Particle in a random magnetic field on a plane

D. V. Khveshchenko* and S. V. Meshkov†
Centre de Recherches sur les Très Basses Températures, CNRS, 25, avenue des Martyrs, 38042 Grenoble, France
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We study the properties of a two-dimensional spinless particle moving in a random magnetic field. This problem arises in the context of a modern theory of strongly correlated systems as well as in the theory of vortex-lines dynamics in high-T_c materials. The problem is investigated with a variety of methods including direct perturbation theory, quasiclassical approximation, the method of an optimal fluctuation, and Monte Carlo simulations. We obtain a shape of the density of states near the unrenormalized lower boundary of the spectrum, a particle mobility, and its diamagnetic orbital susceptibility.

I. INTRODUCTION

A recent development of the theory of strongly correlated systems[1] which was, in turn, stimulated by the discovery of the high-temperature superconductivity put forward a remarkable problem of noninteracting spinless particles in a spatially random static magnetic field. This field is assumed to be coupled with an orbital current and has no Zeeman interaction. This problem arises in the context of the gauge-field description of the doped Mott insulator, which was supposed to be a representative example of a strongly correlated system.[2–6]

As defined, the gauge field describes fluctuations of spin chirality which is the only low-energy spin excitation mode in a spin disordered state. Within the framework of the gauge theory it turns out that normal state properties are governed by a current-current interaction which is mediated by a transverse gauge field.[4,6]

In the strong-coupling regime this interaction appears to be dominating over a fictitious Coulomb-like one in two dimensions (2D) as well as in three dimensions (3D). Its importance was first stated in the case of an ordinary electromagnetic interaction in a 3D metal in Ref. 7. It was conjectured that eventually the current-current interaction could lead to a breakdown of a Fermi-liquid picture and a formation of an essentially non-Fermi-liquid state.

Moreover, at finite temperatures the most singular contributions in all orders of perturbation theory come from elastic processes of scattering via the gauge field with zero energy transfer. Restricting himself onto these contributions only one deals with an above-mentioned problem of a particle in a random background magnetic field. It was found that being calculated within this approximation a normal state resistivity,[8–10] magnetoresistance,[11] and the Hall conductance[12] as functions of temperature, external magnetic field, and doping are in a relatively good agreement with the experimental results on quasi-two-dimensional high-T_c materials.

Another case where the same problem appears to be relevant is a problem of a vortex line fluid phase in a superconductor. By virtue of the duality transformation[13] this phase was mapped onto a quantum ground state of 2D charged bosons which have no Bose condensate even at zero temperature.[14,15] Here an auxiliary gauge field is coupled to a supercurrent.

Regardless of its physical motivation the random magnetic field problem is of great interest itself as a quite nontrivial counterpart of the problem with a random scalar potential. Although the latter was extensively investigated for decades, the former essentially was not considered at all. Obviously the reasons for no interest in this problem are the smallness of an ordinary electromagnetic current interaction as compared with a static Coulomb and Zeeman interactions. It was a novel problem of high correlation induced by strong spin interaction which attracted an interest to orbital particle dynamics in a random magnetic field.

Recent attempts in this direction[16] yielded a first-order calculation of a density of states and diamagnetic susceptibility. As an alternative an exact numerical diagonalization on finite lattices was performed.[16,17] Another approach based on a real-space path-integral representation for the one-particle Green function was developed by Wheatley for both cases of short-range and long-range correlated random field.[18–20]

In the framework of the original Hubbard or some related models numerous attempts to study a dynamics of a one hole in different spin backgrounds were undertaken previously.[21] The comparison of these available results with those in the random magnetic field problem has to be made to conclude about the very applicability of the quasistatic field approximation. Various treatments of the Hubbard model lead to the conclusion of a drastic suppression of one-particle density of states near the edges of the free particle spectrum.[22] In the simplest approximation there is a band narrowing while a more advanced analysis shows an existence of band tails extending up to the boundaries of the unperturbed spectrum.[23] Until now the nature of states in tails of the spectrum remains uncertain, in particular still there exist a tedious question about a particle mobility in these states.

II. PERTURBATION THEORY IN RANDOM MAGNETIC FIELD

In analogy with the case of a static random potential one could try to apply a perturbation theory to get insight into the problem. In the former case the perturbation approach was quite informative. On the basis of this approach it was first conjectured by Abrahams *et al.*[24] that in the 2D case where the corrections to the conductance are logarithmically divergent all states are localized and there is no mobility threshold. This investigation also showed an existence of diffusion and Cooperon collective modes resulting from a particle number conservation and a time-reversal symmetry, respectively.[25] It was also observed that a diffusion mode is responsible for logarithmically divergent contributions in perturbative calculations. Subsequently it served as a starting point for a formulation of effective descriptions by means of various nonlinear σ models.[26] Starting with a formulation of the random magnetic field problem we suppose its simplest white-noise correlation properties

$$\langle B(\mathbf{0})B(\mathbf{r})\rangle = \Gamma\delta(\mathbf{r})\,. \tag{1}$$

In the case of a lattice system an equivalent relation can be assumed for a correlation function of a flux Φ_P through an elementary plaquette P:

$$\langle \Phi_P \Phi'_P\rangle = \Gamma a^2 \delta_{PP'}\,, \tag{2}$$

where a is a lattice constant. The Hamiltonian of a single particle moving in a random field (1) has the form

$$H = \frac{1}{2m}|\nabla\Psi - i\mathbf{a}\Psi|^2\,, \tag{3}$$

where $\mathbf{a}(\mathbf{r})$ is a 2D random vector potential corresponding to the random magnetic field $B(\mathbf{r})$ (modulo gauge transformations). Because of the energy conservation one can use a representation for Green functions in terms of a time-independent fermion field $\Psi(\mathbf{r},\tau)$. To average over different field configurations a replica method[27] or a supersymmetrical approach[28] can be applied. Alternatively one could deal with a time-dependent fermion field in a direct analogy with an old good "cross" technique in the impurity scattering problem.[25]

