz force per unit length of vortex.

More generally, $\tilde{\mu}_v$ must be taken as a tensor. For instance, the viscous drag force is anisotropic[15] as is the pinning force.[16] In this study the introduction of a third tensor will be avoided; in the final results it should be borne in mind that the scalar mobility cannot fully describe the lattice dynamics.

Using the vortex continuity equation it is possible to express the right-hand side of Eq. (2) solely in terms of the rf magnetic induction $\mathbf{b}=\mathbf{B}-\mathbf{B}_0$.[5] The result is conveniently expressed in terms of the complex-valued vortex-motion skin depth (which includes the effect of flux creep) $\tilde{\delta}_{vc}=(2B_0\phi_0\tilde{\mu}_v/\mu_0\omega)^{1/2}$.[2,3] We then have

$$-\lambda^2\nabla\times(\hat{M}\nabla\times\mathbf{b})-\mathbf{b}+\mu_0\lambda^2\nabla\times(\hat{M}\hat{\sigma}_{\rm nf}\mathbf{E})=\frac{i}{2}\tilde{\delta}_{vc}^2\nabla\times(\hat{B}_0\times[(\nabla\times\mathbf{b})\times\hat{B}_0])\ . \quad (3)$$

In the following we considerably simplify the normal-fluid ($\mathbf{E}$ field) term in Eq. (3). Taking the approximation $\hat{\sigma}_{\rm nf}=\text{const}\hat{M}^{-1}$, we have $\hat{M}\hat{\sigma}_{\rm nf}=\sigma_{\rm nf}$, the scalar normal-fluid conductivity, which, in turn, allows us to write the normal-fluid term as $-\mu_0\lambda^2\sigma_{\rm nf}\dot{\mathbf{b}}$. Introducing the normal-fluid skin depth $\delta_{\rm nf}=(2/\mu_0\omega\sigma_{\rm nf})^{1/2}$, this term becomes $2i\lambda^2\delta_{\rm nf}^{-2}\mathbf{b}$ in linear response, as we are considering. More general expressions for an electrical conductivity tensor can be found in Ref. 17. Anisotropy in the material's Fermi surface can lead to anisotropy in the conductivity. The approximation we have employed breaks down in the presence of significant quasiparticle-scattering-time anisotropy. We additionally specialize Eq. (3) to the case of uniaxial anisotropy, taking for the effective-mass tensor and its inverse

$$\begin{aligned}\hat{M}_{ik}&=\delta_{ik}+r\nu_i\nu_k\ ,\\ \hat{M}_{ik}^{-1}&=\delta_{ik}-[r/(1+r)]\nu_i\nu_k\ ,\end{aligned} \quad (4)$$

where r is a measure of the anisotropy. The inverse mass tensor is easily identified with a superelectron density tensor.[9] The form (4) has a ready geometric interpretation since ν_i are the components of the anistropy axis ($|\boldsymbol{\nu}|=1$). When $\boldsymbol{\nu}$ corresponds to the $\hat{c}$ axis in $YBa_2Cu_3O_{7-\delta}$, r is in the approximate range 30–50 while in this case for $Bi_2Sr_2CaCu_2O_{8+\delta}$, r is approximately 3000. These values come from vortex-lattice decoration, transverse magnetization, and oscillator measurements.[18]

Using Eq. (4), we have a single partial differential equation for $\mathbf{b}$ suitable for uniaxial superconductors with scalar vortex mobility,

$$(2i\lambda^2\delta_{\rm nf}^{-2}-1)\mathbf{b}+\lambda^2\nabla^2\mathbf{b}-\lambda^2 r\nabla\times[(\boldsymbol{\nu}\otimes\boldsymbol{\nu})\nabla\times\mathbf{b}]=\frac{i}{2}\tilde{\delta}_{vc}^2\nabla\times(\hat{B}_0\times[(\nabla\times\mathbf{b})\times\hat{B}_0])\ . \quad (5)$$

We illustrate the solution of Eq. (5) in semi-infinite geometry, which is useful in the modeling of surface impedance. In finite thickness geometry our methods are useful in the description of magnetic permeability, vibrating reed, oscillator, or other experiments involving vortex dynamics.[3]

