Higher conservation laws and algebraic Bethe *Ansätze* for the supersymmetric *t-J* model

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(Received 11 May 1992)

We construct the enveloping fundamental spin model of the *t-J* Hamiltonian using the quantum-inverse-scattering method (QISM), and present all three possible algebraic Bethe *Ansätze*. Two of the solutions have been previously obtained in the framework of the coordinate-space Bethe *Ansatz* by Sutherland and by Schlottmann and Lai, whereas the third solution is new. The formulation of the model in terms of the QISM enables us to derive explicit expressions for higher conservation laws.

I. INTRODUCTION

Strongly correlated electronic systems are currently intensely studied in relation with high-T_c superconductivity. Recently there has been a renewed interest in the one-dimensional *t-J* model as an integrable low-dimensional version of a strongly correlated electronic system. The *t-J* model was proposed by Zhang and Rice.[1] Anderson claimed that two-dimensional systems may share features of one-dimensional systems,[2] which could imply a certain relevance of some results obtained for the one-dimensional *t-J* model to high-T_c superconductivity.

The model describes electrons on a one-dimensional lattice with a Hamiltonian that includes nearest-neighbor hopping (t) and nearest-neighbor spin exchange and charge interactions (J). The Hilbert space of the model is constrained to exclude double occupancy of any single site, which corresponds to an infinite on-site repulsion.

Electrons on a lattice are described by operators $c_{j,\sigma}$, $j=1,\ldots,L$, $\sigma=\pm1$, where L is the total number of lattice sites. These are canonical Fermi operators with anticommutation relations given by $\{c^\dagger_{i,\sigma},c_{j,\tau}\}=\delta_{i,j}\delta_{\sigma,\tau}$. The state $|0\rangle$ (the Fock vacuum) satisfies $c_{i,\sigma}|0\rangle=0$. Due to the constraint excluding double occupancy, there are three possible electronic states at a given lattice site i:

$$|0\rangle,\quad |\uparrow\rangle_i=c^\dagger_{i,1}|0\rangle,\quad |\downarrow\rangle_i=c^\dagger_{i,-1}|0\rangle\ . \tag{1.1}$$

By $n_{i,\sigma}=c^\dagger_{i,\sigma}c_{i,\sigma}$ we denote the number operator for electrons with spin σ on site i and we write $n_i=n_{i,1}+n_{i,-1}$. The spin operators $S=\sum_{j=1}^L S_j$, $S^\dagger=\sum_{j=1}^L S^\dagger_j$, $S^z=\sum_{j=1}^L S^z_j$,

$$\begin{aligned} S_j&=c^\dagger_{j,1}c_{j,-1},\\ S^\dagger_j&=c^\dagger_{j,-1}c_{j,1}\ ,\\ S^z_j&=\tfrac{1}{2}(n_{j,1}-n_{j,-1}) \end{aligned} \tag{1.2}$$

form an su(2) algebra and they commute with the Hamiltonian that we consider below. (We shall always give local expressions $\mathcal{O}_j$ for symmetry generators, implying that the global ones are obtained as $\mathcal{O}=\sum_{j=1}^L\mathcal{O}_j$). The Hamiltonian on a lattice of L sites is given by the following expression:

$$\begin{aligned} H=&-t\sum_{j=1}^{L}\sum_{\sigma=\pm1}c^\dagger_{j,\sigma}(1-n_{j,-\sigma})c_{j+1,\sigma}(1-n_{j+1,-\sigma})\\ &+c^\dagger_{j+1,\sigma}(1-n_{j+1,-\sigma})c_{j,\sigma}(1-n_{j,-\sigma})\\ &+J\sum_{j=1}^{L}[S^z_jS^z_{j+1}+\tfrac{1}{2}(S^\dagger_jS_{j+1}+S_jS^\dagger_{j+1})-\tfrac{1}{4}n_jn_{j+1}]\ . \end{aligned} \tag{1.3}$$

The projectors $(1-n_{j,-\sigma})$ in the kinetic-energy terms ensure that H acts within the constrained Hilbert space, i.e., does not create states with double occupancy. An equivalent expression for H is

$$\begin{aligned} H=\sum_{j=1}^{L}\Big\{&-t\mathcal{P}\sum_{\sigma=\pm1}(c^\dagger_{j,\sigma}c_{j+1,\sigma}+\text{H.c.})\mathcal{P}\\ &+J(\mathbf{S}_j\cdot\mathbf{S}_{j+1}-\tfrac{1}{4}n_jn_{j+1})\Big\}\ , \end{aligned} \tag{1.4}$$

where $\mathcal{P}$ is the projector on the subspace of nondoubly occupied states. It was shown in Ref. 3 that the model described by (1.3) is integrable and can be solved by coordinate space Bethe *Ansatz*. In addition, it is possible to map it onto the integrable quantum lattice gas of Lai and Sutherland.[4,5] The thermodynamics of the model were treated in Refs. 3 and 6 and the ground-state properties and excitation spectrum were investigated in Refs. 7–9.

As the number operator for electrons $\hat{N}=\sum_{j=1}^L\sum_{\sigma=\pm1}n_{j,\sigma}$ commutes with H, we can add a term $2\hat{N}-L$ to the Hamiltonian without changing the set of eigenvectors. Physically this just amounts to a shift of the chemical potential. For the special value $J=2t=2$, the shifted Hamiltonian now exhibits a number of interesting properties. (The case $-J=2t$ can be obtained from $J=2t$ via the transformation $c_j\to(-1)^jc_j$, $c^\dagger_j\to(-1)^jc^\dagger_j$.)

First, it is supersymmetric, i.e., it commutes with all nine generators of the superalgebra u(1|2).[6,7] Second, it can be written as a graded permutation operator:[6,7]

$$H_{\text{susy}}=H+2\hat{N}-L=-\sum_{j=1}^{L}\Pi^{j,j+1}$$
$$=\sum_{j=1}^{L}H_{\text{susy}}^{j,j+1}\ . \tag{1.5}$$

The operator $\Pi^{j,j+1}$ permutes the three possible configurations (1.1) between the sites j and $j+1$, picking up a minus sign if both of the permuted configurations are fermionic, i.e.,

$$\Pi^{j,j+1}|0\rangle_j\times|0\rangle_{j+1}=|0\rangle_j\times|0\rangle_{j+1}\ ,$$
$$\Pi^{j,j+1}|0\rangle_j\times|\sigma\rangle_{j+1}=|\sigma\rangle_j\times|0\rangle_{j+1}\ , \tag{1.6}$$
$$\Pi^{j,j+1}|\tau\rangle_j\times|\sigma\rangle_{j+1}=-|\sigma\rangle_j\times|\tau\rangle_{j+1},\quad \sigma,\tau=\uparrow,\downarrow\ .$$

It is clear that this form of interaction conserves the individual numbers $N_\uparrow$ and $N_\downarrow$ of electrons with spin up and spin down, and due to the constraint of no double occupancy the number N_h of empty sites (or "holes") is also conserved.

The outline of the paper is as follows: In Sec. II, we give a discussion of the supersymmetry of the model and express the Hamiltonian (1.5) in a way most suitable for the analysis of Sec. V. In Sec. III, we perform a detailed construction of the algebraic Bethe *Ansatz* of the model. We derive three different forms for the Bethe Ansatz equations (BAE) and the eigenvalues of the transfer matrix. Two of these solutions have previously been obtained by various authors,[3–5,10] whereas our third solution, presented in Sec. IV, is new. Our expression for the BAE seems to be particularly simple and we hope that it will be useful in clarifying the physical features of the model. The direct physical consequences of our solution (like the structure of the ground states and classification of excitations and correlations) are currently under investigation[11] and will be presented in a future publication. The graded quantum-inverse-scattering method (QISM), discussed in Sec. III, enables us to obtain expressions for (an infinite number of) higher conservation laws at the quantum level. These conserved charges are of interest because physical interactions, although of short range, are not generally well approximated by interactions involving only nearest neighbors. The charges under consideration involve interactions of longer range (next-nearest neighbors, next-next-nearest, etc.) and can be added to the Hamiltonian without spoiling the integrability of the model. Thus, it is possible to construct integrable models with longer-range interactions by using higher conservation laws.[12] The first nontrivial higher integral of motion is, for example, given by the expression

$$H_{(3)}=i\sum_{k=1}^{L}[H_{\text{susy}}^{k,k+1},H_{\text{susy}}^{k-1,k}]\ , \tag{1.7}$$

where $H^{k,k+1}$ is the density of the Hamiltonian defined in (1.5). Section V is devoted to the derivation of explicit formulas for higher conservation laws.

II. SUPERSYMMETRY OF THE *t-J* MODEL

For $J=2t$, the *t-J* model exhibits a (global) u(1|2) invariance on the constrained Hilbert space. In the literature this symmetry algebra has frequently been denoted by $spl(2,1)$.[7] The relation between $spl(2,1)$ and su(1|2) [neglecting the trivial u(1) factor for the time being] is as follows.

The algebra su(1|2) is a real form of the complex Lie superalgebra $sl(1,2;\mathbb{C})$, whereas $spl(2,1)$ is an equivalent notation for $sl(1,2;\mathbb{C})$. For our purposes it is more convenient to work with a real base field, so that we will work in a representation of u(1|2). The generators of the u(1|2) algebra are given by[13] (we write $O=\sum_{j=1}^{L}O_j$)

$$J_{j,1}=S_j=c_{j,1}^\dagger c_{j,-1}$$
$$J_{j,2}=S_j^\dagger=c_{j,-1}^\dagger c_{j,1}\ ,$$
$$J_{j,3}=S_j^z=\tfrac{1}{2}(n_{j,1}-n_{j,-1})\ ,$$
$$J_{j,4}=Q_{j,1}=(1-n_{j,-1})c_{j,1}\ ,$$
$$J_{j,5}=Q_{j,1}^\dagger=(1-n_{j,-1})c_{j,1}^\dagger\ , \tag{2.1}$$
$$J_{j,6}=Q_{j,-1}=(1-n_{j,1})c_{j,-1}\ ,$$
$$J_{j,7}=Q_{j,-1}^\dagger=(1-n_{j,1})c_{j,-1}^\dagger\ ,$$
$$J_{j,8}=T_j=1-\tfrac{1}{2}n_j\ ,$$
$$J_{j,9}=I_j=1\ .$$

The operators S, $S^\dagger$, S^z, Q_1, $Q_1^\dagger$, Q_{-1}, $Q_{-1}^\dagger$, and T generate the su(1|2) subalgebra of u(1|2).