In our case a complete investigation of the whole perturbation theory is essentially more difficult because of the presence of interaction terms linear and quadratic in $\mathbf{A}$. An account of both is necessary to preserve the gauge invariance. One can readily see that the perturbative expansions for the Green functions in the random field coincide with those of a scalar 2D quantum electrodynamics

$$L = |\partial_\mu\Phi - ia_\mu\Phi|^2 - M^2|\Phi|^2 + \frac{1}{2\Gamma}(\epsilon_{\mu\nu}\partial_\mu a_\nu)^2 \tag{4}$$

in absence of any matter polarization. The latter condition means that one should not consider any matter field loops and the vector field propagator has no renormalization. To observe this correspondence one has to identify a fixed energy of the particle ε with mass squared of a charged 2D scalar field ($M^2 = 2m\varepsilon$). A straightforward perturbative expansion of any gauge noninvariant Green function exhibits logarithmic divergencies at large momenta. Formally these divergencies originate from a singular behavior of the vector field propagator which preserves in all gauges. For instance, in the transverse Coulomb gauge ($\partial_\mu a_\mu \equiv 0$) the propagator has a form

$$\langle a_\mu(\mathbf{k})a_\nu(-\mathbf{k})\rangle = \mathbf{k}^{-2}\Gamma(\delta_{\mu\nu} - k_\mu k_\nu/\mathbf{k}^2).$$

As an example in the lowest order we find logarithmic contributions into a real part of a self-energy (Fig. 1) and as well as a three-vertex (Fig. 2)

$$\mathrm{Re}\Sigma(\varepsilon,\mathbf{p}) = \frac{\Gamma}{8\pi m}\frac{2m\varepsilon + \mathbf{p}^2}{2m\varepsilon - \mathbf{p}^2}\ln\left(\frac{\Lambda^2}{\max\{2m\varepsilon,\mathbf{p}^2\}}\right) \tag{5}$$

and

$$\Gamma(\omega,\mathbf{p}_1,\mathbf{p}_2) = 1 + \frac{\Gamma}{8\pi m}\frac{2m\varepsilon + \mathbf{p}^2}{(2m\varepsilon - \mathbf{p}^2)^2}\ln\left(\frac{\Lambda^2}{\max\{2m\varepsilon,\mathbf{p}^2\}}\right)\,, \tag{6}$$

where Λ is a short wavelength cut-off. On a lattice it is of order of an inverse lattice constant $\Lambda \sim a^{-1}$. Because of a renormalizability property of a 2D scalar electrodynamics all logarithmic divergencies can be absorbed into a renormalization of the coupling Γ, the mass M, and wave function of the charged scalar field $\Phi(\mathbf{r})$. A consistent renormalization procedure could be formulated on the basis of Ward identities which are similar to those of an impurity scattering problem and originate from the particle number conservation.[29,30] In particular, we expect an appearance of a counterpart of a diffusion pole in a four-point correlation function.

However, on the contrary to the impurity scattering problem, one-particle properties are strongly influenced by the random magnetic field. Consequently one cannot make any definite conclusion about transport properties before an investigation of the energy spectrum. We plan to perform a more detailed analysis of the renormalization problems in a future publication.

On the other hand a calculation of gauge-invariant quantities shows that these are free of divergencies. It could be expected because physically these divergencies are manifestations of an ambiguity in the choice of a vector potential $\mathbf{a}(\mathbf{r})$ at a given distribution of a magnetic field $B(\mathbf{r})$. A density of states provides one of those examples. The first-order corrections to the density of states $\rho(\varepsilon)$ were found by Ioffe and Kalmeyer[16] in the form $\delta\rho(\varepsilon) \propto \varepsilon^{-1}$. Thus the naive result assumes an in-

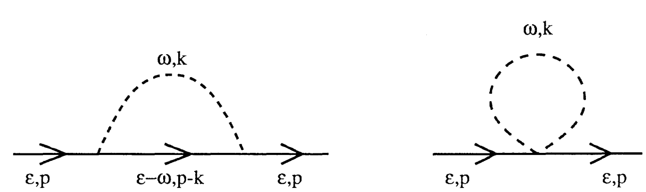

FIG. 1. Lowest-order self-energy corrections.

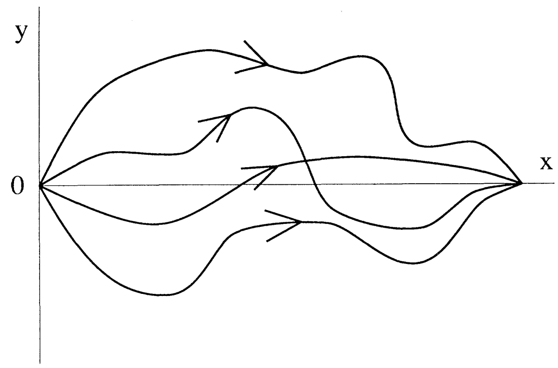

FIG. 2. Trajectories accounted in the quasiclassical approximation.

crease of the density of states near the band edge. It seems to be natural, however, that at smaller energies random field leads to an essential suppression of the density of states because a particle motion becomes more and more difficult as its energy decreases.

The only energy parameter distinguishing between a regime where one could believe the results of the perturbation theory and an essentially nonperturbative regime is Γ/m. At the same time we suppose that a lower bound of the spectrum does not change, because even at $\varepsilon << \Gamma/m$ there always exist configurations of the random field with arbitrary large regions where the field is arbitrary small. Although these realizations of the random field can have an extremely small probability these result in an existence of a tail of the density of states which extends up to $\varepsilon = 0$. In Sec. IV we shall confirm this picture by means of a direct calculation of the tail of the density of states at $\varepsilon << \Gamma/m$. Notice that in the context of the original Hubbard model it is the same problem of the tail of a density of states of holes in a disordered spin background which was first discussed by Brinkman and Rice.[23]

In the remainder of this section we shall demonstrate that an effective suppression of the density of states at $\epsilon < \frac{\Gamma}{m}$ can be seen already in the framework of an improved perturbation theory. First we shall repeat the lowest-order calculation of the imaginary part of Σ performed in Ref. 16. The first diagram on the Fig. 1 yields $\mathrm{Im}\Sigma(\varepsilon,\mathbf{p})$

$$= \frac{\Gamma}{m^2}\int\frac{d^2p'}{(2\pi)^2}\frac{p^2p'^2\sin^2\theta}{(p^2+p'^2-2pp'\cos\theta)^2}\mathrm{Im}G(\varepsilon,\mathbf{p}')\,, \tag{7}$$

where θ is a polar angle between the 2D momenta $\mathbf{p}$ and $\mathbf{p}'$. Substituting the free-particle Green function $G_0(\varepsilon,\mathbf{p}) = [\varepsilon - \frac{p^2}{2m} + i0]^{-2}$ to (7) one obtains

$$\mathrm{Im}\Sigma(\varepsilon,\mathbf{p}) = \frac{\Gamma}{m}\frac{x}{1-x}\,, \tag{8}$$

where $x = \min\{\frac{p^2}{2m\epsilon}, \frac{2m\varepsilon}{p^2}\}$.