Proceeding as in Ref. 5, the rf field is chosen to lie along the z axis, and x measures the distance into the superconductor (see Fig. 1). The time dependence of $\mathbf{b}(\mathbf{x},t)$ has been taken to be $\exp(-i\omega t)$; its spatial dependence is taken to be of the form $\mathbf{b}(x)=f_1(x)\hat{y}+f_2(x)\hat{z}$. Substituting this form into Eq. (5) gives coupled second-order ordinary differential equations for the field components f_1 and f_2, which we write in the form

$$af_1''+\beta f_1+cf_2''=0\ , \quad (6a)$$

$$df_2''+\beta f_2+cf_1''=0\ , \quad (6b)$$

where a prime denotes differentiation with respect to x. The coefficient β is given by $\beta=2i\lambda^2\delta_{\rm nf}^{-2}-1$, reducing to -1 at low temperatures (where the normal-fluid response in negligible). The functions f_1 and f_2 satisfy the four boundary conditions

$$f_1(0)=0,\quad f_2(0)=b_0\ , \quad (7a)$$

$$\lim_{x\to\infty}f_i(x)=0,\quad i=1,2\ . \quad (7b)$$

The coefficients a, c and d in Eq. (6) depend upon the components of the vectors $\hat{B}_0$ and $\boldsymbol{\nu}$. Due to the semi-infinite geometry, it is useful to employ either the Laplace or Fourier sine transform to solve the system (6). It is advantageous to use the sine transform because of the form of the boundary conditions (7). Of immediate interest in this paper are the complex penetration depths $\tilde{\lambda}$ (Refs. 2, 3, 4, and 5) resulting from Eq. (6). Letting s be the transform variable, we write

$$\hat{f}(s)=\int_0^\infty f(x)\sin sx\,dx \quad (8)$$

for the Fourier sine transform. As we identify s^2 with $-1/\tilde{\lambda}^2$,[19] the squares of the *negative reciprocal* complex penetration depths are given from the solution of the quartic equation[5,20]

$$D(s)\equiv(ad-c^2)s^4-\beta(a+d)s^2+\beta^2=0\ . \quad (9)$$

The discriminant of Eq. (9) is $\Delta\equiv[(a-d)^2+4c^2]\beta^2$. Then its solution is

$$s_\pm^2=\frac{\beta(a+d)\pm\Delta^{1/2}}{2(ad-c^2)}\ . \quad (10)$$

The respective sine transforms of f_1 and f_2 are

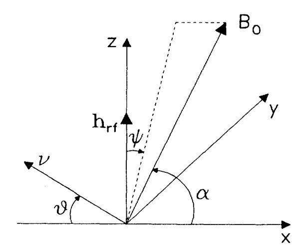

FIG. 1. Geometry of the semi-infinite anisotropic superconductor with an oblique applied static magnetic field. The magnetic flux density $\mathbf{B}_0$, rf magnetic field $\mathbf{h}_{\rm hf}=\hat{z}h_{\rm rf}$, and anisotropy axis $\boldsymbol{\nu}$ are indicated. In this instance the anisotropy axis makes an angle θ with respect to the surface normal. The vector $\mathbf{B}_0$ is inclined at the angle α with respect to the x axis and its projection on the yz plane is at an angle ψ with respect to the z axis.

$$\hat{f}_1(s) = -b_0\beta cs/D(s) ,$$

$$\hat{f}_2(s) = b_0 s[(ad - c^2)s^2 - \beta d]/D(s) . \quad (11)$$

Suppose that the anisotropy axis $\boldsymbol{v}$ lies in the xz plane, $v_x = -\cos\theta$, $v_z = \sin\theta$. (This is the plane formed by the surface normal and the driving magnetic field.) Then the coefficients in Eq. (6) are

$$a = \lambda^2(1 + r\sin^2\theta) + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + B_{0y}^2) , \quad (12a)$$

$$c = \frac{i}{2}\tilde{\delta}_{vc}^2 \hat{B}_{0y}\hat{B}_{0z} , \quad (12b)$$

$$d = \lambda^2 + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0z}^2) , \quad (12c)$$

giving

$$a + d = \lambda^2(2 + r\sin^2\theta) + \frac{i}{2}\tilde{\delta}_{vc}^2(1 + \hat{B}_{0x}^2) , \quad (13a)$$

$$\Delta/\beta^2 = -\tfrac{1}{4}\tilde{\delta}_{vc}^4(1 - \hat{B}_{0x}^2)^2 + \lambda^2 r\sin^2\theta[\lambda^2 r\sin^2\theta + i\tilde{\delta}_{vc}^2(\hat{B}_{0y}^2 - \hat{B}_{0z}^2)] , \quad (13b)$$

$$ad - c^2 = \lambda^4(1 + r\sin^2\theta) + \frac{i}{2}\lambda^2 r\sin^2\theta\,\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0z}^2) + \frac{i}{2}\lambda^2\tilde{\delta}_{vc}^2(1 + \hat{B}_{0x}^2) - \tfrac{1}{4}\tilde{\delta}_{vc}^4\hat{B}_{0x}^2 . \quad (13c)$$

Recall that in the isotropic case ($r=0$) Δ is a perfect square and the denominator in Eq. (10) can be factored, resulting in simplifications in the form of the squared complex penetration depths.[5] In the anisotropic case, note the additional angular combination which occurs in $\Delta \propto \sin^2\alpha\cos 2\psi$ (see Fig. 1 for the components of $\mathbf{B}_0$). In Ref. 9, real penetration depths $\lambda, \lambda(1 + r\sin^2\theta)^{1/2}$ were obtained for an anisotropic one-fluid model in the absence of vortices. When there are no vortices ($\tilde{\delta}_{vc}=0$) and no normal fluid ($\delta_{\rm nf}^{-2}=0$, $\beta=-1$), our result (10) reduces to this special case.

