# Paired Hall states in double-layer electron systems

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We present evidence that a universality class of incompressible electron liquids incorporating $p$-wave pairing has been observed in the recent experiments of Suen and collaborators. Exact diagonalization studies for small numbers of electrons in a spherical geometry with realistic potentials indicate that a uniform incompressible ground state develops at the correct (displaced) flux. We relate states containing two distinguishable species of electrons to ones with indistinguishable electrons, and propose that instances of each have been realized in different double-layer electron systems at filling fraction $\nu=\frac{1}{2}$.

## I. EXPERIMENTAL CONFIGURATION

In a recent paper, Suen *et al.*[1] report the observation of a $\nu=\frac{1}{2}$ fractional quantized Hall state in a low-disorder double-layer electron system. Prior to detailed theoretical analysis of this exciting observation, which we shall argue provides strong evidence for the existence of a qualitatively new universality class of incompressible quantum liquids, the following remarks are essential.

The experiments are performed in strong magnetic fields $B \gtrsim 12$ T, so that the Zeeman energy splitting for the two directions of spin is of order 6 K, which is much larger than the correlation energies (gap $\approx 230$ mK) reported. Thus, the relevant electrons are completely polarized and the spin degree of freedom is frozen out.

The experiments were performed by injecting charge into a 680-Å-wide layer of GaAS—forming the well—bounded by $Al_{0.35}Ga_{0.65}As$ layers. The electrons trapped in the well distribute their charge toward its two extremities, so as to minimize the electrostatic energy. In this way, an effective "double-layer system" develops. In discussing the wave functions of individual electrons, in the spirit of Hartree and Fock, the potential due to the charge density is crucial. This potential is, roughly speaking, of a double-well type, concentrating the electrons in the two layers just mentioned. The energy gap between the symmetric and antisymmetric wave functions (in the direction perpendicular to the GaAs layer, which we shall call the third direction) is estimated[1] to be $\Delta_{SAS} \gtrsim 5$ K for all densities relevant to the experiments. This splitting is much larger than the correlation energies and the temperatures at which the experiments were performed. Although at zero magnetic field the Fermi energy would be sufficiently high to populate both symmetric and antisymmetric states, in the strong magnetic field of interest here, the Landau bands are sufficiently flat that only symmetric states are occupied.

With both the spin degree of freedom and motion in the third dimension frozen out, the problem is reduced to a two-dimensional electron gas restricted to the lowest Landau level. Within this framework, the most important element introduced by the double-layer geometry is simply that it takes us into a new regime of effective interactions between the electrons, as compared to previous experiments. Roughly speaking, the wave function is much more spread out in the third dimension. The primary effect of this spreading out is to soften the short-range component of the Coulomb repulsion among the electrons.

## II. NUMERICAL WORK

The possibility of an incompressible state at $\nu=\frac{1}{2}$ for electrons in this ideally simple configuration, for suitable interactions, has been suggested by Halperin,[2] Moore and Read,[3] and us.[4] Important aspects of the theory include the following.

(1) The existence of pairing correlations, resembling those occurring in the BCS theory of superconductivity. Indeed, the existence of these correlations is the element that enables one to transcend the traditional restriction to odd-denominator filling fractions in the hierarchical construction of fractional quantized-Hall-effect states. In the strong pairing limit one imagines that the pairs are *bosons* with filling fraction $\frac{1}{4}$ that of the original electrons. Even-denominator fractional quantized Hall states for bosons are of course easily constructed; the simplest ones are the Laughlin $1/m$ states with $m$ *even*. For $m=8$, we obtain an incompressible state with $\nu=\frac{1}{2}$ for the electrons.

(2) Since spin-polarized states are assumed, the pairing must involve odd orbital angular momenta. The simplest possibility is $p$ wave. One concrete consequence of this pairing is that, for a finite system of $N$ electrons on a

 

sphere subject to a uniform normal magnetic field with total flux $N_\phi$ Dirac quanta, the incompressible state occurs for

$$N_\phi = 2N - 3 , \tag{2.1}$$

with $N$ even. This relation is very valuable in guiding numerical work.

(3) The charged quasiparticles around the ground state are expected to be charge $e/4$ anyons with statistical parameter $\theta/\pi = \frac{1}{8}$. A pair of these *halberons* will be present if $N_\phi$ differs by one unit from the ground-state value given by (2.1).

