Conductivity of a Luttinger liquid

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We show that the real part of the frequency-dependent conductivity of interacting spinless fermions at incommensurate band fillings in one dimension at $T=0$ is proportional to ω^3 for small positive frequencies, contrary to statements in the recent literature. We argue that the dc conductivity of spinless fermions at incommensurate band fillings diverges faster than any power of $1/T$ as $T\rightarrow 0$. We give the extension of our results to the case of commensurate band fillings and fermions with spin.

I. INTRODUCTION

The proper calculation of the frequency- and temperature-dependent conductivity $\sigma(\omega,T)$ of interacting fermions on a periodic lattice is an old problem in solid-state physics. In this paper we study $\sigma(\omega,T)$ for interacting fermions on a one-dimensional lattice. We distinguish between commensurate band filling, where the Fermi wave vector k_F is a rational fraction of a reciprocal-lattice vector, and incommensurate filling, where it is not. For incommensurate filling we show that for small positive ω the real part of the conductivity of a generic 1D fermion system at an incommensurate band filling varies as ω^3, so that it is finite at any finite frequency. It is not possible to calculate directly the dc conductivity $\sigma(\omega=0,T)$; however, by assuming that one may define the dc conductivity in terms of the time it takes for a state of finite current to decay we argue that in a one-dimensional fermion system at an incommensurate filling $\sigma(\omega=0,T)$ diverges faster than any power of $1/T$ as $T\rightarrow 0$. Our result for $\sigma(\omega,T=0)$ is different from that obtained in several recent papers.[1-3] We argue that the difference is due to the neglect, in Refs. 1 and 2, of irrelevant operators arising from band nonparabolicity and in Ref. 3 to the small size of the system studied numerically. The difference we find between the small ω and small T behaviors of $\sigma(\omega,T)$ implies that a commonly used approximation, the "memory function" approximation,[4] gives incorrect results in this case for $\sigma(\omega=0,T)$ because it does not respect a conservation law. We give the extension of our results to the cases of fermions with spin and to commensurate band fillings. The half-filled band case agrees with previous results.[5]

The rest of this paper is organized as follows. In Secs. II–V we analyze a spinless fermion model. Section II gives the Hamiltonian, the formalism, and an elementary perturbation-theory result; Sec. III gives exact results for $\sigma(\omega,T=0)$ for incommensurate band fillings; Sec. IV treats $\sigma(\omega=0,T)$ for incommensurate band fillings; Sec. V extends the results of Secs. III and IV to the commensurate case. In Sec. VI we consider fermions with spin. Section VII is a conclusion.

II. SPINLESS FERMIONS: FORMALISM AND PERTURBATION THEORY

We consider a model of spinless fermions created by operators c_i^+ hopping via a hopping matrix element t_{ij} between sites i and j of a one-dimensional lattice and interacting via an interaction which is a functional of the fermion density $\hat{n}_i=c_i^+c_i$. We write the Hamiltonian as

$$\hat{H}=\hat{T}+\hat{V}\ , \tag{2.1}$$

where for our explicit calculations we take the kinetic energy $\hat{T}$ to be

$$\hat{T}=\sum_i tc_i^+c_{i+1}+\text{H.c.}\ , \tag{2.2}$$

and the interaction $\hat{V}$ to be (the summation is over nearest-neighbor pairs i,j)

$$\hat{V}=\sum_{\langle ij\rangle} V_{ij}^{(2)}\hat{n}_i\hat{n}_j+\cdots\ . \tag{2.3}$$

We are interested in $\sigma(\omega)$, the real, dissipative part of the frequency-dependent conductivity, at positive frequencies $\omega>0$ and at zero temperature. This is related to the current-current correlation function via[6]

$$\sigma(\omega>0)=\lim_{q\rightarrow 0}\text{Re}\frac{1}{i\omega}\int_0^\infty dt\, e^{i\omega t}\langle\theta|[\hat{\mathbf{J}}_q(t),\hat{J}_{-q}(0)]|\theta\rangle\ . \tag{2.4}$$

Here the expectation value is taken in the ground state $|\theta\rangle$ of the Hamiltonian Eq. (2.1). The current operator $\hat{J}_q$ may be obtained from the density operator $\hat{n}_q=\sum_i e^{iq\cdot R_i}\hat{n}_i$ via the continuity equation, $\partial\hat{n}_q/\partial t+i\mathbf{q}\cdot\hat{J}_q=0$, so that

$$\hat{J}_q=\frac{-1}{q}[\hat{H},\hat{n}_q]\ . \tag{2.5}$$

From (2.5) and (2.2) one may easily verify that if $\hat{V}=0$, then $\lim_{q\rightarrow 0}[\hat{H},\hat{J}_q]=0$. By inserting this result in Eq. (2.4) we see that if $\hat{V}=0$, $\sigma(\omega>0)=0$. In a Galilean-invariant model, the current in the limit $q\rightarrow 0$ is proportional to the momentum and so commutes with the in-

