Quenched bond-mixed cubic ferromagnet in a planar self-dual lattice: Critical behavior

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The critical behavior of the quenched bond-mixed ferromagnetic cubic model, on a planar self-dual hierarchical lattice, is investigated within a simple real-space renormalization group. We obtain the complete phase diagram of the system, exhibiting three phases. This phase diagram is believed to be of high precision for the square lattice. The correlation-length critical exponents and the universality classes are determined as well.

I. INTRODUCTION

The study of the quenched bond (site)-diluted and bond (site)-mixed models is motivated by both theoretical interest and various possible experimental applications on disordered magnetic systems. Several works have been dedicated to these models, very particularly to the Heisenberg model (see Ref. 1 and references therein), the Ising model (see Ref. 2 and references therein), and the cubic model.[3]

The cubic model or discrete N-vector model[3-7] has been conveniently applied to describe phase transitions in rare-earth compounds,[7] molecular oxygen adsorbed on graphite,[8] order-disorder transition in atomic oxygen on tungsten,[8] among many other applications (see Ref. 3 and references therein). The dimensionless Hamiltonian associated with the cubic model is given by

$$\beta H = -NK \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \;, \tag{1}$$

where $\beta \equiv 1/k_B T$, $\langle ij \rangle$ runs over all the couples of first-neighboring sites, the spin $\mathbf{S}_i$ at any given site is an N-component vector which points along the edges of an N-dimensional hypercube, i.e., $\mathbf{S}_i = (\pm 1,0,0,\ldots,0)$, $(0,\pm 1,0,\ldots,0)$, $\ldots$, $(0,0,0,\ldots,\pm 1)$. As we shall see later on, the model is closed, under the renormalization-group (RG) transformation, if a quadrupolar interaction is included into the Hamiltonian, i.e.,

$$\beta H = -NK \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - N^2 L \sum_{\langle ij \rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \;. \tag{2}$$

This Hamiltonian is an interesting one since it contains, as limiting cases, various important statistical models, namely, the Ising model ($N=1$), the $Z(4)$ model ($N=2$), the $2N$-state Potts model ($NL=K$), the N-state Potts model ($K=0$) and the grand-canonical statistics of the self-avoiding walk ($N \to 0$).[9]

The purpose of the present paper is to study, for the first time as far as we know, the criticality of the quenched bond-mixed ferromagnetic cubic model on the square lattice which we approximate here by a self-dual hierarchical lattice. The RG formalism is presented in Sec. II, our results in Sec. III, and our conclusions in Sec. IV.

II. QUENCHED BOND-MIXED MODEL AND RG FORMALISM

The quenched bond-mixed model is defined by Hamiltonian (2) where the following probability law is assumed for the coupling constants K and L:

$$P(K,L) = (1-p)\delta(K-K_1)\delta(L-L_1) + p\delta(K-K_2)\delta(L-L_2) \;, \tag{3}$$

where $0 \le p \le 1$. By imposing the ground state to be ferromagnetic we obtain that $K_1 > 0$, $K_2 > 0$, $K_1 + NL_1 > 0$, and $K_2 + NL_2 > 0$.

Our treatment consists in constructing a real-space RG which associates the probability distribution given by (3) with each bond of the self-dual Wheatstone bridge (Fig. 1), a graph which guarantees an excellent approximation for the square lattice.[10] Let us introduce for convenience

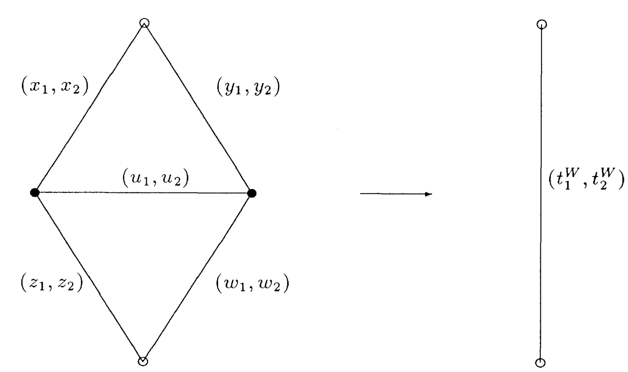

FIG. 1. Iteration associated with the Wheatstone-bridge hierarchical lattice (the full and open circles, respectively, denote the internal and terminal sites of the graph).