Using this lowest-order result and summing over all reducible diagrams with (8) inserted, one finds the density of states in the form

$$\rho(\varepsilon) = \frac{1}{\pi}\mathrm{Im}\int\frac{d^2p}{(2\pi)^2}\frac{1}{\varepsilon-\frac{\mathbf{p}^2}{2m}+i\frac{\Gamma}{m}\frac{x}{1-x}}\,. \tag{9}$$

Estimating the integral in two opposite limits one can see that at $\varepsilon >> \Gamma/m$ the density of states approaches a free-particle value m from above in agreement with the lowest-order calculation performed in Ref. 16. On the contrary, at $\varepsilon << \Gamma/m$ the formula (9) shows that the density of states vanishes as

$$\rho(\varepsilon) \sim \frac{\varepsilon m}{\Gamma}\ln\left(\frac{\Gamma}{\varepsilon m}\right)\,. \tag{10}$$

Although such behavior is in agreement with our general discussion, we remind the reader that strictly speaking the approximation used can be justified only at $\varepsilon > \Gamma/m$.

As another result of the naive perturbation theory we shall estimate a particle mobility. It can be performed within an ordinary relaxation-time approximation as

$$\mu(\varepsilon) = \frac{e}{m}\frac{\tau_{\mathrm{tr}}(\varepsilon)}{1+\varepsilon^2\tau_{\mathrm{tr}}^2}\,. \tag{11}$$

The transport time τ_{tr} can be found in the form analogous to the case of a random potential[29,30]

$$\tau_{\mathrm{tr}}^{-1}(\varepsilon) = \frac{\Gamma}{2\pi\rho(\varepsilon)m^2}\int\frac{d^2p}{(2\pi)^2}\frac{d^2p'}{(2\pi)^2}\Delta G(\varepsilon,\mathbf{p})\Delta G(\varepsilon,\mathbf{p}')\frac{p^2p'^2\sin^2\theta(1-\cos\theta)}{(p^2+p'^2-2pp'\cos\theta)^2}\,, \tag{12}$$

where $\Delta G_0(\varepsilon,\mathbf{p}) = G_R(\varepsilon,\mathbf{p}) - G_A(\varepsilon,\mathbf{p})$. Using the bare Green functions $G_A = G_R^*$ in (12) we obtain a finite scattering rate

$$\tau_{\mathrm{tr}}^{-1}(\varepsilon) = \frac{\Gamma}{\rho(\varepsilon)}\,. \tag{13}$$

Then we obtain that in the limit $\varepsilon >> \Gamma/m$ the mobility $\mu(\varepsilon)$ vanishes as ε^{-2}:

$$\mu(\varepsilon) = \frac{e\Gamma}{m^2\varepsilon^2}\,, \tag{14}$$

while at small ε the expression (11) yields

$$\mu(\varepsilon) = \frac{e\rho(\varepsilon)}{m\Gamma}\,. \tag{15}$$

Notice that at $\varepsilon \sim \Gamma/m$ the mobility has a maximum. This energy dependence seems to be qualitatively correct because it is a random field which is only responsible for a momentum relaxation at high energies (and so $\mu \propto \Gamma$) while at small energies μ vanishes together with ρ.

Relations (14) and (15) can be compared with the results $\rho \propto \varepsilon$ and $\mu \propto T^{-1}$ obtained in the model with a spatially homogeneous random field which does not depend on coordinates.[20] At finite temperatures this model corresponds to the case when a correlation length of the random field exceeds a particle thermal length. Obviously such an assumption becomes invalid as T tends to zero. An alternative method which we shall describe in the next section is based on the approximation of the original model by the Caldeira-Leggett one.[18,19] The consideration based on this model predicts a shift of the lower bound of the spectrum $\delta\varepsilon_0 \sim \frac{\Gamma}{m}\ln(\Lambda^2/\Gamma)$, a divergent density of states $\rho \propto 1/\varepsilon$, and a finite renormalization of the effective mass of a particle. At $\epsilon > \epsilon_0$ the results obtained in the framework of the Caldeira-Leggett model qualitatively coincide with the results of our perturbation theory but this model certainly fails to describe the low-energy part of the spectrum.

Although our perturbative analysis does not enable us to conclude that not all states in the spectrum are extended and there is a mobility threshold at $\varepsilon \sim \Gamma/m$, we suppose that the states near the lower bound of the spectrum $\epsilon = 0$ are localized. These states look very similar to those lying deeply in the localization regime in the random potential problem. In Sec. IV we shall undertake an attempt to study this part of the spectrum by means of the nonperturbative optimal fluctuation method and present additional arguments in favor of this conjecture. However the question about an existence of localized states certainly remains to be understood better.

III. PATH INTEGRAL IN REAL SPACE AND QUASICLASSICAL GREEN FUNCTION

An alternative formulation of the random magnetic field problem can be achieved by using the path-integral representation in a real space and imaginary time. Such an approach was developed by Wheatley *et al.*[18–20] This consideration starts from the following expression for the partition sum for a particle moving in a random field:

$$Z = \int D\mathbf{a}(\mathbf{r}) D\mathbf{r}(\tau) \exp\Bigg[- \int_0^\beta d\tau \frac{1}{2m}\left(\frac{d\mathbf{r}}{d\tau}\right)^2 + i\int d\mathbf{r}\,\mathbf{a}(\mathbf{r}) - \frac{1}{2\Gamma}\int d^2\mathbf{r}(\epsilon_{\mu\nu}\partial_\mu a_\nu)^2\Bigg], \quad (16)$$

where $\beta = 1/T$ and the integral is taken over periodic particle's trajectories with $\mathbf{r}(0) = \mathbf{r}(\beta)$. After integrating the random vector potential out one obtains an effective action for the particle in the form

$$S = \int_0^\beta d\tau \frac{1}{2m}\left(\frac{d\mathbf{r}}{d\tau}\right)^2 + \Gamma \oint d\mathbf{r} \oint d\mathbf{r}' \ln\left(|\mathbf{r}-\mathbf{r}'|\Lambda\right) . \quad (17)$$

This expression has no ambiguities because the integrals are taken over closed trajectories ($\oint d\mathbf{r} = 0$) and the partition sum is obviously gauge invariant. Because of that the argument of the logarithm can be multiplied by any number Λ.

