Uncertainty principle and off-diagonal long-range order in the fractional quantum Hall effect

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(Received 5 October 1992)

A natural generalization of the Heisenberg uncertainty principle inequality holding for non-Hermitian operators is presented and applied to the fractional quantum Hall effect (FQHE). This inequality was used in a previous paper to prove the absence of long-range order in the ground state of several one-dimensional (1D) systems with continuous-group symmetries. In this paper, we use it to rule out the occurrence of Bose-Einstein condensation in the bosonic representation of the FQHE wave function proposed by Girvin and MacDonald. We show that the absence of off-diagonal long-range order in this two-dimensional (2D) problem is directly connected with the q^2 behavior of the static structure function $S(q)$ at small momenta.

Important results on the role of fluctuations in systems with broken symmetries have been obtained in the past using a famous inequality due to Bogoliubov.[1] This inequality provides important constraints on the static response of the system and yields,[2,3] through the use of the fluctuation-dissipation theorem, the following result for the fluctuations of a general physical operator A:

$$\langle \{ A^\dagger, A \} \rangle \langle [B^\dagger,[H,B]] \rangle \geq k_B T |\langle [A^\dagger,B] \rangle|^2 . \quad (1)$$

In Eq. (1), A and B are two arbitrary (non-Hermitian in general) operators and H and T are, respectively, the Hamiltonian and the temperature of the system. Furthermore $\{A^\dagger,B\} = A^\dagger B + BA^\dagger$, $[A^\dagger,B]=A^\dagger B - BA^\dagger$, and $\langle\ \rangle$ is the statistical average (without any loss of generality, here and in the following we assume $\langle A \rangle = \langle B \rangle = 0$).

Inequality (1) was successfully employed in Refs. 4–6 to prove the absence of long-range order at finite temperature in a relevant class of one- (1D) and two-dimensional (2D) systems including Bose superfluids and superconductors, isotropic ferromagnets and antiferromagnets, and crystals. The Bogoliubov inequality is not, however, particularly useful in the study of fluctuations in the low-temperature regime dominated by quantum effects. Actually, Eq. (1) becomes useless in the zero-temperature limit, while the original Bogoliubov inequality, without the use of the fluctuation-dissipation theorem, provides direct information only on the static response[2,3] of the system and not on its fluctuations.

On the other hand, fundamental restrictions on quantum fluctuations are in general provided by the Heisenberg uncertainty principle. This principle, usually formulated for Hermitian operators, can be generalized naturally to the case of non-Hermitian operators starting from the inequality

$$\sqrt{\langle A^\dagger A \rangle \langle B^\dagger B \rangle} + \sqrt{\langle A A^\dagger \rangle \langle B B^\dagger \rangle} \geq |\langle [A^\dagger,B] \rangle| , \quad (2)$$

which can be easily proven applying the Schwartz inequality to the scalar product defined by $(A,B) \equiv \langle A^\dagger B \rangle$, and using the inequality $|\langle [A^\dagger,B] \rangle| \leq |\langle A^\dagger B \rangle| + |\langle B A^\dagger \rangle|$. From Eq. (2), noting that $|a| + |b| \geq 2\sqrt{|a||b|}$, one immediately obtains the result

$$\langle \{ A^\dagger, A \} \rangle \langle \{ B^\dagger, B \} \rangle \geq |\langle [A^\dagger,B] \rangle|^2 \quad (3)$$

already derived in Ref. 7. When A and B are Hermitian, both Eqs. (2) and (3) coincide with the usual uncertainty principle inequality $\langle A^2 \rangle \langle B^2 \rangle \geq \frac{1}{4} |\langle [A,B] \rangle|^2$. In Ref. 7, we derived result (3) using a different method, based on auxiliary operators related to the physical ones through a linear transformation.[8] Other generalizations of the uncertainty principle for non-Hermitian operators have been considered in the literature (see, for example, Refs. 9 and 10). However, differently from such generalizations, our inequalities (2) and (3) are characterized by the occurrence of the commutator on the right-hand side, a typical and important feature of the traditional Heisenberg uncertainty principle.

It is worth comparing the Bogoliubov inequality (1) with the uncertainty principle inequality (3). While the former explicitly accounts for the role of thermal fluctuations, the latter turns out to be particularly powerful at low temperatures where quantum fluctuations become more and more important. Note that result (3) does not involve the Hamiltonian of the system in an explicit way, and hence it expresses fundamental properties of fluctuations, regardless of the explicit form of the interaction and of the energy spectrum of the system. Another important feature of result (3) is that it does not imply statistical equilibrium, and can be consequently used by averaging on arbitrary nonequilibrium states.

