

## Models with inverse-square exchange

Z. N. C. Ha and F. D. M. Haldane

*Department of Physics, Princeton University, Princeton, New Jersey 08544*

(Received 29 April 1992)

A one-dimensional quantum  $N$ -body system of either fermions or bosons with  $SU(n)$  "spins" (or colors in particle physics language) interacting via inverse-square exchange is presented in this paper. A class of eigenstates of both the continuum and lattice version of the model Hamiltonians is constructed in terms of the Jastrow-product-type wave function. The class of states we construct in this paper corresponds to the ground state and the low-energy excitations of the model that can be described by the effective harmonic fluid Hamiltonian. By expanding the energy about the ground state we find the harmonic fluid parameters (i.e., the charge, spin velocities, etc.) explicitly. The correlation exponent and the compressibility are also found. As expected, the general harmonic relation [i.e.,  $v_S = (v_N v_J)^{1/2}$ ] is satisfied among the charge and spin velocities.

### I. INTRODUCTION

In this paper we report an exact treatment of a one-dimensional system of particles with  $SU(n)$  "spins" interacting with the inverse-square, two-body exchange in continuum space. In particular, we show that a class of eigenstates of the Hamiltonian is given by the Jastrow-product type of wave function. We also show that this model can be put on the lattice for all the positive integer values of the dimensionless interaction parameter which is defined in Eq. (2.1). The continuum version is a direct generalization of Sutherland's model of interacting spin-

less fermions (or hard-core bosons).<sup>1</sup> The lattice version corresponds to the  $SU(n)$   $t$ - $J$  model, which, at  $n = 1$ , maps to the Haldane-Shastry model.<sup>2</sup>

Figure 1 shows how various models with the inverse-square exchange have evolved from the original Sutherland model. In the figure the models that already exist are in rectangular boxes and the models that we present in this paper are in ovals. The solid arrows mean "generalized from" or "evolved from." Two equivalent models that are represented in different ways are connected by a dashed arrow. We emphasize that while the bosonic  $t$ - $J$  model ( $\lambda$  odd) is a direct generalization of the bosonic

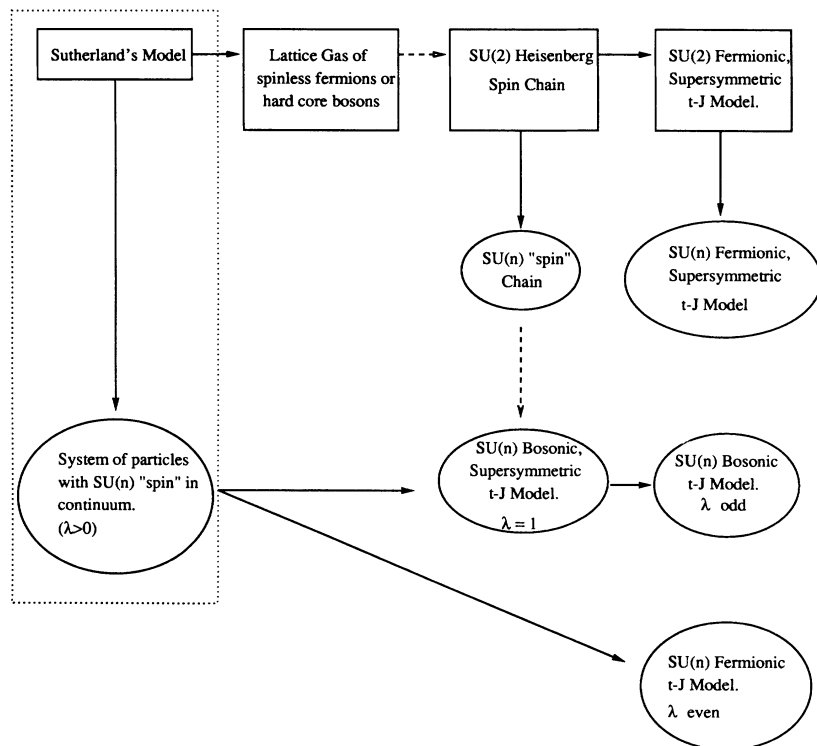


FIG. 1. Evolution of the inverse-square exchange models both in the continuum space and lattice. Models that already exist are in rectangular boxes and new models that are presented in this paper are in ovals. The models enclosed in the dotted box are continuum version and the rest of the models are lattice version. The solid arrows mean "evolved from" or "generalized from." The dashed arrows connect two equivalent models in disguise.

supersymmetric case ( $\lambda = 1$ ), the fermionic  $t$ - $J$  model ( $\lambda$  even) is not directly related to the fermionic supersymmetric  $t$ - $J$  model that Kuramoto and Yokoyama<sup>5</sup> found.

## II. MULTICOMPONENT GENERALIZATION OF SUTHERLAND'S MODEL

For convenience we consider a periodic system, where the Hamiltonian in units of  $\hbar^2/m$  is given by

$$H = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \lambda \sum_{i < j} \frac{\lambda + P_{ij}^\sigma}{d(x_i - x_j)^2}, \quad (2.1)$$

where  $\lambda$  is the dimensionless interaction parameter.  $P_{ij}^\sigma$  is an operator that exchanges particle spins at  $x_i$  and  $x_j$ , and  $d(x) = (L/\pi) |\sin(\pi x/L)|$ .  $d(x_i - x_j)$  is the chord distance between particles at  $x_i$  and  $x_j$  on a circle with circumference  $L$ . If all particles have the same spin, this model reduces to the system of spinless particles studied by Sutherland. Note that Sutherland's coupling parameter  $\lambda'$  corresponds to  $\lambda - 1$  in our notation. For example, the spinless free Fermi gas corresponds to  $\lambda' = 1$ , but in our notation  $\lambda = 0$ .

It is easily found that the wave function  $\Psi$  must vanish as  $|x_i - x_j|^{\lambda+1}$  ( $|x_i - x_j|^\lambda$ ) as  $x_i - x_j \rightarrow 0$  in the case of symmetric (antisymmetric) spin configuration of the two particles at  $x_i$  and  $x_j$ . And if  $0 < \lambda < 1$  the effective interaction strength is attractive for the antisymmetric spin configuration and there is some ambiguity in this case as it must be further specified whether  $\Psi$  vanishes as  $|x_i - x_j|^\lambda$  or as  $|x_i - x_j|^{1-\lambda}$  as particles approach. If we choose the first boundary condition the free fermion limit is obtained as  $\lambda \rightarrow 0$ .

There are two possible interpretations for this system. The first is to consider the system as a ring embedded in a plane. Hence, even though the particles are constrained to move in one dimension, the interaction is two dimensional in nature. The other is to regard the system as strictly one dimensional by taking  $1/d(x_i - x_j)^2$  as the effective interaction after summing the pairwise interaction of the particles around the ring infinite times. The following identity,

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(x + nL)^2} = \frac{1}{d(x)^2}, \quad (2.2)$$

was used by Sutherland<sup>1</sup> to show the validity of the latter interpretation.

In analogy with the states previously constructed for the SU(2) spin chain in Ref. 2, we propose the following Jastrow-product-type wave functions for our Hamiltonian:

$$\Psi(\{z\sigma\}) = \Psi_0 \prod_k z_k^{J_{\sigma_k}}, \quad (2.3)$$

where

$$\begin{aligned} \Psi_0 &= \prod_{n>m} \phi_{nm}, \\ \phi_{nm} &= |z_n - z_m|^{\lambda-x} (z_n - z_m)^{x+\delta_{\sigma_n\sigma_m}} \\ &\quad \times \exp[i\frac{1}{2}\pi \text{sgn}(\sigma_n - \sigma_m)]. \end{aligned} \quad (2.4)$$

Here,  $\delta$  is the Kronecker delta function and  $z_n = \exp(2\pi i x_n/L)$ .  $\sigma_n$  is the ordered spin index and  $J_\sigma$  the global current of particles with  $\sigma$  spin. We take  $J_\sigma$  to be an integer and will discuss the restrictions on the allowed values of  $J_\sigma$  later in this paper. Note also that the wave function with  $\lambda = x = 0$  is the Slater determinant that corresponds to the states of free SU( $n$ ) fermions.