the transmissivity variables[6,11–15]

$$t_1=\frac{1-\exp(-2NK)}{1+2(N-1)\exp[-N(K+NL)]+\exp(-2NK)}\in[0,1]\ , \tag{4}$$

$$t_2=\frac{1-2\exp[-N(K+NL)]+\exp(-2NK)}{1+2(N-1)\exp[-N(K+NL)]+\exp(-2NK)}\in[0,1]\ , \tag{5}$$

hence

$$\exp[-N(K+NL)]=\frac{1-t_2}{1+Nt_1+(N-1)t_2}\ , \tag{6}$$

$$\exp(-2NK)=\frac{1-Nt_1+(N-1)t_2}{1+Nt_1+(N-1)t_2}\ . \tag{7}$$

The transmissivity (t_1,t_2) generalizes the scalar one introduced for the Ising,[16] and Potts,[11] and $Z(4)$ (Ref. 17) models. The equivalent transmissivity $(t_1^{(s)},t_2^{(s)})$ $((t_1^{(p)},t_2^{(p)}))$ of a series (parallel) array of two bonds with transmisivities $(t_1^{(1)},t_2^{(1)})$ and $(t_1^{(2)},t_2^{(2)})$ is given by

$$t_r^{(s)}=t_r^{(1)}t_r^{(2)}\ ,\quad (r=1,2)\ , \tag{8}$$

$$(t_r^{(p)})^D=(t_r^{(1)})^D(t_r^{(2)})^D\ ,\quad (r=1,2)\ , \tag{9}$$

with

$$(t_1^{(q)})^D=\frac{1-Nt_1^{(q)}+(N-1)t_2^{(q)}}{1+Nt_1^{(q)}+(N-1)t_2^{(q)}}\ ,\quad (q=1,2,p)\ , \tag{10}$$

$$(t_2^{(q)})^D=\frac{1-t_2^{(q)}}{1+Nt_1^{(q)}+(N-1)t_2^{(q)}}\ ,\quad (q=1,2,p)\ , \tag{11}$$

where D stands for dual (see Ref. 18 and references therein).

The series and parallel algorithms enable the calculation of the equivalent transmissivity of any (two-terminal) graph which is reducible in series and parallel operations. For those graphs (e.g., the Wheatstone-bridge array of Fig. 1) which are not reducible, the Break-collapse method[12,19] can be used. We have applied this method to Fig. 1 and have obtained the following equivalent transmissivity $(t_1^{(W)},t_2^{(W)})$ where

$$t_1^{(W)}=\frac{N_1^{(W)}}{D^{(W)}}\ , \tag{12}$$

$$t_2^{(W)}=\frac{N_2^{(W)}}{D^{(W)}}\ , \tag{13}$$

with

$$\begin{aligned}N_1^{(W)}=&\,y_1w_1+x_1z_1+u_1y_1z_1+u_1w_1x_1+(N-1)^2(u_1x_2y_1z_1w_2+u_1x_1y_2z_2w_1)\\&+(N-1)(x_2y_1z_2w_1+x_1z_1y_2w_2+u_1y_1z_1w_2+u_1x_1z_2w_1+u_1w_1x_1y_2+u_1z_1x_2y_1\\&\quad+u_2w_2z_1x_1+u_2x_1y_2z_1+u_2w_1z_2y_1+u_2w_1x_2y_1)\\&+(N-1)(N-2)(u_2w_2x_1y_2z_1+x_2y_1z_2w_1w_2)\ ,\end{aligned} \tag{14}$$

$$\begin{aligned}N_2^{(W)}=&\,y_2w_2+x_2z_2+u_2y_2z_2+u_2w_2x_2+N(N-1)x_1y_1z_1w_1u_2\\&+(N-2)(N-3)x_2y_2z_2w_2u_2+N(N-2)(x_1y_1z_2w_2u_1+x_2y_2z_1w_1u_1)\\&+(N-2)(x_2y_2z_2w_2+x_2y_2z_2u_2+y_2z_2w_2u_2+x_2z_2w_2u_2+x_2y_2w_2u_2)\\&+N(x_1y_1z_1w_1+y_2z_1w_1u_1+x_2z_1w_1u_1+x_1y_1w_2u_1+x_1y_1z_2u_1)\ ,\end{aligned} \tag{15}$$

$$\begin{aligned}D^{(W)}=&\,1+N(u_1z_1w_1+u_1y_1x_1)+(N-1)(u_2y_2x_2+u_2w_2z_2)+Nx_1z_1w_1y_1\\&+(N-1)x_2y_2z_2w_2+N(N-1)u_2y_1z_1w_1x_1\\&+(N-1)(N-2)x_2z_2w_2u_2y_2+N(N-1)(u_1x_1y_1z_2w_2+u_1x_2y_2z_1w_1)\ .\end{aligned} \tag{16}$$