teraction, so that $\sigma(\omega>0)=0$ always. However, in the lattice model defined by Eqs. (2.1)–(2.3), if $\hat{V}\neq 0$ then $\lim_{q\to 0}[\hat{H},\hat{J}_q]\neq 0$; in this case $\lim_{q\to 0}\hat{J}_q$ has matrix elements between the ground state $|\theta\rangle$ of $\hat{H}$ and some excited states $|n\rangle$ of $\hat{H}$, so that $\sigma(\omega)$ is nonzero at some finite frequencies. To obtain further information we expand $\sigma(\omega>0)$ in perturbation theory in $\hat{V}$. Because $\sigma(\omega>0)$ vanishes for $\hat{V}=0$, and $\lim_{\omega\to\infty}\omega\sigma(\omega)$ vanishes for any $\hat{V}$, such an expansion is obviously possible at sufficiently high frequencies. Because in the metallic phase $\sigma(\omega)$ does not diverge at a nonzero frequency, and diverges less rapidly than $1/\omega$ as $\omega\to 0$, the expansion turns out to be possible at all finite frequencies for sufficiently weak couplings $\hat{V}$. The most efficient way to generate the expansion is to integrate the expression in Eq. (2.4) by parts and use the relation $\partial\hat{J}_q/\partial t=i[H,\hat{J}_q]$. The result[4,6] is

$$\sigma(\omega)=\lim_{q\to 0}\mathrm{Re}-\left[\frac{1}{i\omega}\right]^3\int_0^\infty dt\, e^{i\omega t}\langle\{[\hat{H},\hat{J}_q(t)],[\hat{H},\hat{J}_{-q}(0)]\}\rangle\ . \tag{2.6}$$

From the previous remarks it is clear that σ is $O(\hat{V})^2$ and that the leading-order term in an expansion in powers of $\hat{V}$ is obtained by evaluating (2.6) using the noninteracting Hamiltonian, Eq. (2.3). We have computed $[\hat{H},\hat{J}_q]$ for the model of Eqs. (2.2) and (2.3). We find

$$\lim_{q\to 0}[\hat{H},\hat{J}_q]=itV\sum_{kk'p}M_{kk'p}c_k^+c_{k+p}c_{k'}^+c_{k'-p}\ , \tag{2.7}$$

with

$$M_{kk'p}=\cos(p)[\sin(k+p)-\sin k+\sin(k'-p)-\sin k']\ . \tag{2.8}$$

Here we have set the lattice constant equal to unity.

One may pass to the continuum or Galilean-invariant limit by considering a very low density of particles. In this case the only relevant momenta are small compared to the lattice constant, so one may set $\sin k\to k$, etc. It is clear that in this limit $M_{kk'p}\to 0$, so $\sigma(\omega)$ vanishes in the continuum limit as it must.

We now evaluate (2.6) by the perturbation expansion in V. The leading term at $T=0$ is

$$\sigma(\omega)=\frac{\alpha}{\omega^3}t^2V^2{\sum_{kk'p}}'M_{kk'p}^2\delta\{\omega-[(\varepsilon_{k'}-\varepsilon_{k'-p})+(\varepsilon_k-\varepsilon_{k+p})]\}\ . \tag{2.9}$$

Here α is a number, ε_p is the energy relative to the Fermi surface of a state of momentum p, and the prime indicates that the sum over the momenta is restricted to the region where k' and k are outside the Fermi surface and $k'-p$ and $k+q$ are inside the Fermi surface. Thus, the δ function explicitly written in Eq. (2.9) ensures that all four energies are within $\sim\omega$ of zero; thus as $\omega\to 0$, all four momenta, k, k', $k'-p$, and $k+p$, must converge on the Fermi momentum. In the small ω limit, two regimes of p are possible: $p\sim|\omega|/v_F$ or $p\sim 2k_F$. In the former case $M_{kk'p}$ is obviously of order ω^2; in the latter case, provided $2k_F$ does *not* equal π the constraints imply k and k' must be on opposite sides of the Fermi surface so that $M_{kk'p}$ again turns out to be second order in the deviations of the wave vectors from the Fermi wave vector. However, if $2k_F=\pi$ then $M_{kk'p}$ turns out to be constant. In either case the k and k' integrals each extend over a range of order ω. Thus one obtains $\sigma(\omega)\sim\omega^3$ in the incommensurate case and $\sigma(\omega)\sim 1/\omega$ in the commensurate case. A similar calculation in two spatial dimensions would yield $\sigma(\omega)\sim\ln\omega$ and in three $\sigma(\omega)\sim$ const for incommensurate fillings.[7]