The symmetry of the wave function with respect to the exchange of particles is given as

$$\begin{aligned} \Psi(\dots, z_i\sigma_i, \dots, z_j\sigma_j, \dots) \\ = (-1)^{x+1} \Psi(\dots, z_j\sigma_j, \dots, z_i\sigma_i, \dots). \end{aligned} \quad (2.5)$$

Hence, for bosons (fermions),  $x = 1$  ( $x = 0$ ).

We write the total Hamiltonian as  $H = H^0 + H^1 + H^2$ , where  $H^0$ ,  $H^1$ , and  $H^2$  are the kinetic, potential, and spin exchange Hamiltonian, respectively. We will show that each operator acting on the wave function gives two types of terms, "wanted" and "unwanted." "Wanted" terms are defined to be the terms that depend only on the global variables such as the total number of particles,  $J_\sigma$ , etc. "Unwanted" terms explicitly depend on the local variables such as  $z_i$ . Since the eigenenergy should depend only on the global variables, the "unwanted" terms for  $H^0$ ,  $H^1$ , and  $H^2$  should cancel or combine to give "wanted" terms.

We first examine  $H^0$  acting on the wave function. We define the following derivatives:

$$\eta_{ij} \equiv \frac{\partial_{z_j} \phi_{ij}}{\phi_{ij}} = -\frac{\lambda}{z_i - z_j} - \frac{\lambda - x}{2z_j} - \frac{\delta_{\sigma_i\sigma_j}}{z_i - z_j}, \quad (2.6)$$

$$\xi_{ij} \equiv \partial_{z_j} \varphi_{ij}, \quad (2.7)$$

$$\eta_j^{(1)} \equiv z_j \frac{\partial_{z_j} \Psi_0}{\Psi_0} = z_j \sum_{i(\neq j)} \varphi_{ij}, \quad (2.8)$$

$$\eta_j^{(2)} \equiv z_j^2 \frac{\partial_{z_j}^2 \Psi_0}{\Psi_0} = z_j^2 \sum_{i(\neq j)} \xi_{ij} + (\eta_j^{(1)})^2. \quad (2.9)$$

In terms of these derivatives we find that  $H^0\Psi = 2(\pi/L)^2 \sum_j [\eta_j^{(2)} + (2J_{\sigma_j} + 1)\eta_j^{(1)} + J_{\sigma_j}^2]\Psi$ . After some algebra one can show that  $H^0$  acting on the wave function gives "wanted" ( $W_K$ ) and "unwanted" ( $U_K$ ) terms. We write these terms as follows:

$$H^0\Psi = 2 \left(\frac{\pi}{L}\right)^2 (W_K + U_K)\Psi, \quad (2.10)$$

where

$$\begin{aligned}
W_K &= \frac{\lambda^2}{12} N(N^2 - 1) + \frac{1}{2}(\lambda + 1) \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1) \\
&+ \frac{1}{3} \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1)(M_{\sigma} - 2) \\
&- \sum_{\sigma} M_{\sigma} J_{\sigma} (\lambda N - M_{\sigma} - J_{\sigma} - \lambda + 1) + E(x),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
U_K &= \lambda \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \{ \lambda - 1 + 2\delta_{\sigma_i \sigma_j} \} \\
&+ \lambda \sum_{i \neq j} \frac{z_i J_{\sigma_i} - z_j J_{\sigma_j}}{z_i - z_j} \\
&+ \lambda \sum_{(i \neq j \neq k)} \frac{z_i(z_i + z_k)}{(z_i - z_j)(z_i - z_k)} \delta_{\sigma_i \sigma_j},
\end{aligned} \tag{2.12}$$

where  $E(x) = x^2 N(N - 1)^2/4 + (x/2)(N - 1) \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1) + x(N - 1) \sum_{\sigma} M_{\sigma} J_{\sigma}$ .  $M_{\sigma}$  is the total number of  $\sigma$  particles in the system. We can now anticipate  $U_K$  to be canceled or combined with the “unwanted” terms from  $H^1$  and  $H^2$  to give local variable independent terms.

$H^1$  acts trivially on  $\Psi$  to give  $\lambda^2 \sum_{i < j} 1/d(x_i - x_j)^2$ . The action of  $H^2$  on the wave function is less trivial. We need to evaluate the following expression:

$$\begin{aligned}
P_{\{M\}} &\equiv \frac{1}{2\Psi} \sum_{i > j} \frac{P_{ij}^{\sigma} \Psi}{d(x_i - x_j)^2} \\
&= -\frac{1}{\Psi(\{z\sigma\})} \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \Psi(\{z\sigma'\}),
\end{aligned} \tag{2.13}$$

where  $\{z\sigma'\}$  is a configuration with  $\sigma_i$  and  $\sigma_j$  exchanged with respect to  $\{z\sigma\}$ . Similarly, we let  $\{z'\sigma\}$  be equal to  $\{z\sigma\}$  with  $z_i$  and  $z_j$  exchanged. Using the identity,  $\Psi(\{z\sigma\}) = (-1)^{x+1} \Psi(\{z'\sigma'\})$ , we may then rewrite the exchange operation as

$$\begin{aligned}
P_{\{M\}} &= (-1)^{x+1} \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \frac{\Psi(\{z\sigma'\})}{\Psi(\{z'\sigma'\})} \\
&= \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} (-1)^{\delta_{\sigma_i \sigma_j}} \left( \frac{z_i}{z_j} \right)^{J_{\sigma_j} - J_{\sigma_i}} \\
&\quad \times \prod_{k \neq ij} \left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma_j \sigma_k} - \delta_{\sigma_i \sigma_k}}.
\end{aligned} \tag{2.14}$$

The action of  $P^{\sigma}$  is the same for both boson and fermion cases and is independent of the interaction parameter  $\lambda$ .

We use the following identity:

$$\begin{aligned}
\left( \frac{z}{z'} \right)^n &\equiv \sum_{q=0}^{|n|} \binom{|n|}{q} \left\{ \left( \frac{z - z'}{z'} \right)^q \theta(n) \right. \\
&\quad \left. + \left( \frac{z' - z}{z} \right)^q \theta(-n) \right\},
\end{aligned} \tag{2.15}$$

to rewrite Eq. (2.14) as

$$P_{\{M\}} = P_0 + \sum_{\sigma, \sigma'} \sum_{q=1}^{|J_{\sigma'} - J_{\sigma}|} P_q^{\sigma \sigma'}. \tag{2.16}$$

$\theta$  in Eq. (2.15) is the step function with  $\theta(0) = \frac{1}{2}$ , and  $P_0$  and  $P_q^{\sigma \sigma'}$  are given by

$$P_0 = \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \left\{ -\delta_{\sigma_i \sigma_j} + (1 - \delta_{\sigma_i \sigma_j}) \prod_{k \neq ij} \left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma_j \sigma_k} - \delta_{\sigma_i \sigma_k}} \right\}, \tag{2.17}$$

$$\begin{aligned}
P_q^{\sigma \sigma'} &= 2 \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \delta_{\sigma \sigma_i} \delta_{\sigma' \sigma_j} (1 - \delta_{\sigma \sigma'}) \prod_{k \neq ij} \left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma_j \sigma_k} - \delta_{\sigma \sigma_k}} \\
&\quad \times \binom{J_{\sigma'} - J_{\sigma}}{q} \theta(J_{\sigma'} - J_{\sigma}) \left( \frac{z_i - z_j}{z_j} \right)^q.
\end{aligned} \tag{2.18}$$

It is useful to consider separately the terms with  $q = 0, 1$  and the terms with  $q \geq 2$ . For the terms with  $q = 0, 1$  we introduce two sets of site indices  $\{\alpha\}$  and  $\{\beta\}$ . The set  $\{\alpha\}$  ( $\{\beta\}$ ) includes all the locations of particles with the spin  $\sigma$  ( $\sigma'$ ). Using the identity

$$\left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma' \sigma_k} - \delta_{\sigma \sigma_k}} \equiv 1 - \delta_{\sigma' \sigma_k} \frac{z_i - z_j}{z_k - z_j} + \delta_{\sigma \sigma_k} \frac{z_i - z_j}{z_k - z_i} \text{ for } \sigma \neq \sigma', \tag{2.19}$$

the products in  $P_{\{M\}}$  may be expanded and the typical terms in  $P_{\{M\}}$  can be simplified using the following two theorems.