In the fundamental representation there exists an invariant, nondegenerate bilinear form $K_{\alpha\beta}$ on u(1|2), which is given as the supertrace over two generators[14]

$$K_{\alpha\beta}=(K^{\alpha\beta})^{-1}=\text{str}(J_{j,\alpha}J_{j,\beta}) \tag{2.2}$$

and which is explicitly computed in Appendix A. Note that the nondegeneracy of $K_{\alpha\beta}$ is a feature of the fundamental representation and does not generally hold for other representations because u(1|2) is not semisimple. For later use we define the structure constants of u(1|2) as

$$[J_{k,\alpha},J_{k,\beta}\}:=J_{k,\alpha}J_{k,\beta}-(-1)^{\epsilon_\alpha\epsilon_\beta}J_{k,\beta}J_{k,\alpha}$$
$$=f_{\alpha\beta}{}^\gamma J_{k,\gamma}\ , \tag{2.3}$$

where ϵ_α are the Grassmann parities of the generators $J_{k,\alpha}$ (i.e., $\epsilon=1$ for the fermionic generators $J_4,\ldots,J_7$ and $\epsilon=0$ for the rest).

The Hamiltonian of the *t-J* model on the constrained Hilbert space can now be expressed in terms of the densities $J_{k,\alpha}$ as

$$H_{\text{susy}}=\sum_{j=1}^{L} H_{\text{susy}}^{j,j+1}=-\sum_{j=1}^{L} K^{\alpha\beta}J_{j,\alpha}J_{j+1,\beta}$$
$$=-\sum_{j=1}^{L}\sum_{\sigma=\pm 1} Q_{j,\sigma}^{\dagger}Q_{j+1,\sigma}+Q_{j+1,\sigma}^{\dagger}Q_{j,\sigma}-\sum_{j=1}^{L}-2S_j^zS_{j+1}^z-S_j^{\dagger}S_{j+1}-S_jS_{j+1}^{\dagger}+2T_jT_{j+1}-I_jI_{j+1}\ , \tag{2.4}$$

$K_{\alpha\beta}$ given in Appendix A. In this form the global u(1|2) invariance of the Hamiltonian is easily confirmed

$$[H,J_\alpha]=0\ ,\quad \alpha=1,\ldots,9\ . \tag{2.5}$$

The "manifestly supersymmetric" expression for the Hamiltonian (2.4) in terms of the Killing form will enable us to express the higher conservation laws we derive in Sec. IV, in a u(1|2) invariant way.

III. GRADED QUANTUM-INVERSE-SCATTERING METHOD

In this section we construct the enveloping spin model of the Hamiltonian of the one-dimensional supersymmetric *t-J* model, using the quantum-inverse-scattering method. Due to the fact that we are dealing with a supersymmetric theory, it is necessary to modify the QISM along the lines discussed in (Refs. 10 and 15). Below we give a summary of the "graded" version of the QISM. We start with an R matrix, obeying a graded Yang-Baxter equation, and construct from it a "fundamental" spin model (i.e., the L operator is constructed directly from the R matrix). We then show that the trace identities of the corresponding transfer matrix give rise to the Hamiltonian of the *t-J* model. Finally, we construct a set of simultaneous eigenstates of the transfer matrix and the Hamiltonian, using a nested algebraic Bethe *Ansatz* (NABA).[10,19–21] (For a general introduction to the algebraic Bethe *Ansatz*, we refer to Refs. 16–18.) Due to the grading there exist three choices for the R matrix, all of them describing the same physical system, but leading to different (yet equivalent as shown in Appendix C) forms of the NABA. Two of these possibilities of performing a Bethe *Ansatz* analysis are equivalent to the coordinate space Bethe *Ansatz* solutions of Lai and Sutherland. We reproduce their respective periodic boundary conditions and expressions for energy eigenvalues of the Hamiltonian.

A. Yang-Baxter equation

Consider the graded linear space $V^{(m|n)}=V^m\oplus V^n$, where m and n denote the dimensions of the "even" (V^m) and "odd" (V^n) parts, and $\oplus$ denotes the direct sum. Let $\{e_1,\ldots,e_{m+n}\}$ be a basis of $V^{(m+n)}$, such that $\{e_1,\ldots,e_m\}$ is a basis of V^m and $\{e_{m+1},\ldots,e_{m+n}\}$ is a basis of V^n. The Grassmann parities of the basis vectors are given by $\epsilon_1=\cdots=\epsilon_m=0$ and $\epsilon_{m+1}=\cdots=\epsilon_{m+n}=1$. Linear operators on $V^{(m|n)}$ can be represented in block form [$M\in\text{End}(V^{(m|n)})$]:

$$M=\begin{bmatrix}A & B\\ C & D\end{bmatrix},\quad \epsilon\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}=0,\quad \epsilon\begin{bmatrix}0 & B\\ C & 0\end{bmatrix}=1\ . \tag{3.1}$$

The supertrace is defined as

$$\text{str}(M)=\text{tr}(A)-\text{tr}(D)\ , \tag{3.2}$$

where the traces on the rhs are the usual operator traces in V^m and V^n. We now define the graded tensor product space $V^{(m|n)}\otimes V^{(m|n)}$ in terms of its basis vectors $\{e_a\otimes e_b\,|\,a,b=1,\ldots,m+n\}$ as follows:

$$v\otimes w=(e_av_a)\otimes(e_bw_b)$$
$$=(e_a\otimes e_b)v_aw_b(-1)^{\epsilon_{v_a}\epsilon_b}\ . \tag{3.3}$$

Compared to the "ordinary" tensor product, the additional factor $(-1)^{\epsilon_{v_a}\epsilon_b}$ occurs, which originates in passing v_a past e_b. The action of the right linear operator $F\otimes G$ on the vector $v\otimes w$ in $V^{(m|n)}\otimes V^{(m|n)}$ is given by

$$(F\otimes G)(v\otimes w)=F(v)\otimes G(w)\ . \tag{3.4}$$

Therefore, its matrix elements are of the form

$$(F\otimes G)_{cd}^{ab}=F_{ab}G_{cd}(-1)^{\epsilon_c(\epsilon_a+\epsilon_b)}\ . \tag{3.5}$$

The identity operator in $V^{(m|n)}\otimes V^{(m|n)}$ is given by $I_{a_2b_2}^{a_1b_1}=\delta_{a_1b_1}\delta_{a_2b_2}$ and the matrix Π that permutes the individual linear spaces in the tensor product space, is of the form

$$\Pi(v\otimes w)=(w\otimes v)\ ,$$
$$(\Pi)_{a_2b_2}^{a_1b_1}=\delta_{a_1b_2}\delta_{a_2b_1}(-1)^{\epsilon_{b_1}\epsilon_{b_2}}\ . \tag{3.6}$$

The physical relevance of the above construction is as follows: If we consider a lattice gas of m species of bosons and n species of fermions, then $V^{(m|n)}$ denotes the space of configurations at every site of the lattice. For the example of the *t-J* model we have $m=1$, $n=2$, and the three allowed configurations are given by (1.1). The tensor product space $V^{(1|2)}\otimes V^{(1|2)}$ describes two neighboring sites and, owing to the fermionic nature of some of the configurations, the tensor product has to carry a grading. Π permutes configurations on neighboring sites, and we pick up a minus sign if we permute two fermions.

A matrix $R(\lambda)$ (depending on a spectral parameter λ) is said to fulfill a graded Yang-Baxter equation if the following identity on $V^{(m|n)}\otimes V^{(m|n)}\otimes V^{(m|n)}$ holds:

$$[I\otimes R(\lambda-\mu)][R(\lambda)\otimes I][I\otimes R(\mu)]$$
$$=[R(\mu)\otimes I][I\otimes R(\lambda)][R(\lambda-\mu)\otimes I]\ . \tag{3.7}$$

In components this identity reads

$$R(\lambda-\mu)^{a_2c_2}_{a_3c_3}R(\lambda)^{a_1b_1}_{c_2d_2}R(\mu)^{d_2b_2}_{c_3b_3}$$
$$=R(\mu)^{a_1c_1}_{a_2c_2}R(\lambda)^{c_2d_2}_{a_3b_3}R(\lambda-\mu)^{c_1b_1}_{d_2b_2}\ . \quad (3.8)$$

Note that, despite the fact that the tensor product in (3.7) carries a grading, there are no extra signs in (3.8) compared to the nongraded case. It is easily checked that the R matrix

$$R(\lambda)=b(\lambda)I+a(\lambda)\Pi\ ,$$
$$a(\lambda)=\frac{\lambda}{\lambda+i},\quad b(\lambda)=\frac{i}{\lambda+i} \quad (3.9)$$

fulfills Eq. (3.8).

B. Construction of the transfer matrix

By multiplying (3.8) by $\Pi^{f_3e_3}_{f_1a_1}\Pi^{e_2a_2}_{e_3a_3}$ from the left, one can derive the equation

$$R(\lambda-\mu)^{f_1c_1}_{e_2c_2}[\Pi R(\lambda)]^{c_1b_1}_{f_3c_3}[\Pi R(\mu)]^{c_2b_2}_{c_3b_3}(-1)^{\epsilon_{c_2}(\epsilon_{c_1}+\epsilon_{b_1})}=[\Pi R(\mu)]^{f_1c_1}_{f_3c_3}[\Pi R(\lambda)]^{e_2c_2}_{c_3b_3}R(\lambda-\mu)^{c_1b_1}_{c_2b_2}(-1)^{\epsilon_{e_2}(\epsilon_{f_1}+\epsilon_{c_1})}\ . \quad (3.10)$$

In matrix notation (3.10) reads

$$R_{12}(\lambda-\mu)\{[\Pi_{13}R_{13}(\lambda)]\otimes[\Pi_{23}R_{23}(\mu)]\}$$
$$=\{[\Pi_{13}R_{13}(\mu)]\otimes[\Pi_{23}R_{23}(\lambda)]\}R_{12}(\lambda-\mu)\ . \quad (3.11)$$

where the indices 1,2,3 indicate in which of the spaces $V^{(m|n)}$ in the tensor product space $V^{(m|n)}\otimes V^{(m|n)}\otimes V^{(m|n)}$ the matrices act nontrivially. The tensor product in (3.11) is between spaces 1 and 2. We now call the third space "quantum space" and the first two spaces "matrix spaces." The physical interpretation of the quantum space is as the Hilbert space over a single site of a one-dimensional lattice. We now consider the situation where intertwining relations of the type (3.11) hold for all sites of a lattice of length L. The quantum space index "3" now gets replaced by an index labeling the number of the site. We define the L operator (on site n) as a linear operator on $\mathcal{H}_n\otimes V^{(m|n)}_{\text{matrix}}$ (where $\mathcal{H}_n\simeq V^{(m|n)}$ is the Hilbert space over the nth site, and $V^{(m|n)}_{\text{matrix}}$ is a matrix space):

$$L_n(\lambda)^{ab}_{\alpha\beta}=\Pi^{ac}_{\alpha\gamma}R(\lambda)^{cb}_{\gamma\beta}=[b(\lambda)\Pi+a(\lambda)I]^{ab}_{\alpha\beta}\ . \quad (3.12)$$

L_n is a quantum operator valued $(m+n)\times(m+n)$ matrix, with quantum operators acting nontrivially in the nth quantum space (of the direct product Hilbert space over the complete lattice $\otimes_{j=1}^{L}\mathcal{H}_j$). The Greek indices are the "quantum indices" and the Roman indices are the "matrix indices." Equation (3.11) for the nth quantum space can now be rewritten as the operator equation

$$R(\lambda-\mu)^{a_1c_1}_{a_2c_2}L_n(\lambda)^{c_1b_1}_{\alpha_n\gamma_n}L_n(\mu)^{c_2b_2}_{\gamma_n\beta_n}(-1)^{\epsilon_{c_2}(\epsilon_{c_1}+\epsilon_{b_1})}=L_n(\mu)^{a_1c_1}_{\alpha_n\gamma_n}L_n(\lambda)^{a_2c_2}_{\gamma_n\beta_n}(-1)^{\epsilon_{a_2}(\epsilon_{a_1}+\epsilon_{c_1})}R(\lambda-\mu)^{c_1b_1}_{c_2b_2}\ . \quad (3.13)$$

In matrix notation (3.13) takes the form

$$R(\lambda-\mu)[L_n(\lambda)\otimes L_n(\mu)]=[L_n(\mu)\otimes L_n(\lambda)]R(\lambda-\mu)\ . \quad (3.14)$$

Here the graded tensor product is between the two matrix spaces and R only acts in the matrix spaces. The intertwining relation (3.14) enables us to construct an integrable spin model as follows.