(4) We also expect the existence of neutral fermion quasiparticles, which correspond to the pair-breaking excitations of BCS theory. One of these quasiparticles will be present when (2.1) is satisfied but $N$ is odd.

(5) There is a beautiful trial wave function for the ground state at $\nu = \frac{1}{2}$, which involves a Pfaffian:

$$\Psi_{1/2} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] \prod_{i<j} (z_i - z_j)^2 \prod_i \exp(-\tfrac{1}{4} eB |z_i|^2) . \tag{2.2}$$

This wave function is known to be exact for some idealized model Hamiltonians involving three-body interactions.

(6) The existence of an incompressible state satisfying (2.1), and of quasiparticles as indicated in the two preceding points, has been demonstrated numerically for simple, quasirealistic interactions.

(7) It is straightforward to generalize the theory to other filling fractions $\nu = \frac{1}{2} n$; the generalization of (2.1) reads

$$N_\phi = 2n(N-1) - 1 . \tag{2.3}$$

The double-layer experimental geometry of Suen *et al.*,[1] although its qualitative effect is clearly to soften the short-range repulsion, does not match the simple parametrization of modified Coulomb potentials adopted in Ref. 4. To make a quantitative comparison, fresh calculations are required.

The charge distribution in the third direction calculated in a self-consistent approximation for the double-layer electron system of Suen *et al.* is modeled rather accurately by

$$\rho(z) \propto \begin{cases} \frac{1}{4} + \left[\frac{2z}{d}\right]^2 + \left[\frac{2z}{d}\right]^4 - \left[\frac{2z}{d}\right]^6 & \text{for } \left|\frac{2z}{d}\right| \le 1.297 , \\ 0 \quad \text{otherwise} , \end{cases} \tag{2.4}$$

as depicted in Fig. 1. Here $d$ is the distance between the two layers, or peaks in the charge density. The effective (two-dimensional) electron interaction is given by

$$v(r) = \int dz_1 \int dz_2 \frac{\rho(z_1)\rho(z_2)}{\sqrt{(z_1 - z_2)^2 + r^2}} . \tag{2.5}$$

The pseudopotential $V_m$ for relative angular momentum $m$ is obtained by projecting down to the first Landau level, and separating the appropriate partial wave. This yields

$$V_m = \int_0^\infty dq\, e^{-l^2 q^2} L_m(-l^2 q^2) F(q) , \tag{2.6}$$

where $l$ is the magnetic length, $L_m$ is the Laguerre polynomial, and

$$F(q) = \int dz_1 \int dz_2 \rho(z_1)\rho(z_2)^{-q|z_1 - z_2|} . \tag{2.7}$$

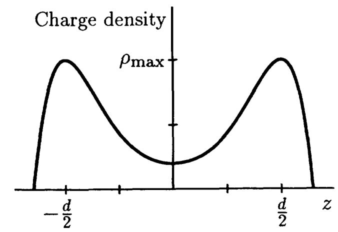


FIG. 1. The charge distribution (2.4) used in our numerical work. It closely resembles the experimental charge distribution of Ref. 1.

Exact diagonalization studies of the appropriate model Hamiltonians for small numbers of electrons in a spherical geometry (we neglect finite-size corrections to the pseudopotentials) allow for the following conclusions.

(1) As displayed in Fig. 2(a), there appears to be a clear energy gap separating a homogeneous ground state from the remaining part of the spectrum. The magnitude of the energy gap is of order 1 K for the values quoted. It should be borne in mind, however, that the simulations are limited to disturbingly small numbers of particles. Thus, while they are certainly suggestive and encouraging, they do not conclusively demonstrate the existence of a gap in the thermodynamic limit. (Indeed, the experimentally observed gap at $\nu = \frac{1}{2}$ is only 250 mK, so a significant decrease is expected.)