Let us now focus the bond-mixed problem. Distribution (3) can be equivalently rewritten as follows:

$$\begin{aligned}P(t_1,t_2)=&\,(1-p)\delta(t_1-t_1^{(1)})\delta(t_2-t_2^{(1)})\\&+p\delta(t_1-t_1^{(2)})\delta(t_2-t_2^{(2)})\ ,\end{aligned} \tag{17}$$

where $(t_1^{(1)},t_2^{(1)})$ are related to (K_1,L_1) through Eqs. (4) and (5) [which also provide the relationship between $(t_1^{(2)},t_2^{(2)})$ and (K_2,L_2)]. If we associate now distribution (17) with each bond of the Wheatstone-bridge array of Fig. 1, we obtain for the equivalent distribution $P_W(t_1,t_2)$ the following expression:

$$\begin{aligned}P_W(t_1,t_2)=&\sum_{i=1}^{14}F_i(1-p)^{n_i}p^{5-n_i}\delta(t_1-t_1^{(i)})\\&\times\delta(t_2-t_2^{(i)})\ ,\end{aligned} \tag{18}$$

where F_i and n_i are, respectively, the weights and exponents associated with the possible bond configurations in the (self-dual) Wheatstone bridge. The $\{F_i\}$ satisfy $\sum_{i=1}^{14}F_i=2^5$, and the $\{(t_1^{(i)},t_2^{(i)})\}$ are straightforwardly obtained from Eqs. (12)–(16). The present scaling operation does not preserve the original binary form since P_W has 14 terms. At this level, we shall introduce an approx-

TABLE I. RG values for some fixed points (exact results), correlation-length and crossover exponents, and limiting slopes. Whenever available, exponents and slopes are compared with the square lattice exact results. ν_t and ν_p are the correlation-length critical exponents; Φ_{tp} is the crossover critical exponent.

	Critical points	Critical exponents		Slopes	
	$(p,t_1^{(1)},t_2^{(1)},t_1^{(2)},t_2^{(2)})$	Present RG	Square lattice (exact)	Present RG	Square lattice (exact)
$N=1$	(1,0.414,0.414,0.414,0.414)	$\nu_t=1.15$	1[a]	$-dt_1/dp=0.45$	$6\sqrt{2}-8\approx0.48$[b]
		$\Phi_{tp}=1.56$	?		
	(1,0.414,0.414,0.414,1)	$\nu_t=1.15$	1[a]	$-dt_1/dp=0.45$	$6\sqrt{2}-8\approx0.48$[b]
	(1,0.414,0.414,0.414,0.036)	$\nu_t=1.15$	1[a]	$-dt_1/dp=0.45$	$6\sqrt{2}-8\approx0.48$[b]
$N=2$	(1,0.414,0.414,0.333,0.333)	$\nu_t=0.95$	$\frac{2}{3}$[c]	$-dt_1/dp=0.44$	$\frac{4}{9}\approx0.44$[d]
		$\Phi_{tp}=2.72$	?		
	(1,0.414,0.414,0.414,0.172)	$\nu_t=1.15$	1[b]	$-dt_1/dp=0.55$	$\frac{1}{2}$
				$-dt_1/dt_2=0.5$	$\frac{1}{2}$[e]
$N=3$	(1,0.34,0.34,0.34,0.21)	$\nu_t=0.92$	1st order?		
$\forall N$	(0.5,0,0,1,1)	$\nu_t=1.43$	$\frac{4}{3}$[f]	$-dt_1/dp=3.04$	$4\ln 2\approx2.77$[g]
		$\nu_p=1.43$	$\frac{4}{3}$[f]		
		$\Phi_{tp}=1$	1[f]		

[a]Reference 21.
[b]Reference 22.
[c]Reference 23.
[d]Reference 24.
[e]Reference 25.
[f]Reference 26.
[g]Reference 27

imation, namely, to approach P_W by the following binary distribution,

$$P'(t_1,t_2)=(1-p')\delta(t_1-t_1^{(1)'})\delta(t_2-t_2^{(1)'})+p'\delta(t_1-t_1^{(2)'})\delta(t_2-t_2^{(2)'})\ , \quad (19)$$