Thus from the elementary perturbation calculation we learn that for $2k_F\neq\pi$ we should expect the conductivity to be nonzero but rapidly vanishing in the low-frequency limit, and that to obtain a nonzero result we must have an underlying lattice (so that the current operator is not simply proportional to k) and we must consider the variation of the current operator as one moves away from the Fermi surface. This variation was neglected in Refs. 1 and 2; the neglect accounts for the incorrect claim that $\sigma(\omega)=0$ for a finite range about $\omega=0$ for incommensurate fillings. However at $2k_F=\pi$ the conductivity behaves differently. Of course, perturbation theory is not valid for one-dimensional Fermi systems. Thus in the next section we present a calculation of $\sigma(\omega)$ based on Haldane's formulation[8] of the low-energy physics of the model specified by Eqs. (2.2) and (2.3).

III. $\sigma(\omega)$ FROM EXACT SOLUTION-INCOMMENSURATE DENSITIES

The low-energy physics of one-dimensional spinless fermion models can be described by a Hamiltonian involving boson variables. One must distinguish two cases: commensurate density, when the Fermi wave vector of the noninteracting system is a rational fraction of the reciprocal-lattice vector; and incommensurate density, when it is not. In the commensurate case umklapp scattering may be important. In the incommensurate case, operators involving umklapp scattering necessarily involve states separated from the ground state by a finite energy gap Δ. The Hamiltonian describing processes involving energies less than this gap may be written in two ways. One is a fermionic form involving two species of fermions: "right movers" (which may be thought of as excitations near the Fermi point on the positive k axis) and "left movers" (excitations near the Fermi point on the negative k axis) plus interactions which conserve the number of right and left movers separately. An alternative formulation is in terms of bosons (which may be thought of as particle-hole pair excitations of the original Fermi system) and a conserved quantum number J (which is a velocity times the number of right movers minus the number of left movers). The boson formula-

tion has been discussed by many authors.[9] We use the version due to Haldane, in which the effects of operators coming from band curvature are discussed.[8] In Haldane's formulation one writes

$$\hat{H}'=\hat{H}_0+\hat{H}_I \ , \tag{3.1}$$

where

$$\hat{H}_0=\sum_q \omega_q b_q^+ b_q+\tfrac{1}{2}V_J J^2 \tag{3.2}$$

and $\hat{H}_I$ will be discussed below. Here b_q^+ creates a boson of energy ω_q and momentum q. ω_q is J dependent. For $J=0$ and small q, $\omega_q=v_s|q|$; in the weak-coupling limit $v_s=v_F$. The ground state of $\hat{H}'$ has $J=0$. V_J is a parameter of the model which approaches v_F in the weak-coupling limit.

The operator creating an electron density fluctuation of momentum q may be written, in the low-energy subspace $\omega<\Delta$, as [8]

$$\rho_q=\left[\frac{L|q|}{2\pi}\right]^{1/2}[b_q^+ +b_{-q}] \ . \tag{3.3}$$

The noninteracting boson model Eq. (3.2) describes the low-energy physics of the interacting fermion model Eq. (2.1) in the approximation in which the fermion dispersion is linearized about the Fermi surface. In the nearest-neighbor hopping model used in the explicit calculations, this would correspond to replacing $\varepsilon_k=-2t\cos(k)-\mu$ by $v_F|k-k_F|$, where $v_F=2t\sin k_F$. Retaining higher-order terms in the expansion of the fermion energy about the Fermi surface leads in the boson language to the interaction term $\hat{H}_I$, which takes[8] the explicit form

$$\hat{H}_I=\sum_{p=\pm 1}\int_0^L \frac{dx}{2\pi}\left[\left(\frac{1}{6m}-\frac{\lambda p(\pi J/L)}{12m^2v_F}\right):\Phi_p(x)^3: +\frac{\lambda}{48m^2v_F}:\Phi_p(x)^4:+\cdots\right] \tag{3.4}$$

with

$$\Phi_p(x)=\sum_q pq\,\alpha(pq,-\phi_q)(e^{iqx}b_q^+ +e^{-iqx}b_q) \ , \tag{3.5}$$

$$\alpha(q,\phi_q)=[\theta(q)\cosh\phi_q+\theta(-q)\sinh\phi_q](2\pi/L|q|)^{1/2} \ . \tag{3.6}$$