*Theorem 1.* Let  $\{\alpha\}$  and  $\{\beta\}$  be two sets of distinct integers between 1 and  $N$ , and let  $0 \leq q \leq 1$  and  $\Delta = (1 - \delta_{\sigma, \sigma'}) \delta_{\sigma \sigma_1} \delta_{\sigma \sigma_2} \cdots \delta_{\sigma \sigma_n} \delta_{\sigma' \sigma_j} \delta_{\sigma' \sigma_{\beta_1}} \cdots \delta_{\sigma' \sigma_{\beta_m}}$ . Then,

$$\sum_{n=1}^{M_\sigma-1} \sum_{m=1-q}^{M_{\sigma'}-1} \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \frac{(-1)^m}{n! m!} \frac{z_i z_j^{1-q} (z_i - z_j)^{n+m-2+q}}{(z_{\alpha_1} - z_i) \cdots (z_{\alpha_n} - z_i) (z_{\beta_1} - z_j) \cdots (z_{\beta_m} - z_j)} \Delta$$

$$= \begin{cases} - \sum_{k=1}^{\min(M_\sigma, M_{\sigma'})} (M_\sigma - k)(M_{\sigma'} - k) & \text{for } q = 0 \\ - \sum_{k=1}^{\min(M_\sigma, M_{\sigma'})} (M_\sigma - k) & \text{for } q = 1. \end{cases}$$

*Theorem 2.* For  $q \geq 2$  the following identity will hold for  $M_\sigma \geq M_{\sigma'}$ :

$$\sum_{i \neq j} \frac{z_i}{z_j} \left( \frac{z_i - z_j}{z_j} \right)^{q-2} (1 - \delta_{\sigma\sigma'}) \delta_{\sigma\sigma_i} \delta_{\sigma'\sigma_j} \prod_{k \neq ij} \left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma'\sigma_k} - \delta_{\sigma\sigma_k}} = \begin{cases} M_{\sigma'} & \text{for } q = 2 \\ 0 & \text{for } 2 < q \leq M_\sigma - M_{\sigma'} + 1. \end{cases}$$

Proofs of these theorems are given in Appendixes A and B.

After reorganizing some terms we obtain the following results:

$$P_{\{M\}} = W_P + U_P, \tag{2.20}$$

where

$$W_P = -\frac{1}{3} \sum_{\sigma} M_\sigma (M_\sigma - 1)(M_\sigma - 2) - \frac{1}{3} \sum_{\sigma < \sigma'} M_{\sigma'} (M_{\sigma'} - 1)(3M_\sigma - M_{\sigma'} - 1),$$

$$- \sum_{\sigma < \sigma'} M_{\sigma'} (M_\sigma - M_{\sigma'}) (J_{\sigma'} - J_\sigma) + \sum_{\sigma < \sigma'} M_{\sigma'} (J_{\sigma'} - J_\sigma)^2, \tag{2.21}$$

$$U_P = \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)^2} \{1 - 2\delta_{\sigma_i \sigma_j}\} + \sum_{i \neq j} \frac{z_i J_{\sigma_j} - z_j J_{\sigma_i}}{z_i - z_j} - 2 \sum_{(i \neq j \neq k)} \frac{z_i z_j}{(z_i - z_j)(z_i - z_k)} \delta_{\sigma_i \sigma_k}. \tag{2.22}$$

Because of symmetry we can choose  $M_1 \geq M_2 \geq \dots \geq M_n$  and  $0 \leq J_{\sigma'} - J_\sigma \leq M_\sigma - M_{\sigma'} + 1$  for  $\sigma' > \sigma$  without loss of generality. And, one can easily check that  $U_K$ ,  $U_P$ , and the potential-energy term combine to give local-variable-independent terms.

Before we give the full expressions for the eigenenergy, we discuss the allowed values of the integer currents. Theorem 2 gives a simple rule for selecting allowed currents. If we choose  $-1 \leq J_{\sigma'} - J_\sigma \leq M_\sigma - M_{\sigma'} + 1$  for  $M_\sigma \geq M_{\sigma'}$ , then  $P_q = 0$  for  $q > 2$ . Otherwise, the energy will not in general be independent of the local variables and the corresponding wave function will not be an eigenfunction. Pictorial illustration of the allowed currents is shown in Fig. 2. Each row of  $M_\nu$  boxes represents  $M_\nu$  "particles" of the same spin. A single box gives a unit of current. Figure 2(a) is the reference state with no current. If the current is  $j > 0$  ( $< 0$ ) for a row, the row is moved  $j$  boxes to the right (left) as shown in Fig. 2(b). In order for a state to be an eigenstate, the following condition must hold true: *For a given pair of rows of boxes, all except the first and the last boxes in the row with smaller number of boxes must be positioned within the other row.* Figure 2(c), for example, cannot be an eigenstate because the last two boxes in the second row are not within the first row.

Finally, the eigenenergies of the model can be given as follow:

$$E = \frac{2\hbar^2 \pi^2}{mL^2} [E_1 + E_2 + E(x)], \tag{2.23}$$

where  $E_1$  and  $E_2$  are energies due to one- and two-spin

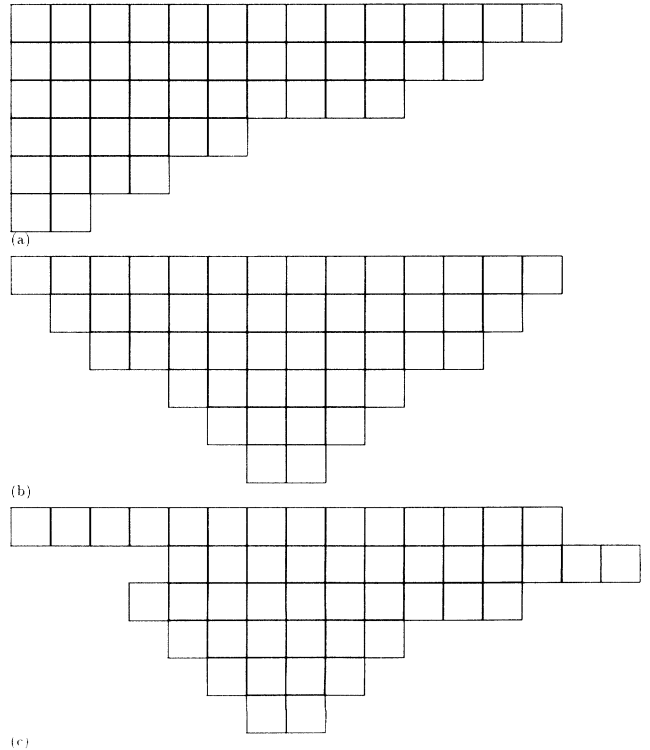


FIG. 2. Illustration of allowed values of the currents for the sector  $\{14, 12, 10, 6, 4, 2\}$ . (a) Allowed. All currents are zero. (b) Allowed. Ground state for the sector. (c) Not allowed.