(2) The gaps become much smaller or zero as $d/l$ is

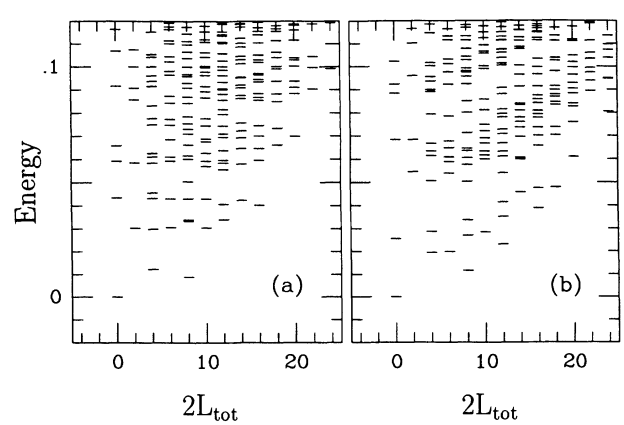


FIG. 2. The spectra for (a) 17 flux units and 10 particles and (b) 19 flux units and 6 particles, in (cgs) units of $e^2/\epsilon l$. The ground state is homogeneous and displays a significant gap. As explained in the text, these are relevant for paired Hall states at $\nu = \frac{1}{2}$ and $\frac{1}{4}$, respectively. Note that the latter could also be interpreted, using particle-hole conjugation, as the spectrum for 19 flux units and 13 particles, relevant to a $\nu = \frac{3}{4}$ state.

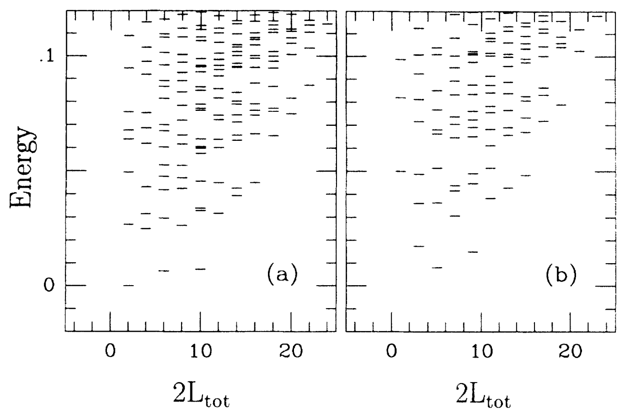


FIG. 3. The spectra for (a) 16 flux units and 10 particles and (b) 15 flux units and 9 particles. The first shows the even-odd alternation to be expected if two identical particles are present; also, the fact that the energy is lowest for small angular momenta is consistent with these particles repelling one another. The second shows a roughly parabolic shape with a minimum at a nonzero angular momentum, consistent with the interpretation of a single pair-breaking excitation around the nominal Fermi surface.

varied outside the range $3 \lesssim d/l \lesssim 10$, for the same flux and particle number.

(3) As displayed in Figs. 3(a) and 3(b), the spectrum is consistent with that expected for two halberons or a neutral fermion, when there is a unit flux deficit or an odd number of electrons, respectively.

(4) The overlap of the exact ground states for $d/l=7$ with the Pfaffian state (2.2) is found to be 0.8684, 0.8785, and 0.8786 for 6, 8, and 10 particles, respectively. Note that the overlap *increases* slightly with the number of particles, much unlike conventional Hall states. This is not really surprising, since we do not expect (2.2) to be such a good trial wave function for very small numbers of particles. For in this case the pairing drastically affects the Laughlin-Jastrow correlations, which are the essential advantage of the state as compared to a Wigner solid or a charge-density wave.

(5) The situation is less clear for paired Hall states at other filling fractions. Exact diagonalization for $N=6$ with the correct flux displacement (2.3) indicates a clear gap for $\nu=\frac{1}{4}$, as shown in Fig. 2(b). However, it is not clear whether this gap is present in the thermodynamic limit. The paired Hall states definitely become less stable as $\nu$ decreases. For the model potentials considered here, no incompressible ground state emerges for small systems ($N=4$) at filling fractions $\frac{1}{6}$ and $\frac{1}{8}$.

### III. INDEXED AND INDEX-FREE STATES

The experiments reported in Ref. 1 were largely inspired by a different class of theoretical proposals.[2,5–7] The primary assumption in these proposals is that the electrons in the two layers are essentially *distinguishable*, so that there is a two-valued "quasispin" variable necessary to specify the state of an electron, in addition to its two-dimensional coordinate. This situation can arise if the tunneling between layers is extremely small, so that the gap $\Delta_{SAS}$ is negligible compared to correlation energies and to temperature, and the symmetric and the antisymmetric states are equally populated. Presumably, intermediate cases are also possible, where the populations are both nonzero but significantly different (see the remarks below).