Here the : : symbol denotes boson normal ordering and ϕ_q is a real even function of q which parametrizes the electron-electron interactions. $\phi_q=0$, if $\hat{V}=0$ in Eq. (2.1) and $\lim_{q\to 0}\phi_q\neq 0$ for an interacting electron model. In the weak coupling ($V\ll t$ limit), the parameter m is determined by the term in the fermion energy dispersion quadratic about $k=k_F$ and the parameter λ by the term proportional to $(k-k_F)$.[3] In a Lorentz-invariant model $m\to\infty$ and $\lambda/m^2\to 0$. In the weak-coupling limit of the nearest-neighbor hopping model defined above in Eq. (2.7), $1/2m=\cos k_F$ and $\lambda/4m^2v_F=-\sin k_F$. In the low-density continuum limit described earlier, v_F (i.e., ω_q/q) and $\lambda/12m^2v_F$ become small relative to $1/2m$. In a true continuum model with a $k^2/2m$ dispersion and a finite density of electrons, $\lambda=0$.

Near half-filling ($k_F=\pi/2$), $1/(2mv_F)\ll 1$ and a perturbation expansion in $\varepsilon=1/(2mv_F)$ would be rapidly convergent. Haldane[8] argues that for general k_F an expansion in $1/2mv_F$ will be well behaved; coefficients appearing in the fixed-point Hamiltonian may change, but the qualitative behavior will not be affected. We shall compute to leading nontrivial order in ε and assume that the qualitative behavior will not change even if ε becomes large.

The current operator defined using Eqs. (3.3) and (3.5) is, in the limit $q=0$,

$$\hat{J}=V_J\frac{\pi J}{L}+\sum_q\left[\frac{q}{2m}-\frac{\lambda}{4m^2v_F}\frac{\pi J}{L}|q|\cosh(2\theta)\right]b_q^+ b_q$$
$$-\frac{\lambda}{8m^2v_F}\frac{\pi J}{L}\sum_q|q|\sinh 2\theta(b_q^+ b_{-q}^+ +b_q b_{-q})$$
$$-\frac{\lambda}{12m^2v_F}\sum_p\int_0^L\frac{dx}{2\pi}:p\Phi_p^3(x): \ . \tag{3.7}$$

We have verified that in the noninteracting model ($\phi=0$) we have $[\hat{H},\hat{J}]=0$ even though both $\hat{H}$ and $\hat{J}$ are nonlinear in Bose operators.

We now compute $\sigma(\omega)$ from Eq. (2.6). Of the many terms generated in $\lim_{q\to 0}[\hat{H},J_q]$ the leading-order term which does not annihilate the ground state is

$$\lim_{q\to 0}[H,\hat{J}_q]=\frac{\lambda}{48m^2v_F}e^{2\phi}\sum_{p=\pm 1}\sum_{q_1\cdot q_3}\delta(q_1+q_2+q_3)(q_1q_2q_3)(\omega_{q_1}+\omega_{q_2}+\omega_{q_3})$$
$$\times\alpha(pq_1,-\phi_{q_1})\alpha(pq_2,-\phi_{q_2})\alpha(pq_3,-\phi_{q_3})b_{q_1}^+ b_{q_2}^+ b_{q_3}^+ \ . \tag{3.8}$$

Inserting this into (2.6) and evaluating the expectation value at $T=0$ yields

$$\sigma(\omega)=\frac{1}{8\pi}\left[\frac{\lambda}{12m^2v_F}\right]^2\frac{\sinh^2 2\phi}{v_s^3}\omega^3+O(\omega^5) \ . \tag{3.9}$$

From this we see that in a "Luttinger-liquid" 1D system in which umklapp scattering is irrelevant, $\sigma(\omega)\sim\omega^3$; further, the conductivity vanishes in the noninteracting limit ($\phi\to 0$) and in either the Lorentz-invariant or the Galilean-invariant limit ($\lambda/m^2\to 0$ in either case). As in the weak-coupling calculation it is necessary to expand the current operator to second order about the Fermi surface to obtain a finite result.

The precise coefficient appearing in Eq. (3.9) is only correct to leading order in an expansion in the parameter $1/(2mv_F)$. We believe the general form of (3.9), $\sigma(\omega)=A\omega^3\sinh^2 2\phi$, with A a coefficient parametrizing the departure from Galilean invariance in the underlying lattice Hamiltonian, is correct for any Luttinger liquid at an incommensurate band filling.

IV. $\sigma(\omega=0,T)$—INCOMMENSURATE DENSITY

We now consider the temperature dependence of the dc conductivity. Because $\sigma(\omega=0,T)$ diverges as $T\to 0$, the low T dependence cannot be calculated by a perturbative expansion of the current-current correlation function. We proceed using arguments similar to those used by Peierls[10] to obtain the thermal conductivity of phonons. The essential idea is to define the conductivity in terms of the decay time of a state of finite current. We show that in the Luttinger-liquid model because of momentum conservation an initial current will never decay to zero, so the conductivity is infinite.