If the anisotropy axis lies in the yz plane ($v_y = \cos\theta$, $v_z = \sin\theta$), we find the coefficients

$$a = \lambda^2(1 + r\sin^2\theta) + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0y}^2) , \quad (14a)$$

$$c = \frac{i}{2}\tilde{\delta}_{vc}^2\hat{B}_{0y}\hat{B}_{0z} - \lambda^2 r\sin\theta\cos\theta , \quad (14b)$$

$$d = \lambda^2(1 + r\cos^2\theta) + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0z}^2) , \quad (14c)$$

whereas if $\boldsymbol{v}$ is in the xy plane ($v_x = \cos\theta$, $v_y = \sin\theta$) we have

$$a = \lambda^2 + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0y}^2) , \quad (15a)$$

$$c = \frac{i}{2}\tilde{\delta}_{vc}^2\hat{B}_{0y}\hat{B}_{0z} , \quad (15b)$$

$$d = \lambda^2(1 + r\sin^2\theta) + \frac{i}{2}\tilde{\delta}_{vc}^2(\hat{B}_{0x}^2 + \hat{B}_{0z}^2) . \quad (15c)$$

By noting the sine transform of the sum and difference of two exponential functions the transforms in Eq. (11) can be inverted for general a, β, c, and d. The results are

$$f_1(x) = -\frac{b_0\beta c}{\Delta^{1/2}}(e^{-is_+x} - e^{-is_-x}) , \quad (16a)$$

$$f_2(x) = \frac{b_0}{2}\left[e^{-is_+x} + e^{-is_-x} + \frac{\beta(a-d)}{\Delta^{1/2}}(e^{-is_+x} - e^{-is_-x})\right] . \quad (16b)$$

Now that the magnetic induction has been found, all other electrodynamics fields and densities can be determined. Ampére's law yields the total rf current density while Faraday's law gives the total electric field $\mathbf{E}$. By knowing $\mathbf{E}$, the surface impedance matrix $Z_s = R_s - iX_s$ can be calculated in terms of the complex penetration depths:

$$Z_s^{yz} = \frac{\mu_0}{b_0}E_y(0) = -\frac{\mu_0\omega}{2}\left[\frac{1}{s_+} + \frac{1}{s_-} + \frac{\beta(a-d)}{\Delta^{1/2}}\left[\frac{1}{s_+} - \frac{1}{s_-}\right]\right] , \quad (17a)$$

$$Z_s^{zz} = \frac{\mu_0}{b_0}E_z(0) = -\frac{\mu_0\omega\beta c}{\Delta^{1/2}}\left[\frac{1}{s_+} - \frac{1}{s_-}\right] . \quad (17b)$$

The vortex density n can be found for anisotropic superconductors from Eq. (2) once $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$ is known.

In this paper our phenomenological theory of high-frequency electrodynamic response was illustrated in the calculation of the complex penetration depths and surface impedance for uniaxially anisotropic type-II superconductors. The geometric generality of the results includes the complicating effect of an oblique static magnetic field. The extension to model biaxial superconductors is straightforward: the effective-mass tensor in Eq. (4) is replaced by

$$\hat{M} = I + r_1\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + r_2\boldsymbol{v}_2 \otimes \boldsymbol{v}_2 , \quad (18)$$

where I is the 3×3 identity matrix and the two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthonormal. The self-consistent coupling of current density and vortex displacement was extended to the case of anisotropic superconductors. The theory accounts for two-fluids effects, including a possible anisotropic normal-fluid contribution, in addition to nonlocal vortex interaction. The normal-fluid contribution allows the results to be continuously valid through the upper critical field or the transition temperature, for then the effective complex skin depth $\tilde{\delta}_{vc}$ associated with vortex motion and flux creep vanishes and the complex penetration depth becomes proportional to the normal-state skin depth. At T_c, the governing tensor equation (3) or (5) itself becomes the normal-state diffusion equation. Although presented here with a scalar dynamic mobility, the vortex dynamics nonetheless simultaneously includes the effects of pinning, flux flow, and flux creep. Our approach is applicable to the modeling of a wide range of complex response functions including the rf magnetic permeability, inductances, conductivity, and transmission and reflection coefficients.

I thank M. Ledvij and A. Simonov for useful discussions.

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[19]I.e., s is a complex propagation constant (see Refs. 1 and 13).

[20]When the Laplace transform is employed, the sign of β is reversed.