It was proposed by Wheatley and Hong[18] to treat the second term in the action (17) self-consistently assuming some sort of a powerlike diffusion law:

$$\langle (\mathbf{r}(\tau) - \mathbf{r}(0))^2 \rangle \sim \tau^\eta .$$

As a result the effective action (17) becomes Gaussian:

$$S = \int_0^\beta d\tau \frac{1}{2m}\left(\frac{d\mathbf{r}}{d\tau}\right)^2 + \Gamma \int_0^\beta d\tau \int_0^\beta d\tau' \frac{d\mathbf{r}}{d\tau}\frac{d\mathbf{r}'}{d\tau'} \ln|\tau - \tau'|^\eta \quad (18)$$

and enables a relatively simple analysis.

In this section we shall discuss another approximation for the gauge-invariant Green function $G(\mathbf{r},\mathbf{0},\tau) = \langle \Psi(\mathbf{r},\tau)\Psi^\dagger(\mathbf{0},0)\exp(i\int_0^{\mathbf{r}} \mathbf{a}\,d\mathbf{r})\rangle$, the integral in the exponent being taken over a straight line connecting points $\mathbf{r}$ and $\mathbf{0}$. It is a sort of the eikonal approximation which is applicable at high energies of a particle or, correspondingly, at short times. In fact the condition to be satisfied has a form

$$\frac{R}{\tau} >> \Gamma^{1/2} . \quad (19)$$

Although the requirement (19) seems to be quite obvious we shall comment on its origin below.

In the eikonal approximation we shall account only those paths in the path integral (16) which can be projected onto a radius vector connecting the initial and the end points without folding. In other words, if we choose the vector $\mathbf{r}$ along the x axis then the selected trajectories can be described as single-valued functions $y(x)$ (see Fig. 2). For all these paths the action (17) can be rewritten in the form

$$S = \int_0^\beta d\tau \left\{ \frac{1}{2m}\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 \right] + \Gamma|y|\frac{dx}{d\tau} \right\} , \quad (20)$$

where the interaction term manifests itself as an area inside the contour formed by the trajectory $y(x)$ and a segment of the x axis. Of course, even in this approximation, which intuitively seems to be correct for fast particles, the problem does not become Gaussian. However, considering (20) as an effective action for the 2D quantum mechanics, one obtains the following equation for the effective Green function:

$$\left(-\frac{\partial}{\partial\tau} + \frac{1}{2m}\nabla^2 + \frac{\Gamma|y|}{2im}\nabla_x\right) G'(x;y,y';\tau) = \delta(x)\delta(y-y')\delta(\tau) . \quad (21)$$

Then after the Fourier transformation of (21) with respect to x and τ one arrives at the 1D equation

$$\left(i\varepsilon - \frac{k^2 + \Gamma k|y|}{2m} - \frac{1}{2m}\frac{d^2}{dy^2}\right) G'(k;y,y';\varepsilon) = \delta(y-y') . \quad (22)$$

At arbitrary values of y and y' the 1D Green function can be easily found by means of the Wronskian method as a sum over bilinear products of the Airy functions. Actually we need its value only for $y = y' = 0$ which is given by

$$G(k,\varepsilon) = G'(k;0,0;\varepsilon) = \frac{|z|}{(\Gamma k)^{1/3}}(J^2_{-1/3}(\xi) - J^2_{1/3}(\xi)), \tag{23}$$

where $\xi = \frac{2}{3}z^{3/2}$ and $z = (2im\varepsilon - k^2)(\Gamma k)^{2/3}$. After some simple algebra we obtain an approximate expression for the 2D Fourier transform of the Green function G:

$$G(\mathbf{k},\varepsilon) = \min\left\{\frac{m^{1/2}}{(\Gamma k)^{1/3}(i\varepsilon - \frac{\mathbf{k}^2}{2m})^{1/2}}, \frac{1}{(i\varepsilon - \frac{\mathbf{k}^2}{2m})}\right\}. \tag{24}$$

We observe that in the case of a random field particle's propagation ceases to be coherent and the Green function acquires cut singularities and branching points in the complex ε plane. This circumstance has to be understood in view of the assumptions that in a finite density system a random field leads to the breakdown of Fermi or Bose coherence.

To get the Green function in the coordinate representation one has to perform an inverse Fourier transformation with respect to $\mathbf{k}$ and ε. The result depends on the relation between R and τ. At $R^{2/3}\tau^{1/3} << \Gamma^{-2/3}$ a free-particle result can be restored $G(R,\tau) \sim \tau^{-1}\exp(-mR^2/4\tau)$, while at $R^{2/3}\tau^{1/3} >> \Gamma^{-2/3}$:

$$G(R,\tau) = \frac{m^{1/3}}{\Gamma^{1/3}\tau^{4/3}}\Phi(5/6,1,-mR^2/4\tau), \tag{25}$$

where $\Phi(\alpha,\beta,z)$ is a degenerate hypergeometric function. Using its asymptotics at $R^2/m\tau >> 1$ we obtain

$$G(R,\tau) \sim \frac{m^{1/6}}{\Gamma^{1/3}R^{1/3}\tau^{7/6}}\exp(-mR^2/4\tau). \tag{26}$$

The latter condition should be satisfied simultaneously with (19). As a result we get a simple condition $R >> \Gamma^{-1/2}$ imposed onto R, which restricts an applicability of the formula (26) by large R. Now we comment on the origin of the condition (19). It means that an action on the classical trajectory $S_0 \sim R^2/m\tau$ is much larger than the contribution caused by the random field $\delta S \sim R^{2/3}\tau^{1/3}\Gamma^{2/3}m^{-1/3}$.