We first define the monodromy matrix $T_L(\lambda)$ as the matrix product over the L operators on all sites of the lattice, i.e.,

$$T_L(\lambda)=L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda)\ ,$$
$$\{[T_L(\lambda)]^{ab}\}_{\substack{\alpha_1\cdots\alpha_L\\ \beta_1\cdots\beta_L}}=L_L(\lambda)^{ac_L}_{\alpha_L\beta_L}L_{L-1}(\lambda)^{c_Lc_{L-1}}_{\alpha_{L-1}\beta_{L-1}}\cdots L_1(\lambda)^{c_2b}_{\alpha_1\beta_1}(-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}\ . \quad (3.15)$$

$T_L(\lambda)$ is a quantum operator valued $(m+n)\times(m+n)$ matrix that acts nontrivially in the graded tensor product of all quantum spaces of the lattice and by construction fulfills the same intertwining relation as the L operators (as can be proven by induction over the length of the lattice):

$$R(\lambda-\mu)[T_L(\lambda)\otimes T_L(\mu)]$$
$$=[T_L(\mu)\otimes T_L(\lambda)]R(\lambda-\mu)\ . \quad (3.16)$$

The transfer matrix $\tau(\lambda)$ of the integrable spin model is now given as the matrix supertrace of the monodromy matrix

$$\tau(\lambda)=\text{str}[T_L(\lambda)]=\sum_{a=1}^{m+n}(-1)^{\epsilon_a}[T_L(\lambda)]^{aa}\ . \quad (3.17)$$

As a consequence of (3.16), transfer matrices with different spectral parameters commute. This condition implies that the transfer matrix is the generating functional of the Hamiltonian and of an infinite number of "higher" conservation laws of the model.

C. Trace identities

Taking logarithmic derivatives of the transfer matrix at a special value of the spectral parameter, one can generate higher conservation laws.[16] For our specific case at hand, i.e., the R matrix (3.9), the corresponding Hamiltonian is obtained by taking the first logarithmic derivative at zero spectral parameter

$$H_{(2)}=-i\frac{\partial \ln[\tau(\lambda)]}{\partial\lambda}\bigg|_{\lambda=0}=-\sum_{k=1}^{L}(\Pi^{k,k+1}-1)\ . \tag{3.18}$$

The proof of this identity can be carried out in the same way as for the ungraded case, the main difference being the grading of the tensor product of the quantum spaces [see (3.15)]. By shifting the energy eigenvalues by a constant, we obtain the expression (1.5) for the Hamiltonian of the t-J model

$$H_{\text{susy}}=-i\frac{\partial \ln[\tau(\lambda)]}{\partial\lambda}\bigg|_{\lambda=0}-L=H_{(2)}-L\ , \tag{3.19}$$

if we choose our underlying graded vector space to have signature (1,2) i.e., to have a basis with two fermionic and one bosonic state. This shows that the transfer matrix constructed from the L operator (3.12) and R matrix (3.9) is indeed the correct transfer matrix for the one-dimensional supersymmetric t-J model. Higher conservation laws are obtained as the coefficients of the power series

$$\ln\{\tau(\lambda)[\tau(0)]^{-1}\}=\sum_{k=1}^{\infty} i\frac{\lambda^k}{k!}H_{(k+1)}\ . \tag{3.20}$$

There exists, however, a simpler method for the construction of higher integrals of motion than taking logarithmic derivatives, which we will discuss in Sec. V.

D. Algebraic Bethe *Ansatz* with a bosonic background (*FFB* grading) Lai solution

Due to the constraint of no double occupancy, there are three different configurations per site for the t-J model. Thus, the Hilbert space at the kth site of the lattice is isomorphic to $\mathbb{C}^3$ and is spanned by the three vectors $e_1=(1\,0\,0)^T$, $e_2=(0\,1\,0)^T$, and $e_3=(0\,0\,1)^T$. In this section we consider a grading such that e_1 and e_2 are fermionic (representing spin-down and spin-up electrons, respectively) and e_3 is bosonic (empty site). In terms of the Grassmann parities this means that $\epsilon_1=\epsilon_2=1$ and $\epsilon_3=0$. We pick the reference state in the kth quantum space $|0\rangle_k$ and the vacuum $|0\rangle$ of the complete lattice of L sites to be purely bosonic, i.e.,

$$|0\rangle_n=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad |0\rangle=\otimes_{n=1}^{L}|0\rangle_n\ . \tag{3.21}$$

This choice of grading implies that $R(\mu)=b(\mu)I+a(\mu)\Pi$ is given by the following expression

$$R(\lambda)=\begin{bmatrix}
b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b(\lambda) & 0 & -a(\lambda) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b(\lambda) & 0 & 0 & 0 & a(\lambda) & 0 & 0\\
0 & -a(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & a(\lambda) & 0\\
0 & 0 & a(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & b(\lambda) & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{3.22}$$

The L operator is defined by (3.12) and is of the form

$$L_n(\lambda)=\begin{bmatrix}
a(\lambda)-b(\lambda)e_n^{11} & -b(\lambda)e_n^{21} & b(\lambda)e_n^{31}\\
-b(\lambda)e_n^{12} & a(\lambda)-b(\lambda)e_n^{22} & b(\lambda)e_n^{32}\\
b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & a(\lambda)+b(\lambda)e_n^{33}
\end{bmatrix}, \tag{3.23}$$

where e_n^{ab} are quantum operators in the nth quantum space with matrix representation $(e_n^{ab})_{\alpha\beta}=\delta_{a\alpha}\delta_{b\beta}$. The monodromy matrix (3.15) is a quantum operator valued 3×3 matrix, which we represent as

$$T_L(\lambda)=L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda)=\begin{bmatrix}
A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda)\\
A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda)\\
C_1(\lambda) & C_2(\lambda) & D(\lambda)
\end{bmatrix}. \tag{3.24}$$

The transfer matrix is then given as

$$\tau(\mu)=\text{str}[T_L(\mu)]=-A_{11}(\mu)-A_{22}(\mu)+D(\mu)\ . \tag{3.25}$$

The action of $L_k(\lambda)$ on the reference state on the kth site is

$$L_k(\lambda)|0\rangle_k=\begin{bmatrix}
a(\lambda) & 0 & 0\\
0 & a(\lambda) & 0\\
b(\lambda)e_k^{13} & b(\lambda)e_k^{23} & 1
\end{bmatrix}|0\rangle_k\ . \tag{3.26}$$

Using (3.15) and (3.26), we determine the action of the monodromy matrix on the reference state to be

$$T_L(\lambda)|0\rangle = \begin{bmatrix} [a(\lambda)]^L & 0 & 0 \\ 0 & [a(\lambda)]^L & 0 \\ C_1(\lambda) & C_2(\lambda) & 1 \end{bmatrix} |0\rangle \ . \tag{3.27}$$

We will now construct a set of eigenstates of the transfer matrix using the technique of the NABA. Inspection of (3.27) reveals that $C_1(\lambda)$ and $C_2(\lambda)$ are creation operators (of odd Grassmann parity) with respect to our choice of reference state. This observation leads us to the following *Ansatz* for the eigenstates of $\tau(\mu)$:

$$|\lambda_1,\ldots,\lambda_n|F\rangle = C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\cdots C_{a_n}(\lambda_n)|0\rangle F^{a_n\cdots a_1} \ , \tag{3.28}$$

where the indices a_j run over the values 1,2, and $F^{a_n\cdots a_1}$ is a function of the spectral parameters λ_j. The action of the transfer matrix on states of the form (3.28) is determined by (3.27) and the intertwining relations (3.16). The components of the intertwining relations (3.16) needed for the construction of the NABA are

$$\begin{aligned} A_{ab}(\mu)C_c(\lambda) &= (-1)^{\epsilon_a\epsilon_p}\frac{r(\mu-\lambda)^{dc}_{pb}}{a(\mu-\lambda)}C_p(\lambda)A_{ad}(\mu) \\ &\quad + \frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_b(\mu)A_{ac}(\lambda) \ , \\ D(\mu)C_c(\lambda) &= \frac{1}{a(\lambda-\mu)}C_c(\lambda)D(\mu) \\ &\quad - \frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_c(\mu)D(\lambda) \ , \\ C_{a_1}(\lambda_1)C_{a_2}(\lambda_2) &= r(\lambda_1-\lambda_2)^{b_1a_2}_{b_2a_1}C_{b_2}(\lambda_2)C_{b_1}(\lambda_1) \ , \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} r(\mu)^{ab}_{cd} &= b(\mu)\delta_{ab}\delta_{cd} - a(\mu)\delta_{ad}\delta_{bc} \\ &= b(\mu)I^{ab}_{cd} + a(\mu)[\Pi^{(1)}]^{ab}_{cd} \ . \end{aligned} \tag{3.30}$$

