Anomalous diffusivity and localization of classical waves in disordered media: The effect of dissipation

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It is argued that, contrary to assertions in the literature, absorption does not provide a cutoff length analogous to an inelastic scattering length for the renormalization of wave transport in a multiply scattering medium. The argument is supported with numerical experiments. A reconciliation between this evidence and the conventional understanding is effected. The error in the latter is identified.

Anomalously weak and slow diffusion has been observed in some interesting recent work on the diffuse transport of microwaves in a multiply scattering medium.[1,2] In the most recent of the Letters[1] reporting this work, it is claimed that a narrow window of localization has been found. As had long been anticipated,[3,4] the presence of dissipation has complicated the analysis of the experimental results. It appears to be widely conceived that the effect of dissipation on the Anderson transition for classical waves is analogous to that of inelastic scattering on the transition for electrons. This conception was employed in the interpretation of the recent microwave results. It is the intention of the present paper to establish that that conception is in fact in error. It will be shown that the transport of wave energy in a disordered medium remains nondiffusive even in the presence of dissipation, and that a diffusion process, whether with a renormalized diffusion constant or not, does not correctly describe that transport. It is further shown that the conventional understanding for the effect of dissipation upon conductance (as opposed to diffusivity) is not inconsistent with the present argument.

Throughout much of the literature on the Anderson localization of classical waves it has been asserted that dissipation provides a classical wave analog to the process of inelastic electron scattering.[1,2,4–6] That the thermal scattering of an electron on a length scale L_i is responsible for many of the temperature-dependent properties of the metal-insulator transition is well appreciated. The classical wave analog for this kind of inelastic electron scattering, however, is provided by random time variations in the propagation medium and not by dissipation. Perhaps because of confusions engendered by the multiple meanings of the word "inelastic" it is widely thought that it is dissipation on a time scale τ_a which provides the analog, and therefore acts to cut off the renormalization group flow. Such reasoning is reflected by a formula describing the scaling of wave diffusion in the critical regime [1,2]

$$D(L)=(1/3)vl^2[1/\xi_0+1/L+1/L_a]\ , \qquad (1)$$

where l is the transport mean free path, v is the transport wave speed, ξ_0 is the correlation length (divergent at criticality), L is the length scale of the experiment (often the system size), and L_a is defined as $(D\tau_a)^{1/2}$.

The conventional understanding implicit in (1) represents that in the presence of dissipation, and on length scales greater than L_a, the dynamics is classical with a diffusion constant D less than, but if dissipation is great, comparable to, the bare diffusion constant. There is additionally implicit in some of the literature around (1), a demonstrably erroneous conception that, even in the absence of dissipation, the behavior at large distances is classical with a small renormalized D which vanishes exponentially at large distances. That the conception is incorrect may be proven as follows.

Any assertion in regard to the scaling of the temporal aspects of diffusion in a closed, undissipative, time-invariant ($L_i=\infty$) system is necessarily incorrect in one or two dimensions if it requires that this D vanish as L goes to infinity and ω goes to zero. It is incorrect in three dimensions if it requires this D to vanish exponentially with system size. This is because no process can take place on a time scale longer than provided by the density of states ρ. (Significant numbers of near degeneracies of eigenfrequencies which could result in longer time scales can be ruled out in a random medium, especially in the presence of level repulsion.) The slowest available time scales therefore vary with system size like ρL^d. The time-scale constraint is inconsistent in one or two dimensions with a diffusion time scale like $\sim L^2/D$ with vanishing D. It is inconsistent in three dimensions with any assertion that the time scales of diffusion should scale like L^2/D with a D which vanishes faster than $1/L$. One therefore concludes that transport time scales do not scale to infinity in the way that conductances are expected to scale to zero. The concept of "break time" ρL^d has also been discussed elsewhere, e.g., [Refs. 7 and 8].

That absorption need not reestablish diffusive behavior by destroying the localization of the eigenfunctions can be seen from a counterexample:[9] An absorption which is diagonal[10] in the natural basis provided by the frequency eigenstates of the undissipative system will affect the eigenmodes only trivially: the eigenfunctions will be unchanged, thereby preserving any condition of localization; their eigenfrequencies will gain small imaginary parts. In consequence all time-domain responses are identical to those of the undissipative system except for a trivial additional time-dependent factor $\exp(-t/\tau_a)$.