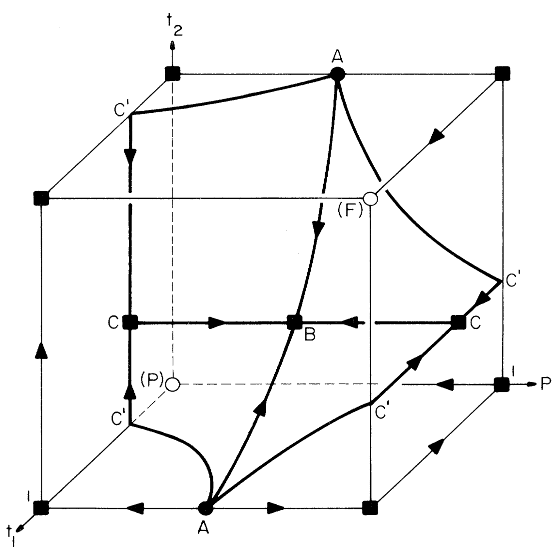

FIG. 2. $N=2$ phase diagram and RG flow in the (p,t_1,t_1,t_2,t_2) subspace ($2N$-state Potts model). ○, ●, ■ denote, respectively, the fully stable, fully unstable, and semistable fixed points. The paramagnetic (P) and ferromagnetic (F) phases are indicated. The critical surface is invariant under the transformation $(p,t_1,t_1,t_2,t_2)\to(1-p,t_2,t_2,t_1,t_1)$. The line ABA lies on the plane $p=\frac{1}{2}$; the line CBC corresponds to $t_1=t_2=1/(\sqrt{2N}+1)$.

where $(p',t_1^{(1)'},t_2^{(1)'},t_1^{(2)'},t_2^{(2)'})$ are functions of $(p,t_1^{(1)},t_2^{(1)},t_1^{(2)},t_2^{(2)})$ to be determined. To do this we impose

$$\langle t_1\rangle_{P'}=\langle t_1\rangle_{P_W}\ , \quad (20)$$

$$\langle t_2\rangle_{P'}=\langle t_2\rangle_{P_W}\ , \quad (21)$$

$$\langle t_1t_2\rangle_{P'}=\langle t_1t_2\rangle_{P_W}\ , \quad (22)$$

$$\langle (t_1)^2t_2\rangle_{P'}=\langle (t_1)^2t_2\rangle_{P_W}\ , \quad (23)$$

$$\langle t_1(t_2)^2\rangle_{P'}=\langle t_1(t_2)^2\rangle_{P_W}\ , \quad (24)$$

where $\langle \cdots \rangle$ denotes the standard mean value (for example,

$$\langle (t_1)^n\rangle_{P'}=(1-p')(t_1^{(1)})^n+p'(t_1^{(2)})^n)\ .$$

This type of approximation (in which the first relevant momenta are preserved) has been successfully used for various problems[2,4,15] which are recovered herein as particular cases.

The set of equations (20)–(24) provides $(p',t_1^{(1)'},t_2^{(1)'},t_1^{(2)'},t_2^{(2)'})$ as explicit functions of $(p,t_1^{(1)},t_2^{(1)},t_1^{(2)},t_2^{(2)})$. Its iteration yields the RG flow in the $(p,t_1^{(1)},t_2^{(1)},t_1^{(2)},t_2^{(2)})$ space [or, equivalently, in the (p,K_1,L_1,K_2,L_2) space], and consequently the phase diagram. The thermal critical exponents are obtained through the calculation of the relevant eigenvalues $\lambda_i(\lambda_i>1)$ of the Jacobian

$$\frac{\partial(p',t_1^{(1)'},t_2^{(1)'},t_1^{(2)'},t_2^{(2)'})}{\partial(p,t_1^{(1)},t_2^{(1)},t_1^{(2)},t_2^{(2)})}$$

associated with the unstable fixed points. More precisely, the correlation-length critical exponent ν_i is given by $\nu_i = \ln(b)/\ln(\lambda_i)$, where b is the RG linear expansion factor ($b=2$ for Fig. 1). The crossover critical exponents Φ_{ij} are defined whenever more than one relevant eigenvalue exists, and are given by $\Phi_{ij} = \ln(\lambda_j)/\ln(\lambda_i)$.