Here $[\Pi^{(1)}]^{ab}_{cd} = -\delta_{ad}\delta_{bc}$ is the 4×4 permutation matrix corresponding to the grading $\epsilon_1=\epsilon_2=1$. $r(\mu)$ can be seen to fulfill a (graded) Yang-Baxter equation on its own

$$\begin{aligned} &r(\lambda-\mu)^{a_2c_2}_{a_3c_3}r(\lambda)^{a_1b_1}_{c_2d_2}r(\mu)^{d_2b_2}_{c_3b_3} \\ &\qquad = r(\mu)^{a_1c_1}_{a_2c_2}r(\lambda)^{c_2d_2}_{a_3b_3}r(\lambda-\mu)^{c_1b_1}_{d_2b_2} \ , \end{aligned} \tag{3.31}$$

and can be identified with the R matrix of a fundamental spin model describing two species of fermions. Using (3.29) we find that the diagonal elements of the monodromy matrix act on the states (3.28) as follows:

$$D(\mu)|\lambda_1,\ldots,\lambda_n|F\rangle = \prod_{j=1}^n \frac{1}{a(\lambda_j-\mu)}|\lambda_1,\ldots,\lambda_n|F\rangle + \sum_{k=1}^n (\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^n C_{b_j}(\lambda_j)|0\rangle F^{a_n\cdots a_1} \ , \tag{3.32}$$

$$\begin{aligned} [A_{11}(\mu)+A_{22}(\mu)]|\lambda_1,\ldots,\lambda_n|F\rangle &= -[a(\mu)]^L\prod_{j=1}^n\frac{1}{a(\mu-\lambda_j)}\prod_{l=1}^n C_{b_l}(\lambda_l)|0\rangle\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}F^{a_n\cdots a_1} \\ &\quad + \sum_{k=1}^n(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^n C_{b_j}(\lambda_j)|0\rangle F^{a_n\cdots a_1} \ , \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} \tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n} &= \mathrm{str}[T^{(1)}_n(\mu)] \\ &= \mathrm{str}[L^{(1)}_n(\mu-\lambda_n)L^{(1)}_{n-1}(\mu-\lambda_{n-1})\cdots L^{(1)}_2(\mu-\lambda_2)L^{(1)}_1(\mu-\lambda_1)] \ , \end{aligned}$$

and

$$\begin{aligned} L^{(1)}_k(\lambda) &= b(\lambda)\Pi^{(1)} + a(\lambda)I^{(1)} \\ &= \Pi^{(1)}r(\lambda) \\ &= \begin{bmatrix} a(\lambda)-b(\lambda)e^{11}_k & -b(\lambda)e^{21}_k \\ -b(\lambda)e^{12}_k & a(\lambda)-b(\lambda)e^{22}_k \end{bmatrix} . \end{aligned} \tag{3.34}$$

$L^{(1)}$ and $r(\mu)$ can be interpreted as the L operator and R matrix of a fundamental spin model [r fulfills the Yang-Baxter equation (3.31)] describing two species of fermions. Hence, $T^{(1)}_n(\mu)$ and $\tau^{(1)}(\mu)$ are the monodromy matrix and transfer matrix of the corresponding inhomogeneous model. Inspection of (3.32) and (3.33) together with (3.25) shows that the eigenvalue condition

$$\tau(\mu)|\lambda_1,\ldots,\lambda_n|F\rangle = \nu(\mu,\{\lambda_j\},F)|\lambda_1,\ldots,\lambda_n|F\rangle \tag{3.35}$$

leads to the requirements that F ought to be an eigenvector of the "nested" transfer matrix $\tau^{(1)}(\mu)$, and that the "unwanted terms" cancel, i.e.,

$$[-(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n} + (\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}]F^{a_n\cdots a_1} = 0 \ . \tag{3.36}$$

The relative sign in (3.36) is due to the supertrace in (3.25) and (3.35). The quantities Λ_k and $\tilde{\Lambda}_k$ are computed in Appendix A. Using their explicit expressions in (3.36), we obtain the following conditions on the spectral parameters λ_j and coefficients F, which are necessary for (3.35) to hold:

$$[a(\lambda_k)]^{-L}\prod_{\substack{l=1\\ l\neq k}}^{n}\frac{a(\lambda_k-\lambda_l)}{a(\lambda_l-\lambda_k)}F^{b_n\cdots b_1}=\tau^{(1)}(\lambda_k)_{a_1\cdots a_n}^{b_1\cdots b_n}F^{a_n\cdots a_1},\quad k=1,\ldots,n\ . \tag{3.37}$$

This completes the first step of the NABA. In the next step we will now solve the nesting. The condition that F ought to be an eigenvector of $\tau^{(1)}(\mu)$ requires the diagonalization of $\tau^{(1)}(\mu)$, which can be carried out by a second, "nested" Bethe *Ansatz*. From (3.31) and (3.34), the following intertwining relation is easily derived:

$$r(\lambda-\mu)[T_n^{(1)}(\lambda)\otimes T_n^{(1)}(\mu)]=[T_n^{(1)}(\mu)\otimes T_n^{(1)}(\lambda)]r(\lambda-\mu)\ . \tag{3.38}$$

If we write

$$T_n^{(1)}(\mu)=\begin{bmatrix}A^{(1)}(\mu) & B^{(1)}(\mu)\\ C^{(1)}(\mu) & D^{(1)}(\mu)\end{bmatrix},\qquad \tau^{(1)}(\mu)=-A^{(1)}(\mu)-D^{(1)}(\mu)\ , \tag{3.39}$$

then (3.38) and (3.30) imply that

$$\begin{aligned}D^{(1)}(\mu)C^{(1)}(\lambda)&=\frac{1}{a(\mu-\lambda)}C^{(1)}(\lambda)D^{(1)}(\mu)+\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C^{(1)}(\mu)D^{(1)}(\lambda)\ ,\\ A^{(1)}(\mu)C^{(1)}(\lambda)&=\frac{1}{a(\lambda-\mu)}C^{(1)}(\lambda)A^{(1)}(\mu)+\frac{b(\mu-\lambda)}{a(\mu-\lambda)}C^{(1)}(\mu)A^{(1)}(\lambda)\ ,\\ C^{(1)}(\lambda)C^{(1)}(\mu)&=C^{(1)}(\mu)C^{(1)}(\lambda)\ .\end{aligned} \tag{3.40}$$

As the reference states, for the nesting we pick

$$|0\rangle_k^{(1)}=\begin{bmatrix}0\\1\end{bmatrix},\quad |0\rangle^{(1)}=\otimes_{k=1}^{n}|0\rangle_k^{(1)}\ . \tag{3.41}$$

The action of the nested monodromy matrix $T_n^{(1)}(\mu)$ on the reference state $|0\rangle^{(1)}$ is determined by (3.34) and we find

$$\begin{aligned}A^{(1)}(\mu)|0\rangle^{(1)}&=\prod_{j=1}^{n}a(\mu-\lambda_j)|0\rangle^{(1)}\ ,\\ D^{(1)}(\mu)|0\rangle^{(1)}&=\prod_{j=1}^{n}[a(\mu-\lambda_j)-b(\mu-\lambda_j)]|0\rangle^{(1)}\\ &=\prod_{j=1}^{n}\frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}|0\rangle^{(1)}\ .\end{aligned} \tag{3.42}$$

We now make the following *Ansatz* for the eigenstates of $\tau^{(1)}(\mu)$

$$|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle=C^{(1)}(\lambda_1^{(1)})C^{(1)}(\lambda_2^{(1)})\cdots C^{(1)}(\lambda_{n_1}^{(1)})|0\rangle\ . \tag{3.43}$$

These states can be related to the coefficients $F^{a_n\cdots a_1}$ in the following way. The state $|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle$ "lives" on a lattice of n sites and is thus an element of a direct product over n Hilbert spaces. In components it reads $|\lambda_1^{(1)}\cdots\lambda_{n_1}^{(1)}\rangle_{a_n\cdots a_1}$, which can be directly identified with $F^{a_n,\ldots,a_1}$.

The action of $\tau^{(1)}(\mu)$ on the states (3.43) can be evaluated with the help of the intertwiners (3.40):

$$D^{(1)}(\mu)|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle=\prod_{j=1}^{n_1}\frac{1}{a(\mu-\lambda_j^{(1)})}\prod_{l=1}^{n}\frac{a(\mu-\lambda_l)}{a(\lambda_l-\mu)}|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle+\sum_{k=1}^{n_1}\tilde{\Lambda}_k^{(1)}C^{(1)}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n_1}C^{(1)}(\lambda_j^{(1)})|0\rangle^{(1)}\ , \tag{3.44}$$

$$A^{(1)}(\mu)|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle=\prod_{j=1}^{n_1}\frac{1}{a(\lambda_j^{(1)}-\mu)}\prod_{l=1}^{n}a(\mu-\lambda_l)|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle+\sum_{k=1}^{n_1}\Lambda_k^{(1)}C^{(1)}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n_1}C^{(1)}(\lambda_j^{(1)})|0\rangle^{(1)}\ . \tag{3.45}$$

From (3.44) and (3.45) one can read off the eigenvalues of $\tau^{(1)}(\mu)$:

$$\tau^{(1)}(\mu)|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle=-\left[\prod_{i=1}^{n_1}\frac{1}{a(\mu-\lambda_i^{(1)})}\prod_{j=1}^{n}\frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}+\prod_{i=1}^{n_1}\frac{1}{a(\lambda_i^{(1)}-\mu)}\prod_{j=1}^{n}a(\mu-\lambda_j)\right]|\lambda_1^{(1)},\ldots,\lambda_{n_1}^{(1)}\rangle\ . \tag{3.46}$$

Inserting this expression for the special value $\mu=\lambda_k$ into (3.37), we obtain the first of Bethe equations

$$[a(\lambda_k)]^L=\prod_{i=1}^{n_1}a(\lambda_k-\lambda_i^{(1)}),\quad k=1,\ldots,n\ . \tag{3.47}$$

The unwanted terms $\Lambda_k^{(1)}$ and $\tilde{\Lambda}_k^{(1)}$ are computed in Appendix A and their cancellation [which ensures that the states (3.43) are eigenstates of the transfer matrix $\tau^{(1)}(\mu)$] leads to the following set of Bethe equations for the nesting:

$$\prod_{i=1}^{n}a(\lambda_i-\lambda_p^{(1)})=\prod_{\substack{j=1\\ j\neq p}}^{n_1}\frac{a(\lambda_j^{(1)}-\lambda_p^{(1)})}{a(\lambda_p^{(1)}-\lambda_j^{(1)})}\ ,\qquad p=1,\ldots,n_1\ . \tag{3.48}$$

Due to our choice of grading, n and n_1 can be identified as the total number of electrons and the number of spin-down electrons, respectively, i.e., $n=N_e=N_\uparrow+N_\downarrow$ and $n_1=N_\downarrow$. If we define shifted spectral parameters according to $\tilde{\lambda}_k=\lambda_k+i/2$, we obtain the Bethe equations in their "generic" form:

$$\left[\frac{\tilde{\lambda}_k-i/2}{\tilde{\lambda}_k+i/2}\right]^L=\prod_{j=1}^{N_\downarrow}\frac{\tilde{\lambda}_k-\lambda_j^{(1)}-i/2}{\tilde{\lambda}_k-\lambda_j^{(1)}+i/2}\ ,\quad k=1,\ldots,N_e$$
$$\prod_{k=1}^{N_e}\frac{\tilde{\lambda}_k-\lambda_p^{(1)}-i/2}{\tilde{\lambda}_k-\lambda_p^{(1)}+i/2}=\prod_{\substack{j=1\\ j\neq p}}^{N_\downarrow}\frac{\lambda_j^{(1)}-\lambda_p^{(1)}-i}{\lambda_j^{(1)}-\lambda_p^{(1)}+i}\ , \tag{3.49}$$
$$p=1,\ldots,N_\downarrow\ .$$