Let the two species of electrons be distinguished by calling the two-dimensional coordinate of electrons of the first species $z$ and the two-dimensional coordinate of electrons of the second species $w$. Then, a natural generalization of the Laughlin trial wave functions for a single species is the set (studied in Ref. 6 for the special case $m_1=m_2$)

$$\Psi^{(m_1,m_2,n)}[z_i,w_i]=\prod_{i<j}(z_i-z_j)^{m_1}\prod_{i<j}(w_i-w_j)^{m_2}\prod_{i,j}(z_i-w_j)^n\prod_i \exp(-\tfrac{1}{4}eB|z_i|^2)\prod_i \exp(-\tfrac{1}{4}eB|w_i|^2) \;. \tag{3.1}$$

At present, the most interesting case is $m_1=m_2=3$, $n=1$. This state, denoted (3,3,1) in an obvious notation, has filling fraction $\nu=\frac{1}{2}$. Indeed, in another very recent experiment on double-layer electron systems, Eisenstein *et al.*[10] also found incompressible $\nu=\frac{1}{2}$ states. The parameters of their experiment, and the nature of the states (as reflected in the response to field tilting and the magnitude of the gap) are quite different from Suen *et al.*[1] Most important, the two layers in the experiment of Eisenstein *et al.* are well separated by a narrow but high-potential barrier, so that to a first approximation one may regard electrons inhabiting the two layers as two distinct species. Indeed, these authors tentatively identify their observation with the distinguishable or indexed (3,3,1) state proposed in Refs. 5 and 6. Assuming this to be correct, we suggest that the two groups, while they have both observed fractional quantized Hall states in double-layer geometries at the same filling fraction $\nu=\frac{1}{2}$, have, in fact, observed distinct universality classes.

Actually, there is a remarkable relationship between the two states. A celebrated identity due to Cauchy (see Ref. 8), which underlies the equivalence between bosons and fermions in (1+1)-dimensional physics, reads as follows:

$$\det\left[\frac{1}{z_i-w_j}\right]=(-1)^{n(n-1)/2}\frac{\prod_{i<j}(z_i-z_j)(w_i-w_j)}{\prod_{i,j}(z_i-w_j)} \;. \tag{3.2}$$

Using Cauchy's identity, it is easy to demonstrate that the result of antisymmetrizing $\Psi^{(3,3,1)}$ between the variables $w$ and $z$ is to produce *precisely* the Pfaffian trial wave function for paired Hall states. A similar identity for $\theta$ functions, due to Frobenius,[9] is useful for numerical work on paired Hall states in the torus geometry.

## IV. POSSIBLE EXTENSIONS

There are two possible extensions to the above experimental work.

(1) Paired Hall states may be used as the starting point for hierarchical constructions similar to those used in the traditional treatment of the fractional quantized Hall state.[11] At the level of identifying the filling fractions for the universality classes of these states, presumably the simplest procedure is to consider the strong pairing limit, wherein fictitious pairs of electrons are regarded as constituting an effective bosonic fluid. The filling fraction for the electrons is four times the filling fraction for the effective bosons, since there are half as many bosons each with twice the charge. The hierarchy may be generated by combining the two elementary operations of adiabatically attaching an even number of flux tubes (which does not alter quantum statistics) and particle-hole conjugation. These operations generate the respective changes $\nu^{-1}\to\nu^{-1}+2p$ and $\nu\to 1-\nu$ in the filling factor. In this way, one generates the continued fractions

$$\nu=4\times[2p_n+1,2p_{n-1},\ldots,2p_1+1]\ , \tag{4.1}$$

whose first and last entries are odd, with all intervening ones even. Thus, for example, for $n=2$ we have the filling fractions

$$\nu=4\frac{1}{2p_2+1+1/(2p_1+1)}=4\frac{2p_1+1}{(2p_1+1)(2p_2+1)+1}\ . \tag{4.2}$$

For $p_1=0$ this is simply $\nu=2/(p_2+1)$, that is, an arbitrary fraction with numerator 2; these arise simply from the primary Laughlin $1/(2p_2+2)$ states for the effective bosons.

(2) As mentioned by one of us in Ref.12 and above, incompressible states with unequal values of the $m_i$ are also possible, and experimentally attainable in the separated geometry of Ref. 10. For these states, the densities in the two layers are locked in a definite, predictable ratio. For narrow layers, only odd values of $m_i$ can be expected, since the pairing states seem to require softening of the short-range part of the Coulomb repulsion. However, one can also consider combining the two experimental geometries, that is, juxtaposing a well of the type described by Suen *et al.* with a narrow well, separated by a barrier, or, for that matter, two wells of the type described by Suen *et al.* We believe there are real prospects for a rich *two-dimensional spectroscopy* in the separate filling fractions of the layers, with plateaus filling areas bounded in both these parameters.

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