We see that the random field indeed leads to the suppression of a probability of particle's propagation with respect to the free case. The character of the propagation resembles a motion in some "viscous media" with a momentum-dependent dephasing time $\tau \sim \frac{m}{(\Gamma k)^{1/3}}$.[35]

IV. OPTIMAL FLUCTUATION METHOD AND DENSITY OF STATES TAIL

As we mentioned above the low-energy region of the spectrum ($\varepsilon << \Gamma/m$) certainly cannot be described firmly within the lowest-order perturbation theory. Moreover, one could expect that in analogy to the random potential problem the density of states becomes exponentially small near the bottom of the spectrum. It implies that the integral over $\Psi(\mathbf{r})$ can be treated by means of some sort of the saddle-point approximation called conventionally as an "optimal fluctuation" method (see, e.g., Refs. 31–33). To proceed with such a method we write down the partition sum as an integral over $\Psi(\mathbf{r})$ and $\mathbf{a}(\mathbf{r})$ fields:

$$Z = \left\langle \int D\Psi(\mathbf{r})D\Psi^\dagger(\mathbf{r})\exp\left[-\frac{\beta}{2m}\int|\nabla\Psi - i(\mathbf{a}+\mathbf{A})\Psi|^2 d\mathbf{r}^2 - \frac{1}{2\Gamma}\int d^2\mathbf{r}(\epsilon_{\mu\nu}\partial_\mu a_\nu)^2\right]\right\rangle_{\mathbf{a}(\mathbf{r})}. \tag{27}$$

We also added a probe homogeneous external magnetic field B with vector potential $\mathbf{A}$ to be able to calculate a diamagnetic susceptibility $\chi(\beta)$. Notice that here $\Psi(\mathbf{r})$ has a meaning of a single-particle wave function. It naturally occurs from the total field operator in the one-particle sector characterized by a finite norm $\int\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})d^2\mathbf{r} < \infty$. Due to this condition there are no polarization corrections, so one can avoid using a replica trick or a supersymmetry even at calculating off-diagonal Green functions. However, our principal goal is a calculation of a diagonal Green function or a density of states $\rho(\varepsilon)$ related directly with the partition sum

$$Z(\beta,B) = \int d\varepsilon\exp(-\beta\varepsilon)\rho(\varepsilon).$$

After averaging over a random field $\mathbf{a}(\mathbf{r})$ in (27) we obtain an integral

$$Z(\beta,B) = \int D\Psi(\mathbf{r})D\Psi^\dagger(\mathbf{r})\exp[-S\{\Psi(\mathbf{r})\}]$$

with the action

$$S\{\Psi(\mathbf{r})\} = \frac{\beta}{2m}\int|\nabla\Psi|^2 d\mathbf{r}^2 + \int d^2r\,d^2r'\left(\beta\mathbf{j}(\mathbf{r}) + \frac{B}{\sqrt{\Gamma}}\nabla\right)K(\mathbf{r},\mathbf{r}')\left(\beta\mathbf{j}(\mathbf{r}') + \frac{B}{\sqrt{\Gamma}}\nabla'\right) - \tfrac{1}{2}\mathrm{Tr}[\ln(K)] + \tfrac{1}{2}\mathrm{Tr}[\ln(K_0)], \tag{28}$$

where we have introduced the "current" density $\mathbf{j}(\mathbf{r}) = \frac{1}{2im}\{\Psi(\mathbf{r})^\dagger[\nabla\Psi(\mathbf{r})] - [\nabla\Psi(\mathbf{r})]^\dagger\Psi(\mathbf{r})\}$ and the operator

$$K(\mathbf{r},\mathbf{r}') = \left\langle\mathbf{r}\left|\left(-\frac{1}{\Gamma}\nabla^2 + \frac{\beta\Psi^\dagger\Psi}{m}\right)^{-1}\right|\mathbf{r}'\right\rangle, \tag{29}$$

which is the inverse kernel of the quadratic form over $\mathbf{a}(\mathbf{r})$ in the exponent in (27), K_0 is the same operator without Ψ coming in (28) from the normalization integral, Tr stands for a trace operation.

We intend to find a normalizable configuration $\Psi(\mathbf{r})$ which is a saddle point of the effective nonlinear theory

(28). In what follows we shall restrict ourselves onto real wave functions $\Psi(\mathbf{r}) = \Phi(r/R)/R$ describing zero-current states ($\mathbf{j}(\mathbf{r}) \equiv 0$). One can check that states with nonzero angular momentum are not favored because of an essential cost of the action. Within the subspace of real normalizable functions the size of the state R appears to be the only relevant collective coordinate with respect to which a minimization has to be done. We first perform this procedure without an external field ($B = 0$), when the second term is absent from the action (28).

Recalling about a momentum cut-off a^{-1} on a lattice we observe that the operator $K(\mathbf{r}, \mathbf{r}')$ is almost diagonal in a coordinate space if the condition

$$\frac{\Gamma\beta a^2}{mR^2} >> 1 \tag{30}$$

is satisfied. Then assuming that it does take place we estimate the trace of this operator in the form

$$\mathrm{Tr}[\ln(K/K_0)]$$

$$\approx \int \frac{d^2p}{(2\pi)^2} \int d^2\mathbf{r} \left\langle \mathbf{r} \left| \ln\left(\frac{\frac{1}{\Gamma}\mathbf{p}^2}{\frac{1}{\Gamma}\mathbf{p}^2 + \frac{\beta\Psi^\dagger\Psi}{m}} \right) \right| \mathbf{r} \right\rangle$$

$$\approx \int \frac{d^2r}{a^2} \ln\left(1 + \frac{\Gamma\beta|\Psi(\mathbf{r})|^2 a^2}{m}\right). \tag{31}$$

Estimating the total action as a function of the collective coordinate R we obtain the following expression:

$$S(R) = \frac{c_1\beta}{mR^2} + \frac{c_2R^2}{a^2} \ln\left(\frac{\Gamma\beta a^2}{mR^2}\right), \tag{32}$$

where c_1 and c_2 are numerical constants of order of unity. The extremal value of R is given by

$$R_{\rm opt} = \left[\frac{c_1\beta a^2}{c_2 m} \ln^{-1}\left(\frac{\Gamma^2\beta a^2}{m}\right)\right]^{1/4}. \tag{33}$$