The eigenvalues of the transfer matrix (3.35) are given by

$$\nu(\mu,\{\lambda_j\},F)=[a(\mu)]^L\prod_{j=1}^{N_e}\frac{1}{a(\mu-\lambda_j)}\nu^{(1)}(\mu)+\prod_{j=1}^{N_e}\frac{1}{a(\lambda_j-\mu)}\ ,$$
$$\nu^{(1)}(\mu)=-\left[\prod_{i=1}^{N_\downarrow}\frac{1}{a(\mu-\lambda_i^{(1)})}\prod_{j=1}^{N_e}\frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}+\prod_{i=1}^{N_\downarrow}\frac{1}{a(\lambda_i^{(1)}-\mu)}\prod_{j=1}^{N_e}a(\mu-\lambda_j)\right]\ . \tag{3.50}$$

Using the trace identities (3.19), it is possible to obtain the energy eigenvalues from the eigenvalues of the transfer matrix and we find

$$E_{\text{susy}}=\sum_{j=1}^{N_e}\frac{1}{\tilde{\lambda}_j^2+1/4}-L$$
$$=-2\sum_{j=1}^{N_e}\cos(k_j)+2N_e-L\ , \tag{3.51}$$

where we have reparametrized $\tilde{\lambda}_j=\frac{1}{2}\cot(k_j/2)$. The periodic boundary conditions (3.49) and the energy (3.51) are in perfect agreement with the expressions derived by Lai[4] and by Schlottmann.[3]

E. Algebraic Bethe *Ansatz* with a fermionic background I (*BFF* grading) Sutherland solution

In this section we consider a grading such that e_2 and e_3 are fermionic (representing spin-down and spin-up electrons, respectively) and e_1 is bosonic (empty site). In terms of the Grassmann parities this means that $\epsilon_2=\epsilon_3=1$ and $\epsilon_1=0$. We pick the reference state in the kth quantum space $|0\rangle_k$ and the vacuum $|0\rangle$ of the complete lattice of L sites to be fermionic with all spins up, i.e.,

$$|0\rangle_n=\begin{bmatrix}0\\0\\1\end{bmatrix}\ ,\quad |0\rangle=\otimes_{n=1}^{L}|0\rangle_n\ . \tag{3.52}$$

This choice of grading implies that R is of the form

$$R(\lambda)=\begin{bmatrix}
1&0&0&0&0&0&0&0&0\\
0&b(\lambda)&0&a(\lambda)&0&0&0&0&0\\
0&0&b(\lambda)&0&0&0&a(\lambda)&0&0\\
0&a(\lambda)&0&b(\lambda)&0&0&0&0&0\\
0&0&0&0&b(\lambda)-a(\lambda)&0&0&0&0\\
0&0&0&0&0&b(\lambda)&0&-a(\lambda)&0\\
0&0&(a\lambda)&0&0&0&b(\lambda)&0&0\\
0&0&0&0&0&-a(\lambda)&0&b(\lambda)&0\\
0&0&0&0&0&0&0&0&b(\lambda)-a(\lambda)
\end{bmatrix}\ . \tag{3.53}$$

The L operator is

$$L_n(\lambda)=\begin{bmatrix}a(\lambda)+b(\lambda)e_n^{11}&b(\lambda)e_n^{21}&b(\lambda)e_n^{31}\\ b(\lambda)e_n^{12}&a(\lambda)-b(\lambda)e_n^{22}&-b(\lambda)e_n^{32}\\ b(\lambda)e_n^{13}&-b(\lambda)e_n^{23}&a(\lambda)-b(\lambda)e_n^{33}\end{bmatrix}\ . \tag{3.54}$$

The action of $L_k(\lambda)$ on the reference state on site k is

$$L_k(\lambda)|0\rangle_k=\begin{bmatrix}a(\lambda)&0&0\\0&a(\lambda)&0\\b(\lambda)e_k^{13}&-b(\lambda)e_k^{23}&a(\lambda)-b(\lambda)\end{bmatrix}|0\rangle_k\ . \tag{3.55}$$

We partition the monodromy matrix as before in (3.24), which implies that the transfer matrix is now given by

$$\tau(\mu)=A_{11}(\mu)-A_{22}(\mu)-D(\mu)\ . \tag{3.56}$$

The action of the monodromy matrix on the reference state follows from (3.55);

$$T_L(\lambda)|0\rangle=\begin{bmatrix}[a(\lambda)]^L & 0 & 0\\ 0 & [a(\lambda)]^L & 0\\ C_1(\lambda) & C_2(\lambda) & [a(\lambda)-b(\lambda)]^L\end{bmatrix}|0\rangle\ , \tag{3.57}$$

and inspection of (3.57) reveals that $C_1(\lambda)$ and $C_2(\lambda)$ are creation operators of odd and even Grasssmann parity, respectively. We make the following *Ansatz* for the eigenstates of $\tau(\mu)$:

$$|\lambda_1,\ldots,\lambda_n|F\rangle=C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)\cdots C_{a_n}(\lambda_n)|0\rangle F^{a_n\cdots a_1}\ . \tag{3.58}$$

The intertwining relations are found to be

$$\begin{aligned}A_{ab}(\mu)C_c(\lambda)&=(-1)^{\epsilon_a\epsilon_p+\epsilon_a+\epsilon_b}\frac{r(\mu-\lambda)^{dc}_{pb}}{a(\mu-\lambda)}C_p(\lambda)A_{ad}(\mu)\\ &\quad+(-1)^{(\epsilon_a+1)(\epsilon_b+1)}\frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_b(\mu)A_{ac}(\lambda)\ ,\\ D(\mu)C_c(\lambda)&=\frac{1}{a(\mu-\lambda)}C_c(\lambda)D(\mu)+\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_c(\mu)D(\lambda)\ ,\\ C_{a_1}(\lambda_1)C_{a_2}(\lambda_2)&=r_{FB}(\lambda_2-\lambda_1)^{a_2b_1}_{a_1b_2}C_{b_2}(\lambda_2)C_{b_1}(\lambda_1)\ ,\end{aligned} \tag{3.59}$$

where

$$\begin{aligned}r(\mu)^{ab}_{cd}&=b(\mu)I^{ab}_{cd}+a(\mu)(\Pi_{BF})^{ab}_{cd}\ ,\\ r_{FB}(\mu)^{ab}_{cd}&=b(\mu)I^{ab}_{cd}+a(\mu)(\Pi_{FB})^{ab}_{cd}\ ,\end{aligned} \tag{3.60}$$

and Π_{BF} and Π_{FB} are the permutation matrices for the gradings $\epsilon_1=0$, $\epsilon_2=1$ and $\epsilon_1=1$, $\epsilon_2=0$, respectively. Using (3.59) we find that the diagonal elements of the monodromy matrix act on the states (3.58) as follows:

$$D(\mu)|\lambda_1,\ldots,\lambda_n|F\rangle=\prod_{j=1}^{n}\frac{1}{a(\mu-\lambda_j)}\left[\frac{a(\mu)}{a(-\mu)}\right]^L|\lambda_1,\ldots,\lambda_n|F\rangle+\sum_{k=1}^{n}(\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)|0\rangle F^{a_n\cdots a_1}\ , \tag{3.61}$$

$$\begin{aligned}[A_{11}(\mu)-A_{22}(\mu)]|\lambda_1,\ldots,\lambda_n|F\rangle&=[a(\mu)]^L\prod_{j=1}^{n}\frac{1}{a(\mu-\lambda_j)}\prod_{l=1}^{n}C_{b_l}(\lambda_l)|0\rangle\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}F^{a_n\cdots a_1}\\ &\quad+\sum_{k=1}^{n}(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}C_{b_k}(\mu)\prod_{\substack{j=1\\ j\neq k}}^{n}C_{b_j}(\lambda_j)|0\rangle F^{a_n\cdots a_1}\ ,\end{aligned} \tag{3.62}$$

where

$$\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}=(-1)^{\epsilon_c}L^{(1)}_n(\mu-\lambda_n)^{cc_{n-1}}_{b_na_n}L^{(1)}_{n-1}(\mu-\lambda_{n-1})^{c_{n-1}c_{n-2}}_{b_{n-1}a_{n-1}}\cdots L^{(1)}_1(\mu-\lambda_1)^{c_1c}_{b_1a_1}(-1)^{\epsilon_c\sum_{i=1}^{n-1}(\epsilon_{b_i}+1)\sum_{i=1}^{n-1}\epsilon_{c_i}(\epsilon_{b_i}+1)}\ . \tag{3.63}$$

Here all indices c_i and c are summed over. The expression for $\tau^{(1)}(\mu)$ is significantly different from the one in the *FBB* case treated in Sec. III D, but $\tau^{(1)}(\mu)$ can nonetheless be interpreted as the transfer matrix of an inhomogeneous spin model on a lattice of n sites. Our reference state $|0\rangle$ is now of fermionic nature and we have to define a graded tensor product reflecting this fact:

$$(F\overline{\otimes}G)^{ab}_{cd}=F_{ab}G_{cd}(-1)^{(\epsilon_c+1)(\epsilon_a+\epsilon_b)}\ . \tag{3.64}$$

Effectively the graded tensor product switches even and odd Grassmann parities, i.e., $\epsilon_a\to\epsilon_a+1$. In terms of this tensor product, the transfer matrix $\tau^{(1)}(\mu)$ given by (3.63) can be obtained as

$$\tau^{(1)}(\mu)^{b_1\cdots b_n}_{a_1\cdots a_n}=\mathrm{str}[T^{(1)}_n(\mu)]=\mathrm{str}[L^{(1)}_n(\mu-\lambda_n)\overline{\otimes}L^{(1)}_{n-1}(\mu-\lambda_{n-1})\overline{\otimes}\cdots\overline{\otimes}L^{(1)}_1(\mu-\lambda_1)]\ ,$$
$$L^{(1)}_k(\lambda)=b(\lambda)\Pi_{BF}+a(\lambda)I^{(1)}=\begin{bmatrix}a(\lambda)+b(\lambda)e^{11}_k & b(\lambda)e^{21}_k\\ b(\lambda)e^{12}_k & a(\lambda)-b(\lambda)e^{22}_k\end{bmatrix}\ . \tag{3.65}$$