III. RESULTS

Within the present RG we verify the existence of three phases, namely the paramagnetic (P), ferromagnetic (F), and intermediate (I), ones. These phases were already present in the pure[6] and diluted[4] cases and are characterized by

$$\langle \mathbf{S}_i \rangle \equiv \langle (S_i^1, S_i^2, \ldots, S_i^N) \rangle = \mathbf{0} ,$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N = 0 , \quad \forall \alpha \in \{1,2,\ldots,N\} , \quad (P) ;$$

$$\langle \mathbf{S}_i \rangle = \mathbf{0} ,$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N > 0 \ \text{ for } \alpha = \alpha_0 \in \{1,2,\ldots,N\} ,$$
$$< 0 \ \text{ for } \alpha \neq \alpha_0 , \quad (I) ;$$

$$\langle \mathbf{S}_i \rangle \neq \mathbf{0} ,$$

and

$$\langle (S_i^\alpha)^2 \rangle - 1/N > 0 \ \text{ for } \alpha = \alpha_0 \in \{1,2,\ldots,N\} ,$$
$$< 0 \ \text{ for } \alpha \neq \alpha_0 , \quad (F) .$$

These phases are associated, in the $(p, t_1^{(1)}, t_2^{(1)}, t_1^{(2)}, t_2^{(2)})$