One can easily see that in agreement with our conjecture the condition (30) is satisfied at $\Gamma^2\beta a^2/m >> 1$. From (32) and (33) we obtain a low-temperature behavior of the partition sum as given by the exponentially decaying asymptotics

$$Z(\beta) \sim \exp\left\{-\left[\frac{c\beta}{a^2m} \ln\left(\frac{\Gamma^2\beta a^2}{m}\right)\right]^{1/2}\right\}, \tag{34}$$

where $c = c_1c_2$. The low-energy behavior of the density of states can be found as an inverse Laplace transform

$$\rho(\varepsilon) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{d\beta}{2\pi} \exp(\beta\varepsilon) Z(\beta). \tag{35}$$

The integration contour in the complex β plane is drawn to the right from all singularities of $Z(\beta)$. Now we estimate the integral (35) by using the asymptotics (34) and obtain the following form of the tail of a density of states at $\varepsilon << \Gamma/m$:

$$\rho(\varepsilon) \sim m \left[\frac{m^2\varepsilon^2}{\Gamma^2} \ln\left(\frac{\Gamma}{m\varepsilon}\right)\right]^{c/\varepsilon m a^2}. \tag{36}$$

Several remarks are in order. First, the physical meaning of the result (36) can be clarified by a simple and general estimations going back to Lifshitz consideration of band tails due to repulsive impurities.[31,33] The density of states $\rho(\varepsilon)$ is essentially the probability to find a configuration of external field (random potential in the impurities problem or random fluxes in our case) in which the energy is ε. The low energy (close to bottom of the band) can be reached if the random field is removed from some region of the system. This depletion region must be large enough to avoid a kinetic energy increase due to dimensional quantization. The optimal size is obviously $R \sim (\varepsilon m)^{-1/2}$. Then the admissible flux ϕ per each of $(R/a)^2$ plaquettes can be estimated as $\phi \sim \varepsilon m a^2$ (the equal fluxes of this scale correspond to a uniform magnetic field $B = \phi a^{-2}$ and produce the diamagnetic energy shift $B/2m \sim \varepsilon$). On each plaquette the probability to find a proper flux is $\sim \varepsilon m/\Gamma$, and on all $(R/a)^2$ independent plaquettes it is correspondingly $\sim (\varepsilon m/\Gamma)^{(R/a)^2}$. As a consequence we obtain the estimation $\rho(\varepsilon) \propto (\varepsilon m/\Gamma)^{(R/a)^2} \sim (\varepsilon m/\Gamma)^{1/\varepsilon m a^2}$ which essentially coincides with our former result (36). One could see that the arguments for this estimation are rather general. For example, one obtains qualitatively similar results in the toy model of the random scalar potential $U(\mathbf{r}) = \phi(\mathbf{r})^2$ being the square of a variable ϕ distributed with a Gaussian probability $\exp[-\phi^2/\Gamma a^2]$. In contrast to an ordinary Gaussian random potential problem, this model obviously has a spectrum bounded from below with the exponentially low density of states $\rho(\varepsilon) \propto (\varepsilon/\Gamma)^{1/\varepsilon m a^2}$ at $\varepsilon << \Gamma/m$.

Notice that the form of the tail demonstrates an explicit dependence on the short-wavelength cut-off $\Lambda = a^{-1}$. At $\Lambda \to \infty$ the tail disappears. We find this circumstance to be in agreement with our assumption that the states responsible for the formation of the Lifshitz tail are localized in space. It should be mentioned that in a continuous theory a finite short-wavelength cut-off Λ has to be introduced to render an averaged squared local magnetic field $\langle B^2(\mathbf{r})\rangle$ finite.

We also note that a complete optimization procedure assumes a search of solutions of the nonlinear Schrödinger equation resulting from a variation of the action of the effective theory (25) over $\Psi(\mathbf{r})$. We performed this procedure to check if the assumption about the unique space scale R of the trial function is really valid. We found the low-energy solution (i.e., at $\Gamma^2\beta a^2 m^{-1} >> 1$) to be close to the Bessel function $J_0[r/R_0(\beta)]/R_0(\beta)$ with an exponential tail starting near its first zero. Such a solution obviously corresponds to a localized state.

Another of our purpose is an estimation of the orbital diamagnetic susceptibility. Here we have to keep in the action (28) the second term containing the weak external field B, and we obtain the quadratic contribution to the extremal action

$$\delta S(\beta, B) = \frac{B^2a^2}{\Gamma} \beta \frac{\partial}{\partial\beta} \mathrm{Tr}[\ln(K)]. \tag{37}$$

Substituting expression (32) we have

$$\delta S(\beta, B) = \frac{B^2 a^2}{\Gamma}\left[\frac{c\beta}{a^2 m}\ln\left(\frac{\Gamma^2\beta a^2}{m}\right)\right]^{1/2}, \qquad (38)$$

what leads us to the susceptibility

$$\chi(\beta) = -\left.\frac{\partial}{\partial B^2}\right|_{B=0}\frac{\partial}{\partial\beta}\ln Z(\beta, B) \sim \left[\frac{ca^2}{\beta\Gamma^2 m}\ln\left(\frac{\Gamma^2\beta a^2}{m}\right)\right]^{1/2}. \qquad (39)$$

The leading low-temperature behavior $\chi(\beta) \propto T^{1/2}$ has to be contrasted with those $\chi(\beta) \sim Ta^2/\Gamma$ resulting from the model of a homogeneous random field[20] and $\chi(\beta) \sim 1/\Gamma m$ from the Caldeira-Leggett model.[19] We find the vanishing behavior of $\chi(\beta)$ to be natural because for an external field B much smaller than the amplitude of the random one there exists an optimal configuration of random fluxes canceling this field out with the probability only slightly dependent B. That leads to an absence of a diamagnetic response at zero temperature. In the next section we shall discuss the results of Monte Carlo simulations which illustrate this statement.