A useful (also rigorous) inequality, yielding the Bogoliubov [Eq. (1)] and the uncertainty principle [Eq. (3)] inequalities in the high- and low-temperature regimes, respectively, is given by

$$\langle \{ A^\dagger, A \} \rangle \langle \{ B^\dagger, B \} \rangle \geq |\langle [A^\dagger,B] \rangle|^2 \operatorname{cotgh} \left[\frac{\beta}{2} \frac{\langle [B^\dagger,[H,B]] \rangle}{\langle \{ B^\dagger, B \} \rangle} \right] , \quad (4)$$

with $\beta = 1/k_B T$. This inequality is stronger than (1) and (3) at all temperatures, and belongs to a general class of inequalities that can be derived using, for example, the formalism of Ref. 11.

Several nontrivial results have been obtained recently starting from the uncertainty principle inequality (3). In particular in Ref. 7, we have proven the absence of long-range order in an important class of 1D systems at zero temperature, such as Bose liquids, isotropic antiferromagnets, and crystals. The proof is based on the study of the infrared divergent behavior, induced by a symmetry breaking in the system, in the fluctuation term $\langle \{ A^\dagger, A \} \rangle$ at zero temperature. These results provide the $T=0$ analog of the Hohenberg-Mermin-Wagner theorem.[4–6] Another useful application is the nonperturbative study of isospin impurities in $N=Z$ atomic nuclei,[12] through the explicit determination of a rigorous lower bound for isospin fluctuations. This is an interesting problem characterized by a nonspontaneously broken symmetry in a finite system.

In this work, we provide another interesting application of the uncertainty principle inequality (3) by ruling out the existence of Bose-like off-diagonal long-range order in the fractional quantum Hall effect (FQHE) at $T=0$. This result is particularly relevant because it concerns the absence of long-range order in the ground state of a 2D system.

The bosonic representation of the many-body wave function

$$\Psi_B(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\exp\left[\frac{i}{\nu}\sum_{i,j}\alpha_{i,j}\right]\Psi_F(\mathbf{r}_1,\ldots,\mathbf{r}_N) \tag{5}$$

was used in Ref. 13 in order to investigate the problem of off-diagonal long-range order in the FQHE and to point out the existence of deep analogies between the behavior of superfluidity and the FQHE. In Eq. (5), Ψ_F is the fermionic wave function of electrons, $\nu=1/2k+1$, where k is an integer, is the usual filling factor, and $\alpha_{i,j}$ is the angle between the vector connecting particles i and j and an arbitrary fixed axis. Using Laughlin's expression[14] for the ground-state wave function Ψ_F, the authors of Ref. 13 concluded that there is not Bose-Einstein condensation in the bosonic wave function Ψ_B, but only algebraic long-range order (see also Ref. 15). The same result was obtained in Ref. 16, starting directly from the Chern-Simons-Landau-Ginzburg theory (CSLG).

An interesting question is whether the absence of Bose-Einstein condensation in Ψ_B follows from the explicit Laughlin's choice for the wave function Ψ_F (or from corresponding assumptions in the CSLG theory) or rather has a more general and fundamental reason. In the following, we will show that the absence of Bose-Einstein condensation is a direct consequence of the uncertainty principle inequality (3) applied to a charged system in an external magnetic field, without specific assumptions for the ground-state wave function Ψ_F. This result differs from the case of neutral 2D Bose systems, which are instead expected to exhibit Bose-Einstein condensation at $T=0$.

Let us apply inequality (3) to an arbitrary Bose system exhibiting Bose-Einstein condensation. We make the choice $A=a_\mathbf{q}^\dagger$ and $B=\rho_\mathbf{q}$, where $a_\mathbf{q}^\dagger$ and $\rho_\mathbf{q}=\sum_\mathbf{k} a_{\mathbf{k}+\mathbf{q}}^\dagger a_\mathbf{k}$ are the usual Fourier components of the particle creation and density operators relative to the Bose system. Using the Bose commutation relation $[a_\mathbf{q},\rho_\mathbf{q}]=a_0$, inequality (3) gives $[2n_B(q)+1]2NS(q)\geq|\langle a_0\rangle|^2$, where $n_B(q)=\langle a_\mathbf{q}^\dagger a_\mathbf{q}\rangle$ and $S(q)=1/N\langle\rho_{-\mathbf{q}}\rho_\mathbf{q}\rangle$ are the momentum distribution and the static structure function, respectively. If gauge invariance is broken, the average value $\langle a_0\rangle$ does not vanish and its modulus coincides with $\sqrt{Nn_0}$, where n_0 is the fraction of particles in the condensate, a quantity characterizing the long-range order in the system. Inequality (3) then becomes[17]

$$n_B(q)\geq\frac{n_0}{4S(q)}-\frac{1}{2}\ . \tag{6}$$

Result (6) very explicitly shows the constraints imposed by the uncertainty principle on the momentum distribution of the system. It is useful to recall here that use of the Bogoliubov inequality (1), with the same choice for the operators A and B, yields a different constraint,[4]

$$n_B(q)\geq\frac{n_0mk_BT}{q^2}-\frac{1}{2}\ , \tag{7}$$

useful only at finite temperature. In particular, result (7) can be used to rule out[4] the existence of Bose-Einstein condensation ($n_0=0$) in 1D and 2D Bose systems at $T\neq0$.