We suppose that at time $t=0$ the system is in a state of small finite current j specified by minimizing the free energy of the system subject to a constraint that the expectation value of the current operator equals j. Thus we consider the free-energy function

$$F(\eta,\mu)=-kT\ln[\mathrm{Tr}\exp-\beta(\hat{H}'-\eta\hat{J}-\mu\hat{P})] . \quad (4.1)$$

Here $\hat{H}'$ is given by Eq. (3.1), $\hat{J}$ by Eq. (3.7), and $\hat{P}$, the momentum operator, by[8]

$$\hat{P}=(\pi J/L)+\sum_q qb_q^+ b_q . \quad (4.2)$$

η and μ are Lagrange multipliers.

The free energy has two terms: the energy associated with a given value of the quantum number J in the absence of boson excitations and the free energy of the Bose system in the presence of a given J. At low temperatures T the Bose free energy may be obtained from the part of $\hat{H}'-\eta\hat{J}$ quadratic in Bose operators; the higher terms may renormalize coefficients in the quadratic part but first give a nontrivial contribution to the free energy at $O(T^6)$. At small J and low temperatures we find

$$F(J,\eta,\mu)=\tfrac{1}{2}v_J\left[\left(\frac{\pi J}{L}\right)^2-2(\eta+\mu)\left(\frac{\pi J}{L}\right)\right]+\alpha T^2\left(\frac{1}{v^+}+\frac{1}{v^-}\right) . \quad (4.3)$$

Here α is a number; to leading order in $1/2mv_F$, $\alpha=\pi/12$. $v^{\pm}=\lim_{q\to 0^{\pm}}\omega_q/q$ are the boson velocities for positive and negative q. Explicitly, to order J^2,

$$\frac{v^{\pm}}{v_s}=1+\tilde{\lambda}\varepsilon'^2\left[\left(\frac{\pi J}{L}\right)^2-2\eta\left(\frac{\pi J}{L}\right)\right]\pm 2\left(\frac{\pi J}{L}-(\eta+2m\mu)\right)\varepsilon' , \quad (4.4)$$

where v_s is the boson velocity for $J=0$ and to leading order in $1/(2v_F)$

$$\tilde{\lambda}=\lambda v_s/v_F , \quad (4.5a)$$

$$\varepsilon'=1/2mv_s . \quad (4.5b)$$

We now choose J to minimize Eq. (4.3); the result is

$$\frac{\pi J}{L}=(\eta+2m\mu) . \quad (4.6)$$

We then determine η from the constraint equation

$$j=-\frac{\partial}{\partial\eta}F(\eta,\mu=0) , \quad (4.7)$$

finding, in the small J limit,

$$\eta=\frac{1}{v_J}\left[1-\frac{4\alpha\tilde{\lambda}\varepsilon'^2T^2}{v_s^2}\right]j . \quad (4.8)$$

The physical content of (4.8) is that in the thermodynamic state of finite current some current is carried by the quantum number J (i.e., by a difference in the number of right-moving and left-moving carriers) and some current is carried by the boson excitations. The total momentum P of the system may be found by differentiating (4.3) with respect to μ and is

$$P=-\left.\frac{\partial F(J,\eta,\mu)}{\partial\mu}\right|_{\mu=0}=\frac{\pi J}{L} . \quad (4.9)$$

Thus although the distribution of boson excitations carries a current it carries no momentum.

The state of the system at long times is also given by minimizing an appropriate free energy. Current is not conserved by Eq. (3.1) (if $\phi\neq 0$) while momentum and the quantum number J are. Thus we minimize Eq. (4.3) at $\eta=0$ subject to the constraint that P and J have the $t=0$ values. We find

$$2m\mu=\frac{\pi J}{L}=\left[1-\frac{4\alpha\tilde{\lambda}\varepsilon'^2T^2}{v_s^2}\right] , \quad (4.10)$$

$$\langle\hat{J}\rangle=j . \quad (4.11)$$

Thus the current does not decay at all, in this approximation, because of the constraint of momentum conservation. We believe that (4.11) holds to arbitrary order in T in the model of Eq. (3.1) because the free energy at small J is a function of $[(\pi J/L)^2-2\eta\pi J/L]$ and $(\pi J/L-\eta-2m\mu)^2$ only.