In the second line of (3.65) we have explicitly written the tensor product $\overline{\otimes}$ between the quantum spaces over the sites of the inhomogeneous model (the L operators are, of course, again multiplied as matrices). As before, $F^{a_n\cdots a_1}$ must be an eigenvector of $\tau^{(1)}(\mu)$ if $|\lambda_1\cdots\lambda_n|F\rangle$ is to be an eigenstate of $\tau(\mu)$. The unwanted terms can be computed in a similar way to the ones described for the *FFB* grading in Appendix B. The condition of the cancellation of the unwanted terms,

$$[(\Lambda_k)^{b_1\cdots b_n}_{a_1\cdots a_n}-(\tilde{\Lambda}_k)^{b_1\cdots b_n}_{a_1\cdots a_n}]F^{a_n\cdots a_1}=0\ , \tag{3.66}$$

leads to the conditions

$$F^{a_n\cdots a_1}=[a(-\lambda_k)]^L[\tau^{(1)}(\lambda_k)F]^{a_n\cdots a_1}\ ,\qquad k=1,\ldots,n\ . \tag{3.67}$$

To solve the nesting we first have to note that, because of our change of tensor product, the L operators $L^{(1)}(\lambda)$ are not intertwined by the R matrix $r(\mu)$ defined in (3.60), but by the R matrix

$$\hat{r}(\mu)^{ab}_{cd}=b(\mu)\delta_{ab}\delta_{cd}+a(\mu)\delta_{ad}\delta_{bc}(-1)^{\epsilon_a+\epsilon_c+\epsilon_a\epsilon_c}\ . \tag{3.68}$$

The intertwining relation

$$\hat{r}(\lambda-\mu)\,T^{(1)}_n(\lambda)\overline{\otimes}T^{(1)}_n(\mu)=T^{(1)}_n(\mu)\overline{\otimes}T^{(1)}_n(\lambda)\hat{r}(\lambda-\mu) \tag{3.69}$$

together with the choice of vacuum,

$$|0\rangle^{(1)}_k=\begin{bmatrix}0\\1\end{bmatrix},\quad |0\rangle^{(1)}=\overline{\otimes}^n_{k=1}|0\rangle^{(1)}_k \tag{3.70}$$

can be analyzed along the same lines as for the nesting in Sec. III D. It can be shown that they represent a model of the permutation type with *BF* grading (describing one species of bosons and one species of fermions). The resulting Bethe equations are

$$[a(-\lambda_l)]^L=\prod_{\substack{m=1\\ m\neq l}}^{n}\frac{a(\lambda_m-\lambda_l)}{a(\lambda_l-\lambda_m)}\prod_{j=1}^{n_1}a(\lambda_l-\lambda^{(1)}_j)\ ,\qquad l=1,\ldots,n$$
$$1=\prod_{j=1}^{n}a(\lambda_j-\lambda^{(1)}_k),\quad k=1,\ldots,n_1\ . \tag{3.71}$$

The choice of grading implies that n and n_1 are the number of holes plus the number of spins down and the number of holes, respectively. If we shift the spectral parameters according to

$$\tilde{\lambda}_j=\lambda_j-\frac{i}{2},\quad \tilde{\lambda}^{(1)}_j=\lambda^{(1)}_j-i\ , \tag{3.72}$$

we obtain Sutherland's form of the periodic boundary conditions[5] [for an odd number of lattice sites Sutherland's equation (62a) should be corrected by a factor of -1]:

$$\left[\frac{\tilde{\lambda}_l+i/2}{\tilde{\lambda}_l-i/2}\right]^L=\prod_{\substack{m=1\\ m\neq l}}^{N_h+N_\downarrow}\frac{\tilde{\lambda}_l-\tilde{\lambda}_m+i}{\tilde{\lambda}_l-\tilde{\lambda}_m-i}\prod_{j=1}^{N_h}\frac{\tilde{\lambda}_l-\tilde{\lambda}^{(1)}_j-i/2}{\tilde{\lambda}_l-\tilde{\lambda}^{(1)}_j+i/2}\ ,\qquad l=1,\ldots,N_h+N_\downarrow\ ,$$
$$1=\prod_{j=1}^{N_h+N_\downarrow}\frac{\tilde{\lambda}_j-\tilde{\lambda}^{(1)}_k-i/2}{\tilde{\lambda}_j-\tilde{\lambda}^{(1)}_k+i/2},\quad k=1,\ldots,N_h\ . \tag{3.73}$$

The eigenvalues of the transfer matrix are

$$\nu(\mu,\{\lambda_j\},F)=[a(\mu)]^L\prod_{j=1}^{N_h+N_\downarrow}\frac{1}{a(\mu-\lambda_j)}\nu^{(1)}(\mu)-\prod_{j=1}^{N_h+N_\downarrow}\frac{1}{a(\mu-\lambda_j)}\left[\frac{a(\mu)}{a(-\mu)}\right]^L\ , \tag{3.74}$$

$$\nu^{(1)}(\mu)=\prod_{i=1}^{N_h}\frac{1}{a(\mu-\lambda^{(1)}_i)}\left[\prod_{j=1}^{N_h+N_\downarrow}a(\mu-\lambda_j)-\prod_{j=1}^{N_h+N_\downarrow}\frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}\right]\ .$$

This results in energy eigenvalues

$$E_{\mathrm{susy}}=L-\sum_{j=1}^{N_h+N_\downarrow}\frac{1}{(\tilde{\lambda}_j)^2+1/4}=L-2(N_h+N_\downarrow)-2\sum_{j=1}^{N_h+N_\downarrow}\cos(k_j)\ , \tag{3.75}$$

where we have reparametrized $\tilde{\lambda}_j=\frac{1}{2}\tan(k_j/2)$.

IV. A SOLUTION OF THE *t-J* MODEL (*FBF* GRADING)

The third and last possibility of choosing the grading is $\epsilon_1=\epsilon_3=1$, $\epsilon_2=0$, with e_1 representing spin down and e_3 spin up. This case can be analyzed in precisely the same way as the *BFF* case so that we simply give the final results for the Bethe equations and eigenvalues of the transfer matrix. The BAE are

$$[a(-\lambda_l)]^L=\prod_{j=1}^{N_\downarrow} a(\lambda_j^{(1)}-\lambda_l)\ ,\quad l=1,\ldots,N_h+N_\downarrow\ ,$$
$$1=\prod_{j=1}^{N_h+N_\downarrow} a(\lambda_k^{(1)}-\lambda_j),\quad k=1,\ldots,N_\downarrow\ . \tag{4.1}$$

We again shift the spectral parameters

$$\tilde{\lambda}_j=\lambda_j-\frac{i}{2} \tag{4.2}$$

to obtain the final expression for the set of periodic boundary conditions for the t-J model:

$$\left[\frac{\tilde{\lambda}_l+i/2}{\tilde{\lambda}_l-i/2}\right]^L=\prod_{j=1}^{N_\downarrow}\frac{\lambda_j^{(1)}-\tilde{\lambda}_l-i/2}{\lambda_j^{(1)}-\tilde{\lambda}_l+i/2}\ ,\quad l=1,\ldots,N_h+N_\downarrow\ ,$$
$$1=\prod_{j=1}^{N_h+N_\downarrow}\frac{\lambda_k^{(1)}-\tilde{\lambda}_j-i/2}{\lambda_k^{(1)}-\tilde{\lambda}_j+i/2}\ ,\quad k=1,\ldots,N_\downarrow\ . \tag{4.3}$$

The equivalence of the BAE (4.3) to the BAE (3.73) is demonstrated in Appendix C. The eigenvalues of the transfer matrix are

$$\nu(\mu,\{\lambda_j\},F)=[a(\mu)]^L\prod_{j=1}^{N_h+N_\downarrow}\frac{1}{a(\mu-\lambda_j)}\left[\nu^{(1)}(\mu)-\left[\frac{1}{a(-\mu)}\right]^L\right],$$
$$\nu^{(1)}(\mu)=\prod_{i=1}^{N_\downarrow}\frac{1}{a(\lambda_i^{(1)}-\mu)}\left[1-\prod_{j=1}^{N_h+N_\downarrow}a(\mu-\lambda_j)\right]. \tag{4.4}$$

The energy eigenvalues can be obtained from (4.4) in the usual way by taking logarithmic derivatives

$$E_{\text{susy}}=L-\sum_{j=1}^{N_h+N_\downarrow}\frac{1}{(\tilde{\lambda}_j)^2+1/4}\ . \tag{4.5}$$

The string solutions of the BAE (4.3) are of a very particular structure: n spectral parameters $\tilde{\lambda}_{n,j}$ $(j=1\cdots n)$ combine with $n-1$ spectral parameters $\lambda_{n-1,k}^{(1)}$ $(k=1\cdots n-1)$ of the nesting into one complex string solution:

$$\tilde{\lambda}_{n,j}=\lambda_n+\frac{i}{2}(n+1-2j),\quad j=1\cdots n\ ,$$
$$\lambda_{n-1,k}^{(1)}=\lambda_n+\frac{i}{2}(n-2k)\ ,\quad k=1\cdots n-1\ . \tag{4.6}$$

Note that, due to the symmetry of the Hamiltonian under the interchange of spin up and spin down, there are only three different NABA solutions, as the other three solutions can be obtained via the substitution $N_\downarrow\leftrightarrow N_\uparrow$.

V. HIGHER CONSERVATION LAWS

In this section we derive explicit expressions for the conservation laws $H_{(3)}$ and $H_{(4)}$ (which involve interactions between three and four neighboring sites, respectively), using a generalization of Tetel'man's method[22,23] to the supersymmetric case.

Let us define the "boost" operator

$$B=\sum_n nH_{(2)}^{n,n+1}\ , \tag{5.1}$$

where $H_{(2)}^{n,n+1}$ is the density of the Hamiltonian given by the right-hand side of (3.18). This operator obviously violates periodicity on the finite chain, but one can use it nonetheless in commutators which "differentiate" the linear n dependence and yield formally periodic expressions.

The integrals of motion can now be successively obtained by commutation with the boost operator

$$H_{(k+1)}=i[B,H_{(k)}]$$
$$=i[\tilde{B},H_{(k)}]\ , \tag{5.2}$$

where

$$\tilde{B}=\sum_n nH_{\text{susy}}^{n,n+1}\ . \tag{5.3}$$

This can be seen as follows.