While diagonal damping is surely improbable in a real system, it does provide a proof of the error in what appears to be a common conception.

Whether nondiagonal damping could act to cut off the scaling flow as indicated by (1) is less clear. A small arbitrary dissipation, while it may introduce a change in phase and will certainly introduce an amplitude diminishment, will do so equally for both a path and its reversed image. In other language, dissipation does not induce the random phases which are introduced by inelastic scattering of electrons off phonons; paths and time-reversed paths have identical phase and amplitude in the presence of dissipation, and remain capable of coherent interference. Weak localization arguments involving the constructive interference of paths and their reversed images therefore remain unaffected by the presence of dissipation.

It is thus apparent that there is a substantial theoretical basis for rejecting the literal interpretation of (1). In order to experimentally investigate the possibility of an absorption related cutoff we now consider the dynamics of a damped two-dimensional $V=1$ Anderson model of the form

$$\partial^2\psi_{\mathbf{n}}/\partial t^2+c_{\mathbf{n}}\partial\psi_{\mathbf{n}}/\partial t+K_{\mathbf{n}}\psi_{\mathbf{n}}-\sum\psi_{\mathbf{m}}=F_{\mathbf{n}}(t) \quad (2)$$

where the bold index $\mathbf{n}$ runs over the sites of a 25×25 square lattice. The sum is over the four nearest-neighbor sites $\mathbf{m}$ of site $\mathbf{n}$, and $K_{\mathbf{n}}$ is five plus a random number taken from the uniform distribution $[0,W]$. $c_{\mathbf{n}}$ is a uniform random number taken from the interval $[0,2/\tau_a]$. The frequency domain version of (2) is, neglecting F and c, precisely equivalent to the tight-binding Anderson model. It has identical eigenfunctions. The eigenfrequencies differ, but in a smooth fashion. The difference in eigenfrequencies is an inevitable consequence of going to a classical model. For a narrow-band disturbance the difference should be irrelevant. In the time-domain equation (2) therefore represents the classical dynamics of a forced Anderson model with damping. It also exactly describes the transverse dynamics of a planar array of masses, random springs, and dashpots coupled solely by in-plane inertialess strings with uniform tension and driven by an external force F. Alternatively, (2) may be thought of as a spatially discrete version of a classical wave equation for a tensioned membrane on a random viscoelastic foundation. The damping coefficients c are chosen randomly in order to ensure that the damping is not diagonal in the natural basis. The damping has, however, been taken as diagonal in the configuration-space representation in order to allow the algorithm to employ an explicit scheme. $F(t)$ was taken to be a ten cycle cosine bell tone burst centered on a frequency near the center of the band (2.81 at $W/V=5$). ψ was fixed along the perimeter. Similar time-domain dynamics, but without dissipation, have been studied by Scher,[7] Weaver and Loewenherz,[11] and Prelovsek.[12]

Equation (2) is solved by central differences with a time step chosen short enough to ensure numerical stability. Inasmuch as the eigenfunctions of the system are independent of the temporal differencing, the precise choice of time step size is expected to be unimportant; such was indeed found to be the case. The forcing was distributed uniformly along one side of the square array, on row number one, next to an edge. The evolving energy density was monitored in three strips parallel to the line force, in the strip near the line force including rows five through seven, in a strip near the center of the array, from rows ten through twelve, and in a strip near the side opposite the force, from rows 19 through 21.

Figure 1 shows the resulting 20 configuration ensemble and spatial average energy densities for an undissipative system, $\tau_a=\infty$. In accord with expectations, the nondiffusive character of the transport is found to increase with increasing values for W/V. At $W/V=5.0$ the asymptotic energy density in row 20 is about half that in row six. One ascribes this difference to appreciable renormalization of the conductance on length scales of order 20. Such an estimate is consistent with the predictions of the theory of weak localization which suggests that renormalization is significant on length scales of order $L_0\exp[2\pi^2D_0\rho]$ where ρ is the modal density and D_0 is the bare diffusion constant, on a bare length scale L_0. ρ was independently found by eigenvalue counting in a 10×10 lattice to be about 0.7 at the frequency of the tone burst. $D_0\approx0.25$ was independently estimated from the early time slope of $R^2(t)=4Dt$, the square of the radius of gyration of the energy distribution resulting from a point excitation, in a manner like that of Prelovsek.[12] MacKinnon and Kramer[13] quote for this degree of disorder in a strip with width 25 a localization length $\Lambda=17$ at a neighboring frequency, and are therefore in agreement with this assessment.