We now argue that in the physical system, at incommensurate densities for which the Luttinger liquid model is the low-energy approximation, the conductivity will diverge faster than any power of T, indeed probably exponentially in $1/T$, at small T. This agrees with Ref. 1; however, we argue below in Eq. (7.1) that the agreement is spurious. In a physical system of fermions moving on a lattice "umklapp processes," in which the total momentum of the system changes by a reciprocal-lattice vector, are possible. In a weakly interacting one-dimensional Fermi system the leading such process is the second-order umklapp, in which, say, two right-moving fermions with wave vectors near $\pi/2$ are converted to two left-moving fermions with wave vectors near $-\pi/2$ in such a way that the total momentum of the system changes by

the reciprocal-lattice vector 2π. Similarly one may define an nth-order umklapp in which n right movers are converted to n left movers, and the total momentum is changed only by an integer multiple of the reciprocal-lattice vector. In the presence of umklapp scattering neither the Luttinger-liquid quantum number J nor the total momentum are conserved, so that one expects that in the long-time limit an initially imposed current would vanish.[11] We may estimate the decay time of a current from the probability of an umklapp event. If the band filling is not a rational fraction of π, the nth-order umklapp process will necessarily involve n excitations, each of which has an energy of order $v_F\delta k_n=\Delta_n$. Here δk_n is the magnitude of the difference between k_F and the nearest rational fraction $\pi m/n$. Thus at large n $\delta k_n\sim 1/n$. For temperatures $T<n\Delta_n$, the nth-order umklapp will be gapped, leading to a contribution to the resistivity $\sim A^{(n)}e^{-n\Delta_n/T}$. Here $A^{(n)}$ is the amplitude of the nth-order umklapp. Now at any temperature T there will exist some order n_T for which $\Delta_{n_T}<T$, so an n_T-order umklapp is thermally allowed, leading to $\rho\sim A^{(n)}T^{n_T}$ where the factors of T are from phase space. But because $\Delta_{n_T}\sim 1/n_T$ we have $n_T\sim 1/T$, so this corresponds to a contribution to $\rho\sim\exp-|\ln T|/T$ that is negligible compared to the lower-order umklapps. The lower-order umklapps clearly lead to $\sigma\sim\exp(\Delta/T)$, with the power m determined by the order of umklapp scattering with the smallest value of $m\Delta_m$.

V. SPINLESS FERMIONS—COMMENSURATE DENSITY

If the fermion density is such that the Fermi wave vector k_F is a rational fraction of a reciprocal-lattice vector $k_F=\pi m/n$, then umklapp processes (in which, say, n fermions are transferred from the right-moving to the left-moving branch) are not gapped and may make a nontrivial contribution to the low-energy properties of the model given in Eq. (2.1). If the interaction $\hat{V}$ in Eq. (2.3) is stronger than a critical value (which depends on n and m) it will cause the ground state to be insulating. If it is weaker than a critical value, the ground state will be of the Luttinger-liquid form discussed in the previous section; however, the umklapp processes will lead to the presence of an irrelevant operator which will not conserve momentum and may be of lower dimension than the irrelevant operators due to the band curvature considered in Sec. III. Thus in the commensurate case $\sigma(\omega=0,T)$ will not have the exponential increase as $1/T\to\infty$ found in the incommensurate case and $\sigma(\omega,T=0)\sim\omega^{\nu}$, where as we shall see ν may be a noninteger and less than 3. $\sigma(\omega,T=0)$ in the lowest-order commensurate case has been analyzed previously,[5] extending an earlier analysis[12] of the scaling of the charge stiffness near the metal-insulator transition. For completeness we rederive these results in the notation used here, and consider also higher-order commensurability. The umklapp operator may be written

$$H_U=g_4\sum_{p=\pm1}\sum_{kkq}{}'c_{kp}^{+}c_{k'p}^{+}c_{k-\pi+qp}c_{k'-\pi-qp}\ . \tag{5.1}$$

Here $p=\pm1$ labels the right- and left-moving branches of the electron dispersion and the prime on the sum denotes the restrictions

$$|k-pk_F|<R^{-1}\ , \tag{5.2a}$$

$$|k'-pk_F|<R^{-1}\ , \tag{5.2b}$$

$$|q|<\xi^{-1}\ . \tag{5.2c}$$

Here R is the short-distance cutoff of the Luttinger liquid; it is of order the lattice constant or k_F^{-1}, while $\xi>R$ is the cutoff of the umklapp operator. By following Ref. 8, this interaction may be written in the formalism of Sec. III. The result is

$$\hat{H}_U=g_3\left[\frac{R}{\xi}\right]^{2e^{2\phi}}\left[\frac{\xi}{L}\right]^{2(2e^{2\phi}-1)}\times\frac{1}{L}\int dx\{U_J^{+}\exp[\psi^{+}(x)]\exp[-\psi(x)]+\text{H.c.}\}\ . \tag{5.3}$$

Here ϕ is the Luttinger-liquid interaction parameter introduced in Sec. III, $\hat{U}_J^{+}$ is the raising operator for the current quantum number J, and

$$\psi^{+}(x)=2e^{\phi}\sum_{q\neq0}\left[\frac{2\pi}{L|q|}\right]^{1/2}\operatorname{sgn}(q)e^{iqx}b_q^{+}\ . \tag{5.4}$$