If we introduce the matrix

$$\tilde{R}(\lambda)=\Pi R(\lambda)=b(\lambda)\Pi+a(\lambda)I\ , \tag{5.4}$$

we can rewrite the intertwining relation (3.14) as

$$\tilde{R}(\lambda-\mu)[L_n(\lambda)\otimes L_n(\mu)]$$
$$=[I\otimes L_n(\mu)][L_n(\lambda)\otimes I]\tilde{R}(\lambda-\mu)\ . \tag{5.5}$$

Here the L operators are multiplied as quantum operators on both sides of (5.5). If we interchange the roles of matrix and quantum spaces in (5.5), we obtain the "90° rotated" intertwining relation

$$\tilde{R}_{n,n+1}(\lambda-\mu)[L_{n,1}(\lambda)\otimes L_{n+1,1}(\mu)]$$
$$=[I_n\otimes L_{n+1,1}(\mu)][L_{n,1}(\lambda)\otimes I_{n+1}]\tilde{R}_{n,n+1}(\lambda-\mu)\ , \tag{5.6}$$

where the index "1" indicates the matrix space (which is the same for both L's, they are now multiplied as matrices) and n and $n+1$ label the quantum spaces. The tensor product is now between the quantum spaces and $R_{n,n+1}$ is a quantum operator acting in both quantum spaces. From now on we will drop the matrix space index on the L operators. In components (5.6) reads

$$\tilde{R}_{n,n+1}(\lambda-\mu)_{\alpha_2\gamma_2}^{\alpha_1\gamma_1}L_n(\lambda)_{\gamma_1\beta_1}^{a_1b_1}L_{n+1}(\mu)_{\gamma_2\beta_2}^{b_1c_1}(-1)^{\epsilon_{\gamma_2}(\epsilon_{\gamma_1}+\epsilon_{\beta_1})}=L_{n+1}(\mu)_{\alpha_2\gamma_2}^{a_1b_1}L_n(\lambda)_{\alpha_1\gamma_1}^{b_1c_1}(-1)^{\epsilon_{\gamma_2}(\epsilon_{\alpha_1}+\epsilon_{\gamma_1})}\tilde{R}_{n,n+1}(\lambda-\mu)_{\gamma_2\beta_2}^{\gamma_1\beta_1}\ . \tag{5.7}$$

Now we note that (5.4) implies that

$$\frac{\partial}{\partial\lambda}[(\lambda+i)\tilde{R}_{n,n+1}(\lambda)]=I^{n,n+1}\,,$$
$$\tilde{R}_{n,n+1}(0)=\Pi^{n,n+1}\,. \tag{5.8}$$

Multiplying (5.7) by $\lambda-\mu+i$, differentiating with respect to λ, setting $\lambda=\mu$, and finally acting on both sides of the equation with $(\Pi^{n,n+1})^{\beta_1\sigma_1}_{\beta_2\sigma_2}(-1)^{\epsilon_{\alpha_1}\epsilon_{\alpha_2}+\epsilon_{\sigma_1}+\epsilon_{\sigma_2}}$ from the right, we find

$$[\Pi^{n,n+1},L_{n+1}(\mu)\otimes L_n(\mu)]$$
$$=-i\dot{L}_{n+1}(\mu)\otimes L_n(\mu)+iL_{n+1}(\mu)\otimes\dot{L}_n(\mu)\,. \tag{5.9}$$

The tensor product is once again between quantum space and the overdot denotes differentiation with respect to μ. From (3.18) it now follows that

$$[H^{n,n+1}_{(2)},L_{n+1}(\mu)\otimes L_n(\mu)]$$
$$=i[\dot{L}_{n+1}(\mu)\otimes L_n(\mu)-L_{n+1}(\mu)\otimes\dot{L}_n(\mu)]\,. \tag{5.10}$$

Using (5.9) it is easy to show that (up to the usual "problems" with periodicity)

$$[B,\tau(\mu)]=-i\dot{\tau}(\mu) \tag{5.11}$$

and thus

$$(B,\ln\{\tau(\mu)[\tau(0)]^{-1}\})$$
$$=-i\frac{\partial}{\partial\mu}\ln\{\tau(\mu)[\tau(0)]^{-1}\}-H_{(2)}\,. \tag{5.12}$$

Expanding both sides of (5.12) in powers of μ, we obtain (5.2). We will now use (5.2) to obtain explicit expressions for higher conservation laws. According to (2.4) we can write the t-J Hamiltonian in terms of u(1|2) generators as

$$H_{\rm susy}=\sum_{j=1}^{L}H^{j,j+1}_{\rm susy}=-\sum_{j=1}^{L}\Pi^{j,j+1}$$
$$=-\sum_{j=1}^{L}K^{\alpha\beta}J_{j,\alpha}J_{j+1,\beta}\,. \tag{5.13}$$

$H_{(3)}$ can be obtained by commutation with the boost operator $\tilde{B}$:

$$H_{(3)}=i[\tilde{B},H_{(2)}]$$
$$=i[\tilde{B},H_{\rm susy}]$$
$$=i\sum_{k=1}^{L}[H^{k+1,k+2}_{\rm susy},H^{k,k+1}_{\rm susy}]$$
$$=-i\sum_{k=1}^{L}K^{\alpha\beta}K^{\gamma\delta}f^{\epsilon}_{\beta\gamma}J_{k-1,\alpha}J_{k,\epsilon}J_{k+1,\delta}\,. \tag{5.14}$$

Using the expressions of the u(1|2) generators (2.1) and the form of $K^{\alpha\beta}=(K_{\alpha\beta})^{-1}$ given in Appendix A, it is possible to rewrite $H_{(3)}$ in terms of fermionic creation and annihilation operators:

$$\begin{aligned}H_{(3)}=i\sum_{j=1}^{L}&-2S^{\dagger}_{j-1}S_jS^z_{j+1}-2S^{\dagger}_{j-1}S^z_jS_{j+1}+S^{\dagger}_{j-1}Q^{\dagger}_{j,1}Q_{j+1,-1}+S^{\dagger}_{j-1}Q_{j,-1}Q^{\dagger}_{j+1,1}\\
&-2S^z_{j-1}S^{\dagger}_jS_{j+1}+(1-n_{j-1,-1})Q_{j,-1}Q^{\dagger}_{j+1,-1}+(1-n_{j-1,1})Q^{\dagger}_{j,1}Q_{j+1,1}\\
&-Q^{\dagger}_{j-1,1}S^{\dagger}_jQ_{j+1,-1}-Q^{\dagger}_{j-1,1}(1-n_{j,-1})Q_{j+1,1}+Q^{\dagger}_{j-1,1}Q_{j,1}(1-n_{j+1,1})+Q^{\dagger}_{j-1,1}Q_{j,-1}S^{\dagger}_{j+1}\\
&-Q^{\dagger}_{j-1,-1}S_jQ_{j+1,1}-Q^{\dagger}_{j-1,-1}(1-n_{j,1})Q_{j+1,-1}\\
&+Q^{\dagger}_{j-1,-1}Q_{j,-1}(1-n_{j+1,-1})+Q^{\dagger}_{j-1,-1}Q_{j,1}S_{j+1}-\text{H.c.}\end{aligned} \tag{5.15}$$

The u(1|2) generators $Q_{j,\sigma}$ and $Q^{\dagger}_{j,\sigma}$ are given by (2.1). The next highest conservation law can be computed along similar lines and we find

$$H_{(4)}=i[\tilde{B},H_{(3)}]$$
$$=-2\sum_{k=1}^{L}K^{\mu\nu}K^{\alpha\beta}K^{\gamma\delta}f^{\epsilon}_{\beta\gamma}f^{\omega}_{\delta\mu}$$
$$\times J_{k-1,\alpha}J_{k,\epsilon}J_{k+1,\omega}J_{k+2,\nu}$$
$$+\sum_{k=1}^{L}P^{k-1,k+1}-2\sum_{k=1}^{L}\Pi^{k,k+1}\,. \tag{5.16}$$

where $P^{k-1,k+1}$ is a graded permutation operator between the sites $k-1$ and $k+1$ with definition

$$P^{k-1,k+1}=\Pi^{k-1,k}\Pi^{k,k+1}\Pi^{k-1,k}\,. \tag{5.17}$$

Inspection of (5.13) reveals the expressibility of the last two terms in (5.16) in terms of u(1|2) generators $J_{k,\alpha}$, but the physical nature of the interactions is less obvious in the resulting expression. The results for $H_{(3)}$ and $H_{(4)}$ given in (5.14) and (5.16) generalize trivially to the case of a lattice gas with a u($m|n$) symmetry.[24]

ACKNOWLEDGMENTS

We thank P. A. Bares, S. Dasmahapatra, B. A. S. Peeters, N. Yu. Reshetikhin, E. K. Sklyanin, and especially K. Schoutens for discussions. This work was supported in part by the National Science Foundation under research Grant Nos. PHY91-07261 and NSF91-08054.

APPENDIX A: THE FUNDAMENTAL REPRESENTATION OF u(1|2)

In this appendix we write down the fundamental matrix representation of u(1|2) and use it to compute the inverse Killing form $K^{\alpha\beta}$. We consider the fundamental matrix representation on site k, where we have chosen the fermionic states to be $(1\ 0\ 0)^T$ (spin-down electrons) and $(0\ 1\ 0)^T$ (spin-up electrons), and the bosonic state to be $(0\ 0\ 1)^T$ (empty site). The generators are given by

$$J_{k,1}=S_k=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\quad J_{k,2}=S_k^{\dagger}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad J_{k,3}=S_k^z=\begin{bmatrix}-\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&0\end{bmatrix},$$

$$J_{k,4}=Q_{k,1}=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},\quad J_{k,5}=Q_{k,1}^{\dagger}=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},\quad J_{k,6}=Q_{k,-1}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \tag{A1}$$

$$J_{k,7}=Q_{k,-1}^{\dagger}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\quad J_{k,8}=T_k=\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix},\quad J_{k,9}=I_k=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

The generators $J_{k,1},\ldots,J_{k,8}$ are seen to be supertraceless, i.e., $\mathrm{str}(J)=J_{33}-J_{22}-J_{11}=0$. The grading of the algebra is described in terms of Grassmann parities ϵ_α, which are given by

$$\begin{aligned}&\epsilon_1=\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_8=\epsilon_9=0\\&\epsilon_4=\epsilon_5=\epsilon_6=\epsilon_7=1\ .\end{aligned} \tag{A2}$$

$K_{\alpha\beta}=(K^{\alpha\beta})^{-1}$ is now given by the following expression:[14]

$$K_{\alpha\beta}=\mathrm{str}(J_\alpha J_\beta)=\begin{bmatrix}0&-1&0&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0&0\\0&0&-\frac{1}{2}&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&-1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&\frac{1}{2}&0\\0&0&0&0&0&0&0&0&-1\end{bmatrix}. \tag{A3}$$