If absorption provides a cutoff to the scaling on a length scale $L_a=(D\tau_a)^{1/2}$, then the dynamics seen here should look very different if L_a is taken to be substantially shorter than the system size and localization length. The choice $\tau_a=100$ results in $L_a\leq5$, depending on whether one uses D_0 or, as in Refs. 1 and 2, a renormal-

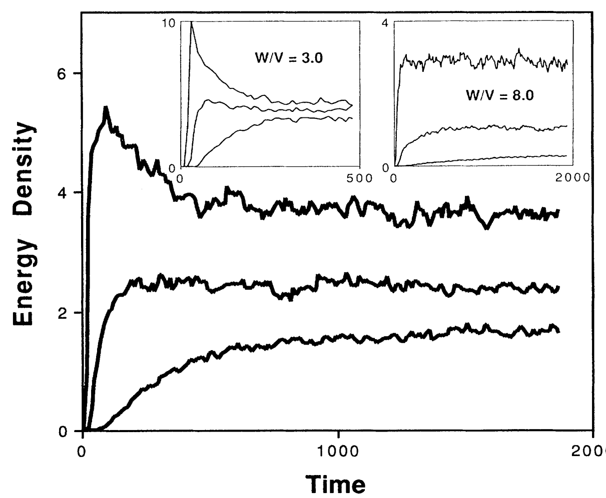

FIG. 1. The transient dynamics of an undamped Anderson model of size $L=25$ at $W/V=5.0$. The upper curve corresponds to the energy density (in arbitrary units) adjacent to the tone burst source; the lower curve to the energy density on a row near the opposite side. The insets show the behavior at other degrees of disorder. The time units are those of Eq. (2).

ized D. The dynamics at $W/V=5.0$ and $\tau_a=100$ are shown in Fig. 2. Except for the trivial $\sim\exp(-t/\tau_a)$ decay, the evolution shown in Fig. 2 is unchanged. Most importantly, the nonclassical effect, whereby the energy density at late time shows a spatial variation, is preserved. The ratio of adjacent and opposite energy densities is unaffected by the introduction of dissipation.

The same behavior is observed at different values of the disorder parameter W/V, system size, and absorption time. It is also observed when the form of the damping is modified to the form $\sum C_{\rm nm}[\partial\psi_{\rm n}/\partial t-\partial\psi_{\rm m}/\partial t]$ where the sum is over nearest neighbors **m**, and C is symmetric. Such terms correspond to resistance elements coupling nearest neighbors. One concludes that numerical studies of the transient dynamics of an Anderson model two-dimensional lattice show precisely no evidence of an absorption cutoff. Wave energy transport is seen to remain nondiffusive even in the presence of absorption. The results of the experiments are in accord with the effect of dissipation on diffuse wave energy transport being fully comprehended by a simple temporal decay in the energy transport propagator.

Having argued theoretically and established experimentally that dissipation has effects which are only trivial and does not affect the renormalization of the energy transport, it becomes appropriate to attempt a reconciliation with the widespread conventional understanding. It is the present contention that the error in assertions such as (1) lies in a too facile identification of conductance σ with a coefficient D in a diffusion equation. Indeed, as argued following Eq. (1) and demonstrated in the numerical results above, the very concept of a diffusion constant is often inappropriate. In those places where it is cleanly defined (see, e.g., McKane and Stone[14]) D does not lend itself to an identification with a coefficient in a diffusion equation. The common identification of σ and D follows from an appeal[14] to an Einstein relation which in turn depends on a circular logic assumption (as argued above necessarily incorrect in a localizing medium without inelastic processes) that the energy flow is in fact governed by a diffusion equation. If equations such as (1) are derived by an appeal to similar, and well accepted, scaling formulas for conductance, followed by an appeal to this identification of D and σ, then this misidentification may be the source of the error.