We may now proceed to calculate $\sigma(\omega,T=0)$ precisely as in Sec. III. We find that the leading contribution to $[H,\hat{J}]$ comes from commuting the current quantum number $\hat{J}$ with $\hat{H}_U$. Evaluation of Eq. (2.6) using this leading contribution gives

$$\sigma(\omega,T=0)=\frac{4\tilde{g}_3^2}{\omega^3}\left[\frac{\xi}{L}\right]^{4(2e^{2\phi}-1)}\frac{1}{L^2}\int dx\,dy\,dt\,e^{i\omega t}\langle\theta|[e^{\psi^{+}(x,t)}e^{-\psi(x,t)},e^{-\psi^{+}(x,0)}e^{\psi(y,0)}]|\theta\rangle\ . \tag{5.5}$$

Here $\tilde{g}_3=g_3(R/\xi)^{2e^{2\phi}}$. The scaling equation for g_3 in Ref. 5 is equivalent to the statement that $\tilde{g}_3$ is invariant under changes in ξ. By noting $\psi(x,t)|\theta\rangle=0$ and using standard methods for evaluating exponentials of boson operators we find that (5.5) becomes

$$\sigma(\omega,T=0)\sim\tilde{g}_3^2\omega^{8e^{2\phi}-5}\ . \tag{5.6}$$

Equation (5.6) expresses $\sigma(\omega,T=0)$ in terms of two parameters: the exponent ϕ which gives the decay of correlation functions in the fixed-point Hamiltonian, and $\tilde{g}_3$ which gives the strength of the leading irrelevant operator which contributes to $[\hat{H},\hat{J}]$. Equation (5.6) is equivalent to the result of Ref. 5 because the coupling constant K^* defined there is given by $K^*=2e^{2\phi}$. Refer-

ence 5 employs a perturbative renormalization-group treatment, valid for small initial g_3 and $e^{2\phi}$ near $\frac{1}{2}$, of the effect of H_U; results for the crossover between the high-frequency regime and the true low-frequency limit are also given. The Luttinger liquid at half-filling is stable against umklapp scattering if $e^{2\phi}>\frac{1}{2}$. The contribution of umklapp scattering to $\sigma(\omega)$ is more important than the band curvature terms considered in Sec. III if $e^{2\phi}<1$. These results may be extended to higher (nth)-order commensurability, where $k_F=\pi m/n$. In this case the umklapp operator involves transferring $2n$ fermions; this leads to a boson expression for $\hat{H}_U$ similar to (5.3) but with $2e^{2\phi}$ replaced by $\frac{1}{2}n^2e^{2\phi}$; the criterion for stability becomes $e^{2\phi}>2/n^2$ and the ω exponent $2n^2e^{2\phi}-5$; thus the nth-order umklapp scattering is less important than the band curvature if $n^2e^{2\phi}>4$. This criterion is satisfied in the $U<0$ Hubbard model at all commensurations except $n=1$. If $e^{2\phi}=2/n^2$, then the umklapp operator is marginal and logarithmic corrections appear; see Ref. 5 for details.

Because the umklapp operator does not conserve the quantum number J or the momentum, the arguments of Sec. IV imply that a state of finite current will decay[11] (since the minimum of the free energy in the absence of any constraints is a state of vanishing current). A straightforward golden-rule estimate of the rate of transitions from a state of one J to another yields,[5] for an nth-order commensurate density,

$$\sigma(T)\sim T^{3-2n^2e^{2\phi}} . \tag{5.7}$$

This is the temperature dependence which would be given by the memory function formalism, if $n^2e^{2\phi}<4$, but if $n^2e^{2\phi}>4$ the memory function formalism would incorrectly predict $\sigma\sim T^{-5}$.

VI. FERMIONS WITH SPIN

There are three cases. The simplest is the insulating ground state. This occurs, e.g., for the repulsive U Hubbard model at half-filling. In this case the conductivity vanishes. The next simplest is that a gap opens in the spin sector but not the charge sector, as occurs, e.g., for the attractive Hubbard model. In this case the low-energy physics are given by a spinless-fermion Luttinger liquid and the results we have obtained in the previous sections would apply. The third case is when neither the spin nor the charge sector is gapped. This would correspond, e.g., to the repulsive U Hubbard model away from half-filling and to the t-J model in regimes where phase separation is absent. In this case the fixed-point Hamiltonian corresponds to two Luttinger liquids: one for spin and one for charge. Further, in models with a spin rotation invariance (such as the Hubbard model in zero magnetic field) the spin Luttinger liquid fixed point possesses an irrelevant operator corresponding to second-order umklapp scattering, while in models without spin rotation invariance (such as the Hubbard model in a magnetic field) it does not. The bosonized Hamiltonian thus involves separate bosonic fields and quantum numbers J for spin and charge excitations. Band curvature leads to cubic and quartic terms coupling spin bosonic excitations to the charged ones. This coupling implies that the momentum of the charge bosons is not conserved. Except at commensurate fillings the charge current quantum number is conserved. In zero magnetic field at $T\neq 0$ the spin current quantum number is not conserved. To order J^2 the spin and charge quantum numbers are not coupled.