V. NUMERICAL SIMULATION

We take a square lattice with the period a and site coordinates designated as $\mathbf{R} \equiv \{x, y\}$. The applied world-line Monte Carlo procedure will be only sketched here with the proper description to be given elsewhere.[34] We consider the partition sum $Z = \mathrm{Tr}(\exp[-\beta\hat{H}])$. The density matrix under the trace sign corresponds to an imaginary time evolution operator from 0 to inverse temperature β. This interval is divided into a very large number N of small time intervals $\Delta\tau = \beta/N$ to define the world line of the particle $\mathbf{R}(\tau)$ on a square lattice. The partition sum is then given by a path summation

$$Z = \sum_{\mathbf{R}(\tau)} W\{\mathbf{R}(\tau)\} \qquad (40)$$

over world lines with coinciding terminal points $\mathbf{R}(0) = \mathbf{R}(\beta)$. At each step $\Delta\tau$ the particle is assumed to either stay at the site or move to one of its closest neighbor. In the statistic weight of the world line $W\{\mathbf{R}(\tau)\}$ each move of the particle by one lattice site costs the factor $\Delta\tau$. These factors can be collected into the weight $W_0\{\mathbf{R}(\tau)\}$, corresponding to a free particle on a lattice. There remains the phase factor $\exp[i\phi\{\mathbf{R}(\tau)\}]$ due to the magnetic flux taken by the world line.

In our problem we have the uniform magnetic field B and the random magnetic field described by fluxes through a plaquette ϕ_{R} distributed in a Gaussian way $w(\phi_{\mathrm{R}}) \propto \exp[-\phi_{\mathrm{R}}^2/4\Gamma]$ independently on each other. For the closed world line the resulting phase factor, averaged over samples of the random flux distribution, is expressed as

$$\langle\exp[i\phi\{\mathbf{R}(\tau)\}]\rangle = \exp[iBS - \Gamma\tilde{S}]$$

through its conventional S and ampere $\tilde{S}$ area evaluated for each world line as

$$S\{\mathbf{R}(\tau)\} \equiv \sum_{\tau} y(\tau)dx(\tau),$$

$$\tilde{S}\{\mathbf{R}(\tau)\} \equiv \sum_{\tau,\tau'} |y(\tau) - y(\tau')|dx(\tau)dx(\tau')\delta(\tau,\tau'),$$

where $dx(\tau) \equiv [x(\tau - \Delta\tau) - x(\tau)]$ and $\delta(\tau,\tau') \equiv \delta([x(\tau + \Delta\tau) + x(\tau)] - [x(\tau' + \Delta\tau) + x(\tau')]$.

Here we are interested in the energy $E(\beta)$ and the static magnetic susceptibility $\chi(\beta)$ at zero magnetic field, which are expressed as a Monte Carlo summation:

$$E(\beta) = -\frac{dZ}{Zd\beta} = -T\langle n\{\mathbf{R}(\tau)\}\rangle, \qquad (41)$$

$$\chi(\beta) = -\left.\frac{d^2}{dB^2}\right|_{B=0}\frac{dZ}{Zd\beta} = -T\langle n\{\mathbf{R}(\tau)\}S^2\{\mathbf{R}(\tau)\}\rangle + T\langle n\{\mathbf{R}(\tau)\}\rangle\langle S^2\{\mathbf{R}(\tau)\}\rangle, \qquad (42)$$

where $n\{\mathbf{R}(\tau)\}$ denotes the number of hole moves on the world line $\mathbf{R}(\tau)$ and $\langle V\rangle$ means the thermodynamical

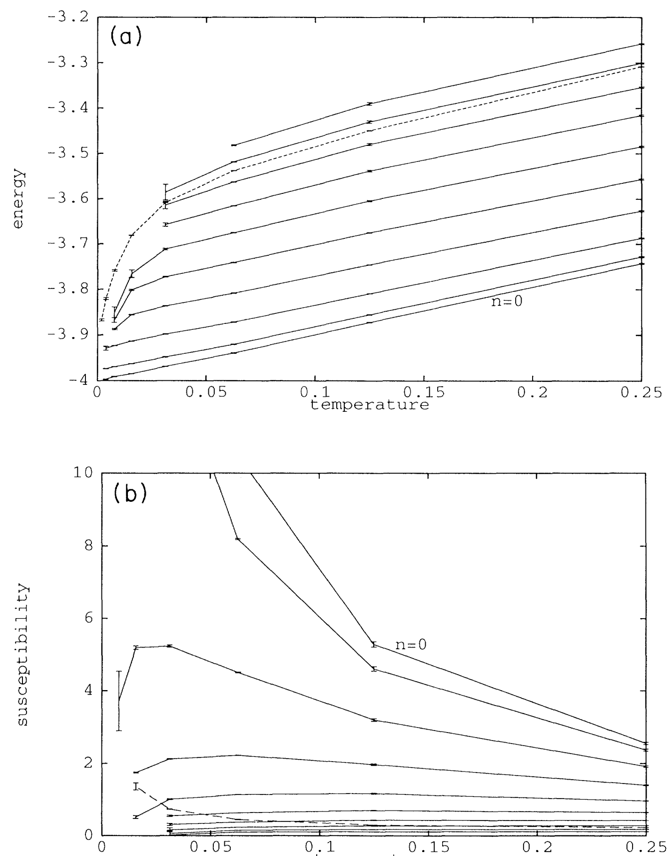

FIG. 3. Quantum Monte Carlo results for the (a) energy and (b) susceptibility as a function of the temperature for $\Gamma = n^2\ln(2)/4$, $n = 0, 1, \ldots, 9$ from (a) bottom to top and (b) from top to bottom. Dashed lines show the results for the infinite-U Hubbard model, whose high-temperature expansion corresponds to $n = 8$.

average of the value $V\{\mathbf{R}(\tau)\}$:

$$\langle V\rangle \equiv \frac{\sum_{\mathbf{R}(\tau)} V\{\mathbf{R}(\tau)\}W_0\{\mathbf{R}(\tau)\}\exp[-\Gamma\tilde{S}\{\mathbf{R}(\tau)\}]}{\sum_{\mathbf{R}(\tau)} W_0\{\mathbf{R}(\tau)\}\exp[-\Gamma\tilde{S}\{\mathbf{R}(\tau)\}]}. \tag{43}$$

The principal thing is to organize the Monte Carlo process generating different world lines of the particle with the probability proportional to the free-particle weight $W_0\{\mathbf{R}(\tau)\}$. Then this factor disappears from Eq. (43). The results of our simulations are presented in Figs. 3(a) and 3(b) for different amplitudes of fluctuation Γ. One sees that both energy and susceptibility go down at low temperature. We are not able to check precisely the square-root asymptotics for $\chi(\beta)$, but we can state that some tendency to this law starts in the same region of T where the energy deviates from the close-to-linear law and starts to drop to the real bottom of the spectrum.