Let us apply result (6) to the bosonic wave function (5). The following comments are in order here: (i) the static structure functions $S(q)$ relative to the Bose and Fermi wave functions of Eq. (5) are identical, as obviously follows from the nature of transformation (5); (ii) though $n_B(q)$ should not be confused with the electronic momentum distribution, its normalization is nevertheless fixed by the total number N of electrons.

The key point to discuss now is the low-q behavior of the static structure factor

$$S(q)=\int S(q,\omega)d\omega\ , \tag{8}$$

where $S(q,\omega)$ is the usual dynamic structure factor. The function $S(q)$ obeys, at $T=0$, the rigorous inequality

$$S(q)\leq\left[\int\omega S(q,\omega)d\omega\int\frac{1}{\omega}S(q,\omega)d\omega\right]^{1/2} = q\left[\frac{1}{2M}G(q)\right]^{1/2}\ , \tag{9}$$

where we have made use of the well-known f-sum rule

$$\int\omega S(q,\omega)d\omega=\frac{q^2}{2m} \tag{10}$$

and introduced the static response function

$$G(q)=\int\frac{1}{\omega}S(q,\omega)d\omega\ . \tag{11}$$

In neutral liquids, $S(q)$ vanishes linearly with q at zero temperature as a consequence of the finite value of the compressibility $\chi=\lim_{q\to0}2mG(q)$, and this behavior ensures, in particular, the absence of long-range order ($n_0=0$) in the ground state of 1D Bose systems.[7] Charged liquids in an external magnetic field H are characterized by a suppression of density fluctuations, resulting in the quadratic law

$$S(q)_{q\to 0}=\frac{q^2}{2m\omega_c} \tag{12}$$

for the static structure function where $\omega_c=eH/m$ is the usual cyclotron frequency [for a discussion of the low-q limit of $S(q)$ in the FQHE, see Ref. 18]. Result (12) can be straightforwardly obtained starting from the Kohn's theorem[19] stating that the leading behavior of the dynamic structure function at small q is given by

$$\lim_{q\to 0}S(q,\omega)=\frac{q^2}{2m\omega_c}\delta(\omega-\omega_c)\ , \tag{13}$$

and hence that the cyclotron resonance exhausts the energy-weighted sum rule (10). With the assumption that the system has no gapless excitations,[20] one immediately finds that the same is also true for the non-energy-weighted sum rule (9), and hence one recovers Eq. (12).

From Eq. (12) and our inequality (6), we conclude that $n_B(q)$ diverges as $n_0 m\omega_c/2q^2$, and consequently the normalization condition $\sum_q n_B(q)=N$ cannot be fulfilled in this 2D problem unless $n_0=0$. The physical interpretation of this result is very clear (see also Ref. 15): the magnetic field suppresses the fluctuations of the electronic density and, according to the uncertainty principle, it increases the bosonic field fluctuations that destroy the condensate. The logarithmic divergency resulting from the $1/q^2$ behavior in $n_B(q)$ emphasizes in an explicit way the analogies between this problem and the problem of 2D neutral Bose superfluids at finite temperature. While in the latter case the absence of long-range order was proven by Hohenberg[4] at $T\neq 0$, employing the Bogoliubov inequality (1), yielding result (7) for the momentum distribution, in this work we have shown the corresponding result for the fractional quantum Hall effect at $T=0$. Of course our results do not exclude algebraic off-diagonal long-range order, whose occurrence was explicitly found by Girvin and MacDonald[13] and by Zhang,[16] nor the occurrence of other types of long-range order such as recently discussed by Read.[21]

Starting from Eq. (13), one can also calculate the static response function (9) in the low-q limit. In the absence of gapless excitations, one finds

$$G(q)_{q\to 0}=\frac{q^2}{2M\omega_c^2}\ . \tag{14}$$

The vanishing of $G(q)$ for $q\to 0$ expresses the incompressibility of the system, a peculiar property of the FQHE. The q^2 behavior of $G(q)$ is expected to be preserved by the addition of a small amount of impurities in the system, whose effect is to broaden the cyclotron resonance (13). Also in this case, using the rigorous inequalities (6) and (9), one can consequently rule out the presence of long-range order in the system.

Useful discussions with A. Aronov, E. Seiler, and S. Zhang are acknowledged. L.P. wishes to thank the hospitality of the Dipartimento di Fisica dell'Università di Trento.

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$$n(q)\geq\frac{n_0}{4S(q)}-\frac{1}{2}+\frac{S(q)}{4n_0}$$

for the momentum distribution, valid for $n_0\geq S(q)$. One can easily show that this stronger inequality becomes an equality for all values of q in the Bogoliubov approximation to the dilute Bose gas.

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