From these considerations it is easy to see how the results of the previous sections can be adapted to the case of fermions with spin. By following the arguments of Sec. III we obtain an extra contribution to the current operator $\hat{J}$ involving spin bosonic operators. These lead to an extra contribution to $[H,\hat{J}]$ and so to extra absorption. Parametrizing the Luttinger-liquid interaction among the spin bosons by the angle ϕ_σ we find that $\sigma(\omega)$ is given by Eq. (3.9) but with "$\sinh^2 2\phi$" replaced by a more complicated expression involving both ϕ and ϕ_σ and the ratio of the velocities of density and spin excitations. This expression vanishes when ϕ and $\phi_\sigma\rightarrow 0$. In particular the frequency dependence is still ω^3. The $\sigma(T)$ estimate of Sec. IV is not significantly changed. The current quantum number J is still conserved so $\sigma(T)$ is still infinite in the absence of umklapp scattering in the charge sector. However, because of spin umklapp scattering (in zero applied field) the momentum in the bosons is no longer conserved, so to obtain the long-time limit of the current we should minimize the free energy of Eq. (4.1) subject only to the constraint of finite J. Thus in contrast to the simple case considered in Sec. IV the current will decay, albeit to a finite value and not to zero.

VII. CONCLUSION

In conclusion, we comment briefly on some implications of our results. The one-dimensional calculation provides a concrete illustration of the general principle that in systems of interacting fermions on a lattice the current operator is not identical to the momentum operator and does not commute with the Hamiltonian, so that in general one expects the current operator to have nonvanishing matrix elements connecting the ground state to other low-lying states. Of course exceptions may arise, as for example in the case of an s-wave superconductor. It is not necessary to have umklapp scattering to have a finite $\sigma(\omega)$ at low frequencies. Of course if umklapp scattering is present it will contribute to $\sigma(\omega)$,[5] but unless the umklapp scattering is stronger than the critical value obtained in Sec. VI it will not give rise to a smaller power of ω than calculated here.

Our results for $\sigma(\omega=0,T)$ suggest that the commonly used "memory function" approximation is dangerous. In this approach one writes

$$\sigma(\omega,T)\sim\frac{1}{i\omega+M(\omega,T)} , \tag{7.1}$$

and *defines* the memory function $M(\omega,T)$ as ω^2 times the right-hand side of Eq. (2.6). Using this definition as $\omega\rightarrow 0$ for $T\neq 0$ corresponds to making a selective resummation of the (formally divergent) perturbation expansion in $\hat{V}$ for σ. The manipulations leading to Eqs. (2.9) or (3.9) may be performed at finite temperature, yielding $M(\omega=0,T)\sim T^5$. Thus the memory function method would incorrectly predict that even for incommensurate densities $\sigma_{dc}\sim T^{-5}$. (In Ref. 1 the memory function was

used and an exponentially divergent $\sigma(T)$ was found [as was a gap in $\sigma(\omega)$] because the band curvature terms were neglected. Had they been correctly included a correct power-law behavior for $\sigma(\omega)$ and an incorrect power-law behavior for $\sigma(T)$ would have been found.) Our results suggest that the difficulty is that the memory function formalism does not respect the conservation of J and of boson momentum which enforced the infinite dc conductivity of the Luttinger liquid.

Our arguments may be extended to higher dimensions. We have already mentioned that the perturbation theory argument gives at small ω $\sigma(\omega, T=0) \sim \ln\omega$ (in 2D) and $\sigma(\omega, T=0)$const (in 3D). The perturbation theory argument will give the correct result whenever the ground state is Fermi-liquid-like.[7] As in one dimension, the dc resistivity at finite temperature is different: a state of finite current is a state of finite momentum, which can only decay via umklapp processes. In high dimensions these are generally not activated, and will lead to a temperature dependence determined by the band filling. Within a weak-coupling model, one may argue[7,10,11] that if the filling is such that $2k_F$ is a reciprocal-lattice vector then a two-particle excitation may lead to momentum decay, so $\rho \sim T$;[2] while if $2k_F$ is less than a reciprocal-lattice vector but $3k_F$ is greater, then a three-particle excitation is required and $\rho \sim T^3$, etc.

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