interactions, respectively, and are given by

$$E_1 = \frac{\lambda^2}{12} N(N^2 - 1) + \frac{1}{2} \lambda N \sum_{\sigma} M_{\sigma} (M_{\sigma} - 1) + \frac{1}{6} (1 - \lambda) \sum_{\sigma} M_{\sigma} (M_{\sigma} - 1) (2M_{\sigma} - 1) + \sum_{\sigma} J_{\sigma} M_{\sigma} (M_{\sigma} + J_{\sigma} - 1), \quad (2.24)$$

$$E_2 = -\frac{1}{3} \lambda \sum_{\sigma < \sigma'} M_{\sigma'} (M_{\sigma'} - 1) (3M_{\sigma} - M_{\sigma'} - 1) - \lambda \sum_{\sigma < \sigma'} M_{\sigma'} (M_{\sigma} - M_{\sigma'}) (J_{\sigma'} - J_{\sigma}) + \lambda \sum_{\sigma < \sigma'} M_{\sigma'} (J_{\sigma'} - J_{\sigma})^2. \quad (2.25)$$

The ground state for fermions (bosons) is given by the following two conditions: (i)  $M_{\sigma'} = M_{\sigma} = M$  and (ii)  $J_{\sigma'} = J_{\sigma} = J = -(M - 1)/2$ ,  $-(N + M - 1)/2$  for fermions and bosons, respectively. The currents should be integers. If condition (ii) does not give an integer current, the ground state is degenerate and the ground-state currents are two integers closest to the half odd integer value.

### III. SU( $n$ ) $t$ - $J$ MODEL

In Ref. 2, Sutherland's model for a system of spinless particles in continuum space was extended to the lattice. Using similar procedures we put our model on the lattice. We propose the following Hamiltonian:

$$H = \mathcal{P} \left( \sum_{j=1}^N \sum_{n=1}^{N_a-1} \frac{t_j^n}{d(n)^2} + \sum_{i>j} \frac{l(l + P_{ij}^{\sigma}) n_i n_j}{d(x_i - x_j)^2} \right) \mathcal{P}. \quad (3.1)$$

$l$  is a positive integer.  $N_a$  is the total number of sites and  $N$  the total number of particles.  $d(n) = (N_a/\pi) |\sin(n)|$ .  $t_j^n$  hops a particle at the  $j$ th site to the  $(j+n)$ th (mod  $N_a$ ) site.  $n_j$  is equal to 1 (0) if the  $j$ th site is occupied (empty).  $P_{ij}^{\sigma}$  is, as before, the spin exchange operator.  $\mathcal{P}$  is the projection operator that insures the absence of multiply occupied sites.

In analogy with Ref. 2 we propose the following eigen-

state for the model Hamiltonian:

$$|\Psi_{\{M\}}\rangle = \sum_{\{z\sigma\}} \phi(\{z\sigma\}) |\{z\sigma\}\rangle. \quad (3.2)$$

Since the total number of each spin is a good quantum number, we classify the eigenstates into sectors labeled by  $\{M\} \equiv (M_1, M_2, \dots, M_n)$ , where  $M_{\nu}$  is the number of  $\nu$ th spin and  $\sum_{\nu=1}^n M_{\nu} = N$ . We also represent the particle configuration as  $\{z\sigma\} \equiv (z_1\sigma_1, \dots, z_i\sigma_i, \dots, z_j\sigma_j, \dots, z_N\sigma_N)$ , where  $z_i$  and  $\sigma_i$  are the location and spin of the  $i$ th particle. The sum in Eq. (3.2) is over  $N_a!/(M_h!M_1! \dots M_n!)$  distinct spin configurations of the sector  $\{M\}$ , where  $N_a$  and  $M_h$  are the total number of sites and the number of holes, respectively. The function  $\phi$  in Eq. (3.2) is given by the following Jastrow-product:

$$\phi(\{z\}|\{\sigma\}) = \prod_{i<j} (z_i - z_j)^{l+\delta_{\sigma_i\sigma_j}} e^{i\frac{\pi}{2} \text{sgn}(\sigma_i - \sigma_j)} \prod_k z_k^{J_{\sigma_k}}. \quad (3.3)$$

Considering the symmetry of the wave function we take odd (even)  $l$  to be a bosonic (fermionic) state.

As in the continuum case we break up the total Hamiltonian into the kinetic, potential, and spin exchange parts ( $H_L^0$ ,  $H_L^1$ , and  $H_L^3$ , respectively). The actions of  $H_L^1$  and  $H_L^3$  are the same as the corresponding operators in the continuum case. For  $H_L^0$  we have the following relations:

$$\begin{aligned} \langle H^0 \rangle &\equiv \frac{\langle z_1\sigma_1, \dots, z_N\sigma_N | H^0 | \Psi_{\{M\}} \rangle}{\langle z_1\sigma_1, \dots, z_N\sigma_N | \Psi_{\{M\}} \rangle} \\ &= 4 \sum_{i=1}^N \sum_{n=1}^{N_a-1} z^{nJ_{\sigma_i}} (1 - z^n)^{-1} (1 - z^{-n})^{-1} \\ &\quad \times \prod_{j(\neq i)} \left( \frac{z_i z^n - z_j}{z_i - z_j} \right)^{l+\delta_{\sigma_i\sigma_j}} \end{aligned} \quad (3.4)$$

where  $|z_1\sigma_1, \dots, z_N\sigma_N\rangle$  is one of the basis states of the sector  $|\Psi_{\{M\}}\rangle$ . In order to evaluate  $\langle H_L^0 \rangle$  we need the following theorem (a slightly different version of this theorem can be found in Ref. 2).

*Theorem 3.* Let  $J$  and  $p$  be non-negative integers and  $z = \exp(2\pi i/N_a)$ . Then,

$$\begin{aligned} \sum_{n=1}^{N_a-1} z^{nJ} (1 - z^{-n})^{-1} (1 - z^n)^{p-1} &= \frac{N_a^2 - 1}{12} - \frac{J(N_a - J)}{2} \text{ for } p = 0, \\ &= -J + \frac{N_a - 1}{2} \text{ for } p = 1, \\ &= +1 \text{ for } p = 2, 0 \leq J \leq N_a - 2 \\ &= -(N_a - 1) \text{ for } p = 2, J = N_a - 1 \\ &= \sum_{m>0} (-1)^{mN_a - J} \binom{p-2}{mN_a - J - 1} N_a \text{ for } p \geq 3. \end{aligned}$$

$m$  is a positive integer and the restriction on the current is  $0 \leq J \leq N_a - 1$  unless specified otherwise.

If we take  $l > 0$ , we can multiply out the product in Eq. (3.4). The resulting expression will be a sum of terms containing a factor  $(1 - z^n)^p$ . Since the maximum value of  $p$  is  $l(N - 1) + M_\sigma - 1$ , if we choose  $0 \leq J_\sigma \leq N_a - l(N - 1) - M_\sigma + 1$ , we may ignore terms with  $p \geq 3$ . We combine the remaining terms with those from  $H_L^1$  and  $H_L^2$  and obtain the following eigenenergies for the lattice model:

$$E^L = E_1^L + E_2^L, \quad (3.5)$$

where

$$\begin{aligned} E_1^L &= \frac{1}{6}N(N_a^2 - 1) + \frac{l^2}{3}N(N - 1)(N - 2) - \frac{l}{2}(N_a - l)N(N - 1) \\ &\quad - \left( \frac{N_a - 1}{2} - l(N - 1) \right) \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1) + \frac{1}{3}(1 - l) \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1)(M_{\sigma} - 2) \\ &\quad - \sum_{\sigma} M_{\sigma}J_{\sigma}(N_a - lN - M_{\sigma} - J_{\sigma} + l + 1), \end{aligned} \quad (3.6)$$

$$E_2^L = E_2, \quad (3.7)$$

where  $E_2$  is given by Eq. (2.25). We emphasize at this point that the above expressions for the eigenenergies for the lattice model are valid only for  $lN + M_\sigma \leq N_a + l + 1$ .