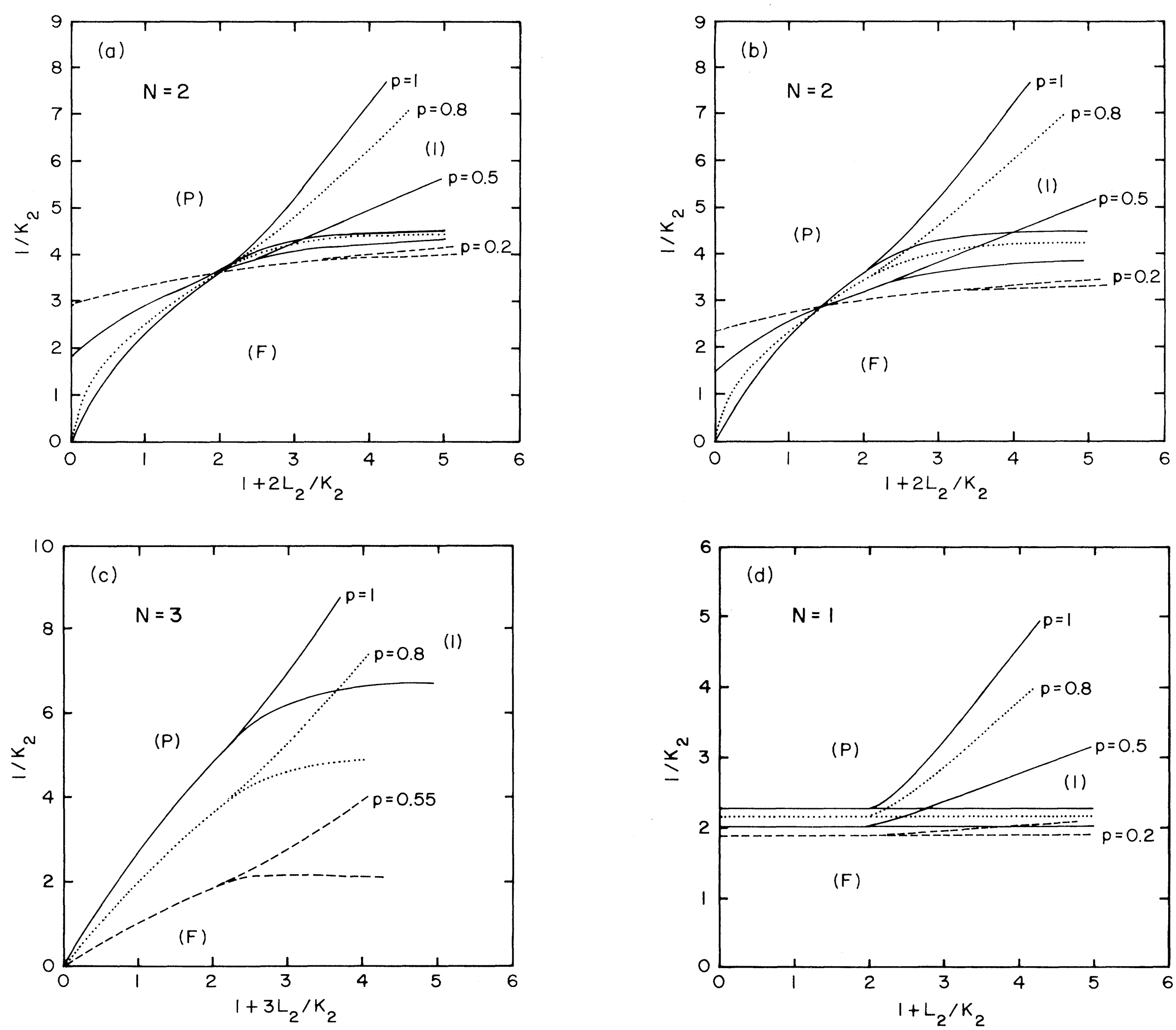

FIG. 3. Typical cuts of the full five-dimensional space $(p, 1/K_2, L_2/K_2, K_1/K_2, L_1/K_1)$: (a) $N=2$, $K_1=2L_1$, $K_1/K_2=1$; (b) $N=2$, $K_1=2L_1$, $K_1/K_2=0.8$; (c) $N=3$, $K_1=L_1=0$; (d) $N=1$, $K_1=L_1, K_1/K_2=0.8$. (P), (F), and (I), respectively, refer to the paramagnetic, ferromagnetic, and intermediate phases.

space, with the following fully stable fixed points:

$(0,0,0,0,0)$ and $(1,0,0,0,0)$ (P) ,

$(1,1,1,1,1)$ and $(0,1,1,1,1)$ (F) ,

$(1,0,1,0,1)$ and $(0,0,1,0,1)$ (I) .

Notice that, in the (I) phase, the system has chosen one of the axes, but not a sense within that axis.

The model we are considering here contains, in many different ways, the Ising, the Potts, and the bond percolation models as particular cases. In all of them, the square lattice exact critical points are recovered within the present RG. Furthermore, various slopes are almost exactly reproduced (see Table I). We have, consequently, good confidence that the RG critical surfaces we obtain here can be considered as a high-precision approximation of the corresponding ones in the square lattice. This satisfactory fact is clearly related to the self-duality of the Wheatstone-bridge array we have used, thus preserving the self-duality of the square lattice. The various critical exponents we have obtained (see Table I) are exact for the hierarchical lattice but clearly not for the square lattice.

The subspace (p,t_1,t_1,t_2,t_2) is closed under RG; it corresponds to the particular case $NL=K$ ($2N$-state Potts model with dimensionless coupling constant $2NK$). For the case $N=1$ (Ising model) the present approach reproduces the solution obtained in Ref. 2. The phase diagram associated with $N=2$ is presented in Fig. 2. The points marked A in this figure are fully unstable fixed ones and correspond to the bond percolation limit. The axes CBC and $C'CC'$ correspond to the pure $2N$-state Potts model. The point B is unstable out of the critical surface; within this surface it is fully stable if $N \leq N^* \approx 2.6$, and semistable if $N>N^*$; indeed, at $N=N^*$, new fixed points appear through a bifurcation.[20]

In Fig. 3 we show typical cuts of the phase diagram in the five-dimensional full space. In Figs. 3(a) and 3(b) we present cuts for $N=2$. These diagrams share with the pure case ($p=1$) the presence of three phases. However, when the value of p decreases, the intermediate phase region shrinks. Fig. 3(c) corresponds to the $N=3$ diluted case, and we verify that for $p<p_c=0.5$ (critical percolation probability) the ferromagnetic phase disappears. The $N=1$ case is shown in Fig. 3(d); let us stress that the P-I critical line must be considered as a mathematical artifact.[6,19]

The behavior for arbitrary values of N has been analyzed and the corresponding diagrams present the same characteristics mentioned above.

IV. CONCLUSION

In this paper we have studied, within a real-space renormalization-group method, the criticality of the quenched bond-mixed discrete ferromagnetic cubic model (N-vector model) in a square lattice, herein approached by a self-dual hierarchical lattice. The five parameters of the present problem ensure considerable freedom for renormalization, thus leading to results whose precision is quite higher than that obtained for the diluted model.[4]

The present treatment provides an efficient method with few mathematical requirements. In fact, the present procedure is practically as simple as a mean-field approximation, providing nevertheless quite superior results. As a final remark, it is worth stressing that the criticality of the present model can be understood in terms of competitions between the Ising, N-state, and $2N$-state Potts, cubic, and percolation models.

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