VI. CONCLUSION

In this paper we undertook an attempt to get insight into an exciting problem of single-particle dynamics in a 2D spatially random magnetic field. In contrast to the problem of a random scalar potential the random magnetic field has a significant effect already onto one-particle spectrum. It leads to a drastic suppression of the density of states near an unshifted lower bound of the spectrum, the density of states in the Lifshitz-like tail being exponentially small. Even in the quasiclassical regime the particle's propagation resembles a motion in some "viscous media" and can be characterized by a momentum-dependent dephasing time. As another consequence of the spectrum transformation we obtain that an orbital diamagnetic susceptibility vanishes at zero temperature as $T^{1/2}$. We also show that within a self-consistent perturbative calculation the particle's mobility $\mu(\omega)$ goes to zero at $\omega \to 0$. On the basis of these observations one could suppose that the states below some threshold $\epsilon_0 \sim \frac{\Gamma}{m}$ are localized. However, the real nature of the states in the band tail remains to be clarified. A primary question is about a possible coexistence of extended and localized states in the spectrum.

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*Permanent address: Landau Institute for Theoretical Physics, 2, st. Kosygina, 117940 Moscow, Russia.
†On leave from Institute of Solid State Physics, 142432 Chernogolovka, Moscow District, Russia.

[1] P.W. Anderson, Science **235**, 1196 (1987).
[2] G. Baskaran, Z. Zou, and P.W. Anderson, Solid State Commun. **63**, 973 (1987).
[3] G. Baskaran and P.W. Anderson, Phys. Rev. B **37**, 580 (1988).
[4] L.B. Ioffe and A.I. Larkin, Phys. Rev. B **39**, 8988 (1989).
[5] P.B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988).
[6] P.A. Lee, Phys. Rev. Lett. **63**, 680 (1989).
[7] M. Yu. Reizer, Phys. Rev. B **40**, 11 571 (1989).
[8] N. Nagaosa and P.A. Lee, Phys. Rev. Lett. **64**, 2450 (1990); Phys. Rev. B **43**, 1233 (1991).
[9] L.B. Ioffe and P.B. Wiegmann, Phys. Rev. Lett. **65**, 653 (1990).
[10] L.B. Ioffe and G. Kotliar, Phys. Rev. B **42**, 10 348 (1990).
[11] L.B. Ioffe and P.B. Wiegmann, Phys. Rev. B **45**, 519 (1992).
[12] L.B. Ioffe, V. Kalmeyer, and P.B. Wiegmann, Phys. Rev. B **43**, 1219 (1991).
[13] M.P.A. Fisher and D.-H. Lee, Phys. Rev. B **39**, 2756 (1989).
[14] M.V. Feigelman, Physica A **168**, 319 (1990).
[15] M.V. Feigelman, A.I. Larkin, V. Geshkenbein, and L.B. Ioffe (unpublished).
[16] L.B. Ioffe and V. Kalmeyer, Phys. Rev. B **44**, 750 (1991).
[17] C. Pryor and A. Zee, Phys. Rev. B **46**, 3116 (1992).
[18] J.M. Wheatley and T.M. Hong, Phys. Rev. B **43**, 6288 (1991).
[19] J.M. Wheatley and A.J. Schofield, Int. J. Mod. Phys. B **6**, 655 (1992).
[20] G. Gavazzi and J.M. Wheatley, Phys. Rev. B **45**, 8791 (1992).
[21] Y. Nagaoka, Solid State Commun. **3**, 409 (1965); Phys. Rev. **147**, 392 (1966).
[22] L.N. Bulaevskii and D.I. Khomskii, Zh. Eksp. Teor. Fiz. **52**, 1603 (1967) [Sov. Phys. JETP **25**, 1067 (1967)]; Fiz. Tver. Tela (Leningrad) **9**, 3070 (1968) [Sov. Phys. Solid State **9**, 2422 (1968)].
[23] W.F. Brinkman and T.M. Rice, Phys. Rev. B **5**, 1324 (1970).
[24] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[25] L.P. Gor'kov, A.I. Larkin, and D.E. Khmel'nitskii, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 (1979)].
[26] F. Wegner, Z. Phys. B **35**, 207 (1979).
[27] S.F. Edwards, J. Phys. C **8**, 1660 (1975).
[28] K.B. Efetov, Zh. Eksp. Teor. Fiz. **82**, 872 (1982) [Sov. Phys. JETP **55**, 514 (1982)].
[29] S.V. Maleev and B.P. Toperverg, Zh. Eksp. Teor. Fiz. **69**, 1440 (1975) [Sov. Phys. JETP **42**, 734 (1975)].
[30] D. Vollhardt and P. Wolfle, Phys. Rev. Lett. **45**, 842 (1980); Phys. Rev. B **22**, 4666 (1980).
[31] I.M. Lifshitz, Usp. Fiz. Nauk **83**, 617 (1965) [Sov. Phys. Usp. **7**, 549 (1964)].
[32] B.I. Halperin and M. Lax, Phys. Rev. **148**, 722 (1966).
[33] I.M. Lifshitz, Zh. Eksp. Teor. Fiz. **53**, 743 (1967) [Sov. Phys. JETP **26**, 462 (1968)]; I.M. Lifshitz, S.A. Gredeskul, and L.A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley-Interscience, New York, 1988), Chap. 17.
[34] S.V. Meshkov and J.C. Anglès-d'Auriac (unpublished).
[35] We recently became aware of the work by B.L. Altshuler and L.B. Ioffe, Phys. Rev. Lett. **69**, 2979 (1992) containing some results similar to those obtained in Sec. III of this paper.