For  $l = 1$  (bosonic supersymmetric case) we can find another expression for the energy for  $N + M_\sigma \geq N_a + 2$ . We proceed by rewriting the kinetic-energy term as

$$\langle H^0 \rangle = 2 \sum_{\sigma} \sum_{i \neq j} (1 - \delta_{\sigma\sigma_h}) \delta_{\sigma\sigma_i} \delta_{\sigma_h\sigma_j} \left( \frac{z_i}{z_j} \right)^{J_{\sigma}+1} \frac{z_i z_j}{(z_i - z_j)^2} \prod_{k \neq ij} \left( \frac{z_k - z_i}{z_k - z_j} \right)^{\delta_{\sigma_h\sigma_k} - \delta_{\sigma\sigma_k}}, \quad (3.8)$$

where  $\sigma_h$  represents the hole. The above expression is almost identical with the expression for the spin exchange operator and can easily be evaluated using theorems 1 and 2. The energy is then given by

$$E(l = 1) = \frac{\pi^2}{2N_a^2} [E_k(l = 1) + W_P], \quad (3.9)$$

where

$$\begin{aligned} E_k(l = 1) &= -\frac{M_h(N_a^2 - 1)}{6} - \frac{1}{3}M_h(M_h - 1)(M_h - 2) + M_h \sum_{\sigma} J_{\sigma}(J_{\sigma} - M_{\sigma} - 1) \\ &\quad + \frac{N_a - 1}{2} \left( \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1) + M_h(M_h - 1) \right) - \frac{1}{3}M_h(M_h - 1)[3N - n(M_h + 1)] \\ &\quad + M_h N + \sum_{\sigma} (J_{\sigma} - 1)[2M_{\sigma}M_h - M_h(M_h + 1)], \end{aligned} \quad (3.10)$$

where the currents are restricted to  $-(M_\sigma - M_h) \leq J_\sigma \leq 0$ .  $W_P$  in Eq. (3.9) is given by Eq. (2.21).

For  $l = 0$  the wave function in Eq. (3.2) is that of Gutzwiller projected free fermions and no longer an eigenstate of the Hamiltonian in Eq. (3.1). However, the wave function is an eigenstate of the following Hamiltonian:

$$H = \mathcal{P} \left( -\sum_{j,n} \frac{t_j^n}{d(n)^2} + \sum_{i>j} \frac{(P_{ij}^\sigma - 1)n_i n_j}{d(x_i - x_j)^2} \right) \mathcal{P}. \quad (3.11)$$

This Hamiltonian is the  $SU(n)$  generalization of the  $SU(2)$  fermionic supersymmetric  $t$ - $J$  model which Kuramoto and Yokoyama<sup>5</sup> solved. If we employ a similar technique to that used for the  $l = 1$  case we may find the eigenenergy given by Eq. (3.9) with  $E_k(l = 1)$  replaced with  $E_k(l = 0)$  which is given by

$$\begin{aligned} E_k(l = 0) &= -\frac{N(N_a^2 - 1)}{6} + \frac{1}{2}(N_a - 1) \sum_{\sigma} M_{\sigma}(M_{\sigma} - 1) \\ &\quad + M_h \sum_{\sigma} J_{\sigma}(M_{\sigma} + J_{\sigma} - 1). \end{aligned} \quad (3.12)$$

The currents are integers restricted to  $-M_\sigma \leq J_\sigma \leq 0$ . The ground-state energy is obtained if  $J_\sigma = -(M_\sigma - 1)/2$  and  $M_\sigma$  odd. If  $M_\sigma$  are not all odd, the ground state is degenerate.

#### IV. $SU(n)$ SPIN CHAIN

If we take the spin-exchange Hamiltonian as our full Hamiltonian for a system of spins on a lattice ring, we obtain the  $SU(n)$  spin chain model. This is equivalent to taking the ‘‘half-filling’’ limit ( $N \rightarrow N_a$ ) of the lattice  $t$ - $J$  model. In this model the eigenenergies for the boson and

fermion systems are identical. Since every site is occupied with a spin,  $U_P$  in Eq. (2.22) can now be summed and is equal to  $-N_a(N_a^2-1)/12 + \{(N_a-1)/2\} \sum_{\sigma} M_{\sigma}(M_{\sigma}-1)$ . Hence,  $U_P$  is now local variable independent, and the eigenenergy is given by Eq. (2.20).

## V. THE HARMONIC FLUID DESCRIPTION

We now turn our attention to the low-energy excitation spectrum of our model in the thermodynamic limit. The low-energy excitation spectrum of a one-dimensional quantum fluid may be described by a sum of two independent harmonic fluid Hamiltonians, one for the charge and the other for the spin. This is a slight generalization of the spinless particle system in Ref. 3. In general the effective Hamiltonian can be written as follows:

$$H = \sum_{\sigma, \sigma'} \int dx [A_{\sigma\sigma'} \Pi_{\sigma}(x) \Pi_{\sigma'}(x) + B_{\sigma\sigma'} \nabla \phi_{\sigma}(x) \nabla \phi_{\sigma'}(x)]. \quad (5.1)$$

$\Pi_{\sigma}(x)$  is the local-density-fluctuation field as in Ref. 3.  $\phi_{\sigma}(x)$  is the canonical conjugate field to  $\Pi(x)$ ,  $[\phi_{\sigma}(x), \Pi_{\sigma'}(x')] = i\delta_{\sigma\sigma'}\delta(x-x')$ . Because of the  $SU(n)$  symmetry we have  $A_{\sigma\sigma'} = a_1 + a_0\delta_{\sigma\sigma'}$  and  $B_{\sigma\sigma'} = b_1 + b_0\delta_{\sigma\sigma'}$ . We now express the Hamiltonian in terms of the Fourier-transformed fields,  $\Pi_{\sigma k}$  and  $\phi_{\sigma k}$ ,

$$H = \sum_k \sum_{\sigma, \sigma'} A_{\sigma\sigma'} \Pi_{\sigma k} \Pi_{\sigma' -k} + B_{\sigma, \sigma'} k^2 \phi_{\sigma k} \phi_{\sigma' -k}. \quad (5.2)$$

We now construct normal mode fields as  $\Pi'_k = \sum_{\sigma} a'_{\sigma} \Pi_{\sigma k}$  and  $\phi'_k = \sum_{\sigma} b'_{\sigma} \phi_{\sigma k}$ . If we choose  $\sum_{\sigma} b'_{\sigma} a'_{\sigma} = \delta_{\nu\nu'}$ , then  $[\phi'_k, \Pi'_{-k}] = i\delta_{kk'}\delta_{\nu\nu'}$ . We have the following equations of motion:  $[H, [H, \Pi'_k]] = -(v^{\nu}k)^2 \Pi'_k$ ,  $[H, [\phi'_k]] = -(v^{\nu}k)^2 \phi'_k$ . For the  $\Pi$  field we obtain the following equation:

$$[a_0 b_0 - (v^{\nu})^2] a'_{\sigma} + (a_1 b_0 + a_0 b_1 + n a_1 b_1) \sum_{\beta} a'_{\beta} = 0. \quad (5.3)$$