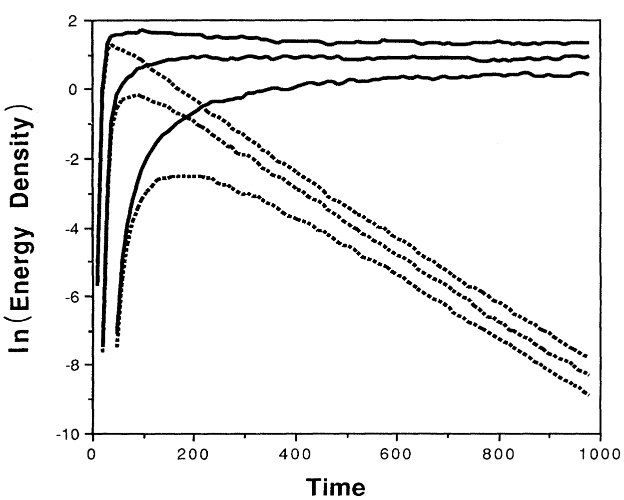

FIG. 2. The transient dynamics of a damped Anderson model are compared at $W/V=5.0$ with dissipation, $\tau_a=100$ (dashed line), and without dissipation (solid line) using the same units used in Fig. 1.

In order to explore the apparent discrepancy between scaling formulas like (1) applied to conductance and the present results it is necessary to make use of a relationship between σ and some concept more robust than D. The appropriate concept is the multiply scattered wave energy propagator $G(L,t)$. G represents the energy density at time t resulting on the far side (L) of a slab of disordered media excited on the near side by a transient addition of wave energy at time zero. This propagator is not known in general, though it may be noted that Vollhardt and Wolfle[15] have suggested a possible form for G for an unbounded medium.

The effect of dissipation on G is, according to the evidence presented above, merely the insertion of a factor $\exp(-st)$, where $s=1/\tau_a$. The effect of dissipation according to the conventional wisdom appears to be (see, e.g., Refs. 1 and 2 for a clear example of this conception) a G which on long length scales $L \gg L_a$ is $\exp(-st)$ times a classical diffusion propagator corresponding to a renormalized diffusion constant $D<D_0$ renormalized according to a formula like (1).

Conductance is given by total transmission, which is essentially the integral of $G(L,t)$ over all time. Hence we write, for σ,

$$\sigma(L,s)=\int \exp(-st)G(L,t)dt\ , \tag{3}$$

where G is the actual nonclassical propagator for wave energy in an undissipative disordered medium attached to leads and the integral is over all time $t>0$. On the other hand, the conventional understanding would have us set (for $L>L_a$)

$$\sigma(L,s)=\int \exp(-st)G^{\rm cl}(D,L,t)dt\ , \tag{4}$$

where $G^{\rm cl}$ is a classical diffusion propagator corresponding to diffusion at a renormalized rate D. It is the solution of a diffusion equation.

While the integrands clearly differ, the factor $\exp(-st)$ renders points corresponding to late times unimportant for the integration. At large s, therefore, the integrals are virtually equal if the short time behavior of the actual propagator is identical to the short time behavior of $G^{\rm cl}$. That identity is a well accepted consequence of simple weak Anderson localization enhanced backscatter arguments. We therefore see that, in the limit of very large s such that Eq. (1) would predict $D\approx D_0$, σ becomes insensitive to the nonclassical aspects of the energy propagator; and, as is predicted by scaling equations like (1) when applied to conductance, dissipation has effectively cut off the scaling.

At slightly smaller s (larger L_a), where Eq. (1) would predict classical diffusion at large L but with a renormalized $D<D_0$, there will be points of the integrands that differ significantly; it is nevertheless plausible that the integrations will continue to agree and equations like (1)

continue to correctly describe the scaling of conductance. It is plausible because G^{cl} will be less than the actual G at early times but greater than the actual G at later times.

Therefore in spite of the very different consequences of dissipation upon the multiply scattered wave energy propagator in the two conceptions, there is apparently no consequent difference in their predictions for the effect of s on σ. Hence the present understanding is not inconsistent with the conventional understanding for the dissipation dependence of the scaling of conductances. It does refute more problematic formulas like (1).

Recent microwave experiments[1,2] which appear to have observed the Anderson transition have employed Eq. (1) for their data analysis. Inasmuch as the analysis relied on Eq. (1)'s implications for transport time scales as well as transmission coefficients (which are equivalent to conductances), that work must now be regarded as requiring new scrutiny. Such a reanalysis, though, requires a trustworthy expression for the as yet unknown G and cannot be carried out without further theoretical and numerical work. Work toward a correct description of the diffuse wave energy propagator is indicated. Further laboratory work is indicated also, with, perhaps, transient excitations and time resolved measurements like those in Weaver's demonstration[9] of the Anderson localization of ultrasound in a two-dimensional disordered medium.

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