The same equation with  $a'_{\sigma}$  replaced with  $b'_{\sigma}$  holds for the  $\phi$  field. There are only two possible values for  $(v^{\nu})^2$ :  $a_0 b_0$  and  $(a_0 + n a_1)(b_0 + n b_1)$ . The first value corresponds to the case  $\sum_{\beta} a'_{\beta} = 0$  and the second  $\sum_{\beta} a'_{\beta} \neq 0$ . Hence, the first would be the spin velocity and the second the charge velocity. The Hamiltonian can now be written as

$$H = \sum_k \left[ (a_0 + n a_1) \Pi'_k \Pi'_{-k} + (b_0 + n b_1) k^2 \phi'_k \phi'_{-k} + \sum_{s=1}^{n-1} \{ a_0 \Pi'_k \Pi'_{-k} + b_0 k^2 \phi'_k \phi'_{-k} \} \right]. \quad (5.4)$$

We have one charge mode and  $n-1$  spin modes; and because of the  $SU(n)$  symmetry all the spin modes have the same velocity. We define the following velocities associated with  $\Pi$  and  $\phi$  fields for the charge (denoted as  $c$ ) and spin (denoted as  $s$ ) modes:  $v_N^c = a_0 + n a_1$ ,  $v_j^c = b_0 + n b_1$ ,  $v_N^s = a_0$ ,  $v_j^s = b_0$ . The charge and spin veloci-

ties are then  $v_c = (v_N^c v_j^c)^{1/2}$  and  $v_s = (v_N^s v_j^s)^{1/2}$ .

Since the eigenstates for our model correspond to the low-energy excitations of the model, we can easily obtain the harmonic fluid parameters by examining the energy expanded about the ground state. By Taylor expansion we obtain the energy for the continuum model as follows:

$$E = e_0 N + \mu N + \frac{\pi \hbar \rho_0}{m} \frac{\pi \hbar}{L} \sum_{\sigma\sigma'} \frac{1}{2} A_{\sigma\sigma'} \Delta M_{\sigma} \Delta M_{\sigma'} + 2 B_{\sigma\sigma'} \Delta J_{\sigma} \Delta J_{\sigma'}, \quad (5.5)$$

where

$$e_0 = \frac{(\pi \hbar \rho_0)^2}{6m} \left( \lambda + \frac{1}{n} \right)^2, \quad (5.6)$$

$$\mu = \frac{(\pi \hbar \rho_0)^2}{2m} \left( \lambda + \frac{1}{n} \right)^2, \quad (5.7)$$

$$A_{\sigma\sigma'} = \lambda \left( \lambda + \frac{1}{n} \right) + \left( \lambda + \frac{1}{n} \right) \delta_{\sigma\sigma'}, \quad (5.8)$$

$$B_{\sigma\sigma'} = -\frac{\lambda}{n} + \left( \lambda + \frac{1}{n} \right) \delta_{\sigma\sigma'}, \quad (5.9)$$

where  $\rho_0 = N/L$ . We can now just read off the velocities,

$$v_s = v_N^s = v_j^s = \frac{\pi \hbar \rho_0}{m} \left( \lambda + \frac{1}{n} \right), \quad (5.10)$$

$$v_N^c = \frac{\pi \hbar \rho_0}{m} n \left( \lambda + \frac{1}{n} \right)^2, \quad (5.11)$$

$$v_j^c = \frac{\pi \hbar \rho_0}{m} \frac{1}{n}, \quad (5.12)$$

$$v_c = v_s. \quad (5.13)$$

The charge and spin velocities are the same for all  $\lambda$  and  $n$ . As expected from the singlet nature of the ground state, the ratio  $v_j^c/v_N^c$  does not get renormalized. The coefficient  $v_j^c$  is independent of the interaction term, as a consequence of Galilean invariance, but  $v_N^c$  gets renormalized due to nonlinear interaction terms not included in the linear form of the harmonic Hamiltonian. The renormalization coupling constant for the charge is  $\exp(-2\varphi) \rightarrow (v_N^c/v_j^c)^{1/2} = n\lambda + 1$  in the limit of long wavelength (i.e.,  $k \rightarrow 0$ ). Because of the scale invariance of the model the dimensionless coupling constant  $\varphi$  is independent of the particle density.

The compressibility per particle is  $(\pi^2 \hbar^2 \rho_0^2/m)(\lambda + 1/n)^2$ . We also find the chemical potential and the ground-state energy to be  $m^* v_F^2/2$  and  $(N/3)(m^* v_F^2/2)$ , respectively. The chemical potential (or the Fermi energy at zero temperature) and the ground-state energy of this one-dimensional system are that of free fermions with the renormalized mass per particle  $m^* = m/(n\lambda + 1)^2$ . The Fermi velocity is given by  $v_F = \pi \hbar \rho_0/(nm^*)$ .

The low-energy excitation of the lattice version is given by Eq. (5.5) as well if we set  $m = \hbar = 1$  and replace  $L$ ,  $e_0$ , and  $\mu$  with  $N_a$ ,  $-\pi^2/3 + 2e_0$ , and  $-\pi^2/3 + 2\mu$ , respectively. The crucial difference is that this expression for the lattice model is true only for  $\rho_0 (= N/N_a) \leq \rho_0^{\max} [= (l+1/n)^{-1}]$ . Hence, the charge velocity is linear in  $\rho_0$  only

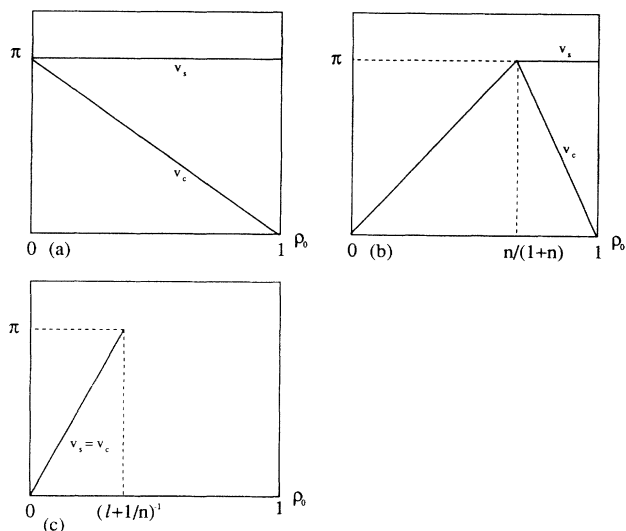


FIG. 3. Charge and spin velocities of the lattice models. (a) Fermionic supersymmetric  $t$ - $J$  model. The velocities are independent of the number of spin species. (b) Bosonic supersymmetric  $t$ - $J$  model. Up to the density  $n/(1+n)$ ,  $v_c$  and  $v_s$  are equal. (c) Bosonic and fermionic  $t$ - $J$  models with  $l \geq 2$ . The charge and spin velocities are the same up to the density  $(l+1/n)^{-1}$ . Beyond the density the ground state is no longer the Jastrow-product type.

up to  $\rho_0^{\max}$ . We expect  $v_c$  to vanish as  $\rho_0 \rightarrow 1$  due to the metal-insulator transition at the density.  $v_c$ , therefore, should exhibit nonanalyticity at  $\rho_0^{\max}$ . This behavior is attributed to the long-range interaction in our model. In Ref. 4 this type of nonanalyticity in the spinon velocity was observed for the  $SU(2)$  spin chain model. In the nearest-neighbor interaction models the sharp change in the charge velocity is smoothed out.

For  $l = 1$  the energy for  $\rho_0 \geq (1 + 1/n)^{-1}$  is explicitly found and the harmonic fluid Hamiltonian is given by Eq. (5.5) with  $A_{\sigma\sigma'} = (1 - \rho_0)(n + 2) - \rho_0/n + \delta_{\sigma\sigma'}$  and  $B_{\sigma\sigma'} = -\rho_0/n + \delta_{\sigma\sigma'}$ . Therefore,  $v_c = \pi(1 - \rho_0)(n - 1)$  and  $v_s = \pi$ . The spin velocity is independent of the density in this region. Similarly, for  $l = 0$  we have  $A_{\sigma\sigma'} = B_{\sigma\sigma'} = -\rho_0/n + \delta_{\sigma\sigma'}$  for  $0 \leq \rho_0 \leq 1$ . Thus,  $v_c = \pi(1 - \rho_0)$  and  $v_s = \pi$ . This agrees with the results obtained by Kuramoto and Yokoyama.<sup>5</sup> However, it is unexpected that the charge and spin velocities are independent of the number of spin species. Figure 3 shows the charge and spin velocities for the lattice models with various interaction parameters.

## VI. CONCLUSION

We have shown in this paper that Sutherland's model of spinless particles interacting with inverse-square exchange can be generalized to the multicomponent system of particles. We further generalized the continuum model to lattice for the integer values of the dimensionless interaction parameter. We explicitly constructed a class of eigenstates for these models and calculated the corre-

sponding eigenenergies. We claimed in this paper that the class of eigenstates corresponds to the ground state and the low-energy excitations of the model. We checked this by the exact numerical diagonalization of the lattice version of Hamiltonian for small systems. The systematic construction of the other eigenstates, thermodynamics, and the general  $n$ -point correlation functions for these models are yet to be found. As Minahan showed,<sup>7</sup> the Hamiltonian of the type given by Eq. (2.1) also arises in a matrix model as a representation of one-dimensional open string theory.

*Note added.* After the completion of this paper we became aware of work by N. Kawakami,<sup>8</sup> whose work is related to ours, and the Hamiltonian given by Eq. (2.1) is also independently identified by Polychronakos.<sup>6</sup>

## ACKNOWLEDGMENTS

This work is supported in part by NSF Grant No. DMR-91-96212.

## APPENDIX A: PROOF OF THEOREM 1

We first discuss a diagrammatic way of writing the exchange operation. A labeled diagram is shown in Fig. 4. The amplitude of the labeled diagram shown in Fig. 4 can be evaluated using the following rules: (i) the dashed line connecting the indices  $i$  and  $j$  gets a factor  $z_i z_j (z_i - z_j)^{-2} (1 - \delta_{\sigma, \sigma'}) \delta_{\sigma\sigma_i} \delta_{\sigma\sigma_j}$ ; (ii) each solid line connecting the indices  $i$  and  $\alpha_k$  gets a factor  $(z_i - z_j)(z_{\alpha_k} - z_i)^{-1} \delta_{\sigma\sigma_{\alpha_k}}$ , (iii) each solid line connecting the indices  $j$  and  $\beta_k$  gets a factor  $-(z_i - z_j)(z_{\beta_k} - z_j)^{-1} \delta_{\sigma\sigma_{\beta_k}}$ . Weight of the dashed line vanishes whenever the spins at the sites  $i$  and  $j$  are the same. On the other hand, weights of the solid lines that are connected to  $i(j)$  vanish whenever the spins at the sites  $\{\alpha\}$  ( $\{\beta\}$ ) and  $i(j)$  are different. Hence, we may think of the diagram as a two-spin interaction diagram. Since the Hamiltonian has only the two-spin exchange operator  $P_{ij}$ , three-spin and higher number of spin interactions are missing.

The amplitude of the labeled diagram is undesirable since it depends on the local variables. A more desirable,

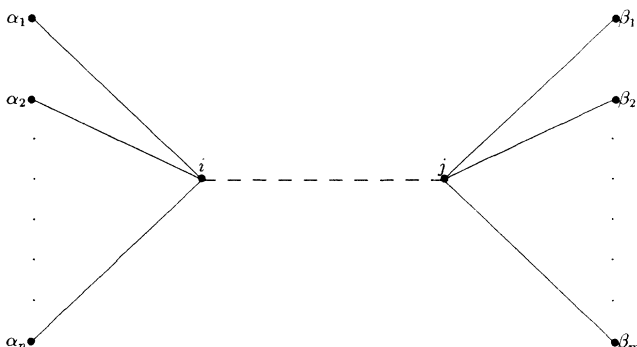


FIG. 4. Labeled diagram for  $n$   $\sigma$  spins and  $m$   $\sigma'$  spins.



local-variable-independent diagram (unlabeled diagram) can be obtained by summing  $(n+1)(m+1)$  cyclically permuted labeled diagrams. This fact will be proved in this appendix.

We note that the indices  $\{i, \alpha_1, \dots, \alpha_n\}$  and  $\{j, \beta_1, \dots, \beta_m\}$  are dummy indices. Consequently, the sum will be invariant under any permutation of the indices. We further observe that  $\Delta$  is invariant un-

der any permutation of the indices except when there is interchange between the two sets  $\{i, \alpha_1, \dots, \alpha_n\}$  and  $\{j, \beta_1, \dots, \beta_m\}$ . On the other hand, the prefactor to  $\Delta$  remains invariant with respect to the permutations of indices  $\{\alpha\}$  and  $\{\beta\}$ . It is convenient to consider only the cyclic permutations which give  $(n+1)(m+1)$  distinct prefactors. We then sum the  $(n+1)(m+1)$  factors as follow:

$$\begin{aligned} P_{nm}^{\sigma\sigma'} &= \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \frac{(-1)^m}{n! m!} \frac{z_i z_j^{1-q} (z_i - z_j)^{n+m-2+q}}{(z_i - z_{\alpha_1}) \cdots (z_i - z_{\alpha_n}) (z_j - z_{\beta_1}) \cdots (z_j - z_{\beta_m})} \Delta \\ &= \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \sum_{k,l} \frac{(-1)^m}{(n+1)! (m+1)!} \frac{z_{\gamma_k^\alpha} z_{\gamma_l^\beta}^{1-q} (z_{\gamma_k^\alpha} - z_{\gamma_l^\beta})^{n+m-2+q}}{(z_{\gamma_k^\alpha} - z_{\alpha_1}) \cdots (z_{\gamma_k^\alpha} - z_i) \cdots (z_{\gamma_l^\beta} - z_{\alpha_n})} \\ &\quad \times \frac{1}{(z_{\gamma_l^\beta} - z_{\beta_1}) \cdots (z_{\gamma_l^\beta} - z_j) \cdots (z_{\gamma_l^\beta} - z_{\beta_m})} \Delta, \end{aligned} \quad (\text{A1})$$

where  $\{\gamma^\alpha\} \equiv \{i, \alpha_1, \dots, \alpha_n\}$  and  $\{\gamma^\beta\} \equiv \{j, \beta_1, \dots, \beta_m\}$ .

We let  $D_\alpha = \prod_{k < k'} (z_{\gamma_k^\alpha} - z_{\gamma_{k'}^\alpha})$  and  $D_\beta = \prod_{l < l'} (z_{\gamma_l^\beta} - z_{\gamma_{l'}^\beta})$ . Using the binomial theorem we expand the numerator in Eq. (A1). We then multiply and divide the expression by  $D_\alpha D_\beta$  and obtain the following:

$$P_{nm}^{\sigma\sigma'} = \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \sum_{s=1}^{n+m-1+q} \frac{(-1)^{m-1+q-s}}{(n+1)! (m+1)!} \binom{n+m-2+q}{s-1} \frac{V_\alpha^s V_\beta^{n+m-s}}{D_\alpha D_\beta} \Delta, \quad (\text{A2})$$

where

$$V_\alpha^s = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_i & z_{\alpha_1} & \cdots & z_{\alpha_n} \\ \vdots & \vdots & & \vdots \\ z_i^{n-1} & z_{\alpha_1}^{n-1} & \cdots & z_{\alpha_n}^{n-1} \\ z_i^s & z_{\alpha_1}^s & \cdots & z_{\alpha_n}^s \end{vmatrix}, \quad (\text{A3})$$

and  $V_\beta^k$  is defined the same way. We further note that  $V_\alpha^s = 0$  if  $0 \leq s \leq n-1$  and  $V_\beta^k = 0$  if  $0 \leq k \leq m-1$ . Therefore, if  $0 \leq q \leq 1$ , the only nonzero contribution in (A2) is the term with  $s = n$ . Since  $V_\alpha^n = D_\alpha$  and  $V_\beta^m = D_\beta$ , we obtain

$$\begin{aligned} P_{nm}^{\sigma\sigma'} &= \frac{(-1)^{n+m-1+q}}{(n+1)! (m+1)!} \binom{n+m-2+q}{n-1} \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \Delta \\ &= (-1)^{n+m-1+q} \binom{n+m-2+q}{n-1} \binom{M_\sigma}{n+1} \binom{M_{\sigma'}}{m+1}. \end{aligned} \quad (\text{A4})$$

Evaluating  $\sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \Delta$  in (A4) is the same as calculating the total number of ways of putting  $n+1$  out of  $M_\sigma$  blue balls in a box and  $m+1$  out of  $M_{\sigma'}$  red balls in another box. The sum, therefore, is equal to  $M_\sigma (M_\sigma - 1) \cdots (M_\sigma - n) M_{\sigma'} (M_{\sigma'} - 1) \cdots (M_{\sigma'} - m)$ .

It is straightforward to prove the following useful identities:

$$\binom{n+m-2+q}{n-1} = \sum_{s=0}^n \binom{n-1}{s} \binom{m-1+q}{s}, \quad (\text{A5})$$

$$\begin{aligned} (-1)^s \binom{m-1+q}{s} &= \sum_{k=0}^s (-1)^k \binom{m+q}{k}, \\ &= \sum_{k=0}^s (-1)^k (s+1-k) \binom{m+1+q}{k}, \end{aligned} \quad (\text{A6})$$

$$\sum_{m=1-q}^{M_\sigma-1} (-1)^m \binom{M_\sigma}{m+1} \binom{m+1}{k} = \begin{cases} -M_\sigma + 1 & \text{if } k = 0 \\ -M_\sigma & \text{if } k = 1 \\ 0 & \text{if } 2 \leq k < M_\sigma \end{cases} \text{ for } q = 0, \tag{A7}$$

$$\begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \leq k < M_\sigma \end{cases} \text{ for } q = 1.$$

Using (A5) we may sum the terms depending on  $n$  and  $m$  in  $P_{nm}^{\sigma\sigma'}$  separately. Using (A6) and (A7) the following relation can easily be obtained:

$$P^{\sigma\sigma'} = \begin{cases} - \sum_{k=1}^{\min(M_\sigma, M_{\sigma'})} (M_\sigma - k)(M_{\sigma'} - k) & \text{for } q = 0, \\ - \sum_{k=1}^{\min(M_\sigma, M_{\sigma'})} (M_\sigma - k) & \text{for } q = 1. \end{cases} \tag{A8}$$

Q.E.D.

**APPENDIX B: PROOF OF THEOREM 2**

We replace the product over  $k$  with two new sets of integer indices,  $\{\alpha\}$  and  $\{\beta\}$ , and rewrite the expression in theorem 2 as

$$Q = \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \frac{1}{(M_\sigma - 1)!(M_{\sigma'} - 1)!} \frac{z_i}{z_j} \left( \frac{z_i - z_j}{z_j} \right)^{q-2} \frac{z_{\alpha_1} - z_j}{z_{\alpha_1} - z_i} \frac{z_{\alpha_2} - z_j}{z_{\alpha_2} - z_i} \dots \frac{z_{\alpha_{M_\sigma-1}} - z_j}{z_{\alpha_{M_\sigma-1}} - z_i} \times \left\{ \sum_{r=0}^{M_{\sigma'}-1} (z_i - z_j)^r f_r(\{z_\beta\}) \right\} \Delta. \tag{B1}$$

Here,  $\Delta$  is the same as in theorem 1. The factor in the curly brackets in (B1) is obtained by writing the terms like  $(z_{\beta_i} - z_i)/(z_{\beta_i} - z_j)$  as  $\{1 - (z_i - z_j)/(z_{\beta_i} - z_j)\}$  and by multiplying out all the factors that depend on  $\{\beta\}$ .  $f_r$  in (B1) is some function of  $\{z_\beta\}$ , however  $f_0 = 1$ .

As in Appendix A, we rewrite the expression by summing over the factors obtained by the cyclic permutations of the indices  $\{i, \alpha_1, \dots, \alpha_{M_\sigma-1}\}$  and obtain

$$Q = \sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \sum_{r=0}^{M_{\sigma'}-1} \frac{(-1)^{M_\sigma}}{M_\sigma!(M_{\sigma'} - 1)!} (z_i - z_j)(z_{\alpha_1} - z_j) \dots (z_{\alpha_{M_\sigma-1}} - z_j) f_r \frac{W_\alpha^{M_\sigma}}{D_\alpha^{M_\sigma}}, \tag{B2}$$

where  $W_\alpha^{M_\sigma}$  is given by the Vandemonde determinant whose last row is modified to  $z_{\gamma_l}(z_{\gamma_l} - z_j)^{q+r-3}/z_j^{q-1}$ ,  $\{\gamma_l\} \equiv \{i, \alpha_1, \dots, \alpha_{M_\sigma-1}\}$ . It is straightforward to show that

$$W_\alpha^{M_\sigma} = \begin{cases} \frac{(-1)^{M_\sigma} D_\alpha^{M_\sigma}}{(z_i - z_j)(z_{\alpha_1} - z_j) \dots (z_{\alpha_{M_\sigma-1}} - z_j)} & \text{if } q = 2 \text{ and } r = 0 \\ 0 & \text{if } 0 \leq q + r - 3 \leq M_\sigma - 3 \end{cases}, \tag{B3}$$

$$\sum_{i \neq j} \sum_{\{\alpha\}, \{\beta\}} \Delta = M_\sigma! M_{\sigma'}!. \tag{B4}$$

The only nonzero contribution to  $Q$  is given by the terms with  $q = 2$  and  $r = 0$ . Since  $0 \leq r \leq M_{\sigma'} - 1$ , the sufficient condition for  $W_\alpha^{M_\sigma} = 0$  is  $3 \leq q \leq M_\sigma - M_{\sigma'} + 1$ . Therefore, we have for  $M_\sigma \geq M_{\sigma'}$

$$Q = \begin{cases} M_{\sigma'} & \text{for } q = 2, \\ 0 & \text{for } 3 \leq q \leq M_\sigma - M_{\sigma'} + 1. \end{cases} \tag{B5}$$

Q.E.D.

<sup>1</sup>B. Sutherland, Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972).

<sup>2</sup>F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); B. Sriram Shastry, *ibid.* **60**, 639 (1988).

<sup>3</sup>F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1982); **48**, 569(E) (1982).

<sup>4</sup>F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

<sup>5</sup>Y. Kuramoto and H. Yokoyama, Phys. Rev. Lett. **67**, 1338 (1991).

<sup>6</sup>A. P. Polychronakos (unpublished).

<sup>7</sup>J. A. Minahan (unpublished).

<sup>8</sup>N. Kawakami, Phys. Rev. B **46**, 1005 (1992); **46**, 3191 (1992).