APPENDIX B: COMPUTATION OF THE "UNWANTED TERMS" FOR THE *FFB* GRADING

In this section we compute the so-called "unwanted terms" in the expressions (3.32), (3.33), (3.44), and (3.45). The "unwanted terms" are characterized by containing a creation operator C with a spectral parameter (SP) μ in place of a creation operator with SP λ_k (or $\lambda_k^{(1)}$ for the nesting). The condition of cancellation of these unwanted terms leads to the Bethe equations. In order to obtain the expression for $\tilde{\Lambda}_k$, we first move the creation operator with SP λ_k to the first place in (3.28), using (3.29),

$$\begin{aligned}&\prod_{i=1}^{n}C_{a_i}(\lambda_i)=C_{b_k}(\lambda_k)\prod_{i=1}^{k-1}C_{b_i}(\lambda_i)\prod_{j=k+1}^{n}C_{a_j}(\lambda_j)S(\lambda_k)_{a_1\cdots a_k}^{b_1\cdots b_k},\\&S(\lambda_k)_{a_1\cdots a_k}^{b_1\cdots b_k}=r(\lambda_{k-1}-\lambda_k)_{c_{k-1}a_{k-1}}^{b_{k-1}a_k}r(\lambda_{k-2}-\lambda_k)_{c_{k-2}a_{k-2}}^{b_{k-2}c_{k-1}}\cdots r(\lambda_1-\lambda_k)_{b_k a_1}^{b_1 c_2}.\end{aligned} \tag{B1}$$

To get an unwanted term, we now have to use the second term in (3.29) to move D past $C_{b_k}(\lambda_k)$, and then always use the first term in (3.29) to move D (which now carriers SP λ_k) to the very right, until it hits the vacuum, on which it acts according to (3.27). This way we obtain

$$(\tilde{\Lambda}_k F)^{b_1\cdots b_n}=S(\lambda_k)^{b_1\cdots b_k}_{a_1\cdots a_k}F^{b_n\cdots b_{k+1}a_k\cdots a_1}\left[-\frac{b(\lambda_k-\mu)}{a(\lambda_k-\mu)}\right]\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{a(\lambda_i-\lambda_k)}\ . \tag{B2}$$

The computation of Λ_k is more complicated. We first derive an expression for the contribution of $A_{11}(\mu)$, which we denote by $\Lambda_{k,1}$. Proceedings along the same lines as in the computation of $\tilde{\Lambda}_k$, we find

$$\begin{aligned}(\Lambda_{k,1}F)^{b_1\cdots b_n}=&S(\lambda_k)^{c_1\cdots c_k}_{a_1\cdots a_k}F^{a_n\cdots a_1}\left[\frac{b(\mu-\lambda_k)}{a(\mu-\lambda_k)}\right]\delta_{b_k,1}\\ &\times\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{a(\lambda_k-\lambda_i)}r(\lambda_k-\lambda_1)^{d_1c_1}_{b_1c_k}r(\lambda_k-\lambda_2)^{d_2c_2}_{b_2d_1}\cdots r(\lambda_k-\lambda_{k-1})^{d_{k-1}c_{k-1}}_{b_{k-1}d_{k-2}}\\ &\times r(\lambda_k-\lambda_{k+1})^{d_ka_{k+1}}_{b_{k+1}d_{k-1}}r(\lambda_k-\lambda_{k+2})^{d_{k+1}a_{k+2}}_{b_{k+2}d_k}\cdots r(\lambda_k-\lambda_n)^{d_{n-1}a_n}_{b_nd_{n-2}}\\ &\times[a(\lambda_k)]^L\delta_{d_{n-1},1}(-1)^{n-1}\ .\end{aligned} \tag{B3}$$

$\delta_{d_{n-1},1}$ stems from the action of $A_{1,d_{n-1}}(\lambda_k)$ (which is what we get after moving A past all the C's) on the vacuum. We also had to include a $\delta_{b_k,1}$ due to the fact that, in (3.33), we denoted by b_k the index of C with SP μ [this means that, on the lhs of (B3) b_k is really equal to 1, too]. The contribution of $A_{22}(\mu)$ differs only in factors $\delta_{b_k,2}\delta_{d_{n-1},2}$ instead of $\delta_{b_k,1}\delta_{d_{n-1},1}$, so that the result for $\Lambda_k=\Lambda_{k,1}+\Lambda_{k,2}$ is found to be

$$\begin{aligned}(\Lambda_k F)^{b_1\cdots b_n}=&S(\lambda_k)^{c_1\cdots c_k}_{a_1\cdots a_k}F^{a_n\cdots a_1}\left[\frac{b(\mu-\lambda_k)}{a(\mu-\lambda_k)}\right]\\ &\times\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{a(\lambda_k-\lambda_i)}[a(\lambda_k)]^L r(\lambda_k-\lambda_1)^{d_1c_1}_{b_1c_k}r(\lambda_k-\lambda_2)^{d_2c_2}_{b_2d_1}\cdots r(\lambda_k-\lambda_{e-1})^{d_{k-1}c_{k-1}}_{b_{k-1}d_{k-2}}(-1)^{n-1}\\ &\times r(\lambda_k-\lambda_{k+1})^{d_ka_{k+1}}_{b_{k+1}d_{k-1}}r(\lambda_k-\lambda_{k+2})^{d_{k+1}a_{k+2}}_{b_{k+2}d_k}\cdots r(\lambda_k-\lambda_n)^{b_ka_n}_{b_nd_{n-2}}\ .\end{aligned} \tag{B4}$$

This expression can (and must be) simplified by carrying out the contractions over the summation indices $c_1,\ldots,c_k$. Noting that

$$r(\lambda_1-\lambda_2)^{b_1a_1}_{b_2a_2}r(\lambda_2-\lambda_1)^{c_1b_1}_{c_2b_2}=I^{a_1c_1}_{a_2c_2}=\delta_{a_1c_1}\delta_{a_2c_2}\ , \tag{B5}$$

we are able to perform all c_i summations with the result

$$S(\lambda_k)^{c_1\cdots c_k}_{a_1\cdots a_k}r(\lambda_k-\lambda_1)^{d_1c_1}_{b_1c_k}\cdots r(\lambda_k-\lambda_{k-1})^{d_{k-1}c_{k-1}}_{b_{k-1}d_{k-2}}=\prod_{i=1}^{k-1}\delta_{a_i,b_i}\delta_{d_{k-1},a_k}\ . \tag{B6}$$

Now we transform the remaining r matrices into L operators, using the identity

$$r(\lambda)^{ab}_{cd}=[r(\lambda)\Pi^{(1)}]^{ad}_{cb}(-1)^{\epsilon_a\epsilon_c}=-[r(\lambda)\Pi^{(1)}]^{ad}_{cb}=-L^{(1)}(\lambda)^{ad}_{cb}\ . \tag{B7}$$

The second equality holds because $\epsilon_1=\epsilon_2=1$. Thus, we obtain our final form for the unwanted terms due to $A_{11}(\mu)+A_{22}(\mu)$:

$$\begin{aligned}(\Lambda_k F)^{b_1\cdots b_n}=&\frac{b(\mu-\lambda_k)}{a(\mu-\lambda_k)}\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{a(\lambda_k-\lambda_i)}[a(\lambda_k)]^L F^{a_n\cdots a_kb_{k-1}\cdots b_1}(-1)^{k+1}L^{(1)}_n(\lambda_k-\lambda_n)^{b_kd_{n-2}}_{b_na_n}\\ &\times L^{(1)}_{n-1}(\lambda_k-\lambda_{n-1})^{d_{n-2}d_{n-3}}_{b_{n-1}a_{n-1}}\cdots L^{(1)}_{k+1}(\lambda_k-\lambda_{k+1})^{d_ka_k}_{b_{k+1}a_{k+1}}\ .\end{aligned} \tag{B8}$$

We now insert (B8) and (B2) into the condition (3.36) for the cancellation of the unwanted terms and multiply the resulting equation by the inverse of $S(\lambda_k)^{b_1\cdots b_k}_{a_1\cdots a_k}$, which satisfies

$$[S^{-1}(\lambda_k)]^{p_1\cdots p_k}_{b_1\cdots b_k}S(\lambda_k)^{b_1\cdots b_k}_{a_1\cdots a_k}=\prod_{i=1}^{k}\delta_{a_i,p_i}\ ,$$

and which is computed via (B5). After some trivial rearrangements we arrive at

$$\prod_{\substack{i=1\\ i\neq k}}^{n}\frac{a(\lambda_k-\lambda_i)}{a(\lambda_i-\lambda_k)}[a(\lambda_k)]^{-L}F^{b_n\cdots b_{k+1}p_k\cdots p_1}$$

$$=L_n^{(1)}(\lambda_k-\lambda_n)_{b_n a_n}^{b_k d_n-2}\cdots L_{k+1}^{(1)}(\lambda_k-\lambda_{k+1})_{b_{k+1}a_{k+1}}^{d_k a_k}$$

$$\times r(\lambda_k-\lambda_{k-1})_{p_{k-1}s_{k-1}}^{p_k b_{k-1}}\cdots r(\lambda_k-\lambda_1)_{p_1 b_k}^{s_2 b_1}F^{a_n\cdots a_k b_{k-1}\cdots b_1}(-1)^{k+1}$$

$$=L_{k-1}^{(1)}(\lambda_k-\lambda_{k-1})_{p_{k-1}b_{k-1}}^{p_k s_{k-1}}L_{k-2}^{(1)}(\lambda_k-\lambda_{k-2})_{p_{k-2}b_{k-2}}^{s_{k-1}s_{k-2}}\cdots L_1^{(1)}(\lambda_k-\lambda_1)_{p_1 b_1}^{s_2 b_k}$$

$$\times L_n^{(1)}(\lambda_k-\lambda_n)_{b_n a_n}^{b_k d_n-2}\cdots L_{k+1}^{(1)}(\lambda_k-\lambda_{k+1})_{b_{k+1}a_{k+1}}^{d_k a_k}F^{a_n\cdots a_k b_{k-1}\cdots b_1}$$

$$=[\tau^{(1)}(\lambda_k)F]^{b_n\cdots b_{k+1}p_k\cdots p_1}\ . \tag{B9}$$

This implies (3.37).

The unwanted terms (3.4.4) for the nesting can be computed along similar lines and we easily find that

$$\Lambda_k^{(1)}=\frac{b(\mu-\lambda_k^{(1)})}{a(\mu-\lambda_k^{(1)})}\prod_{\substack{j=1\\ j\neq k}}^{n_1}\frac{1}{a(\lambda_j^{(1)}-\lambda_k^{(1)})}\prod_{l=1}^{n}a(\lambda_k^{(1)}-\lambda_l)\ ,$$

$$\tilde{\Lambda}_k^{(1)}=\frac{b(\lambda_k^{(1)}-\mu)}{a(\lambda_k^{(1)}-\mu)}\prod_{\substack{j=1\\ j\neq k}}^{n_1}\frac{1}{a(\lambda_k^{(1)}-\lambda_j^{(1)})}\prod_{l=1}^{n}\frac{a(\lambda_k^{(1)}-\lambda_l)}{a(\lambda_l-\lambda_k^{(1)})}\ . \tag{B10}$$

The cancellation of the unwanted terms for the nesting implies (3.48).

APPENDIX C: EQUIVALENCE OF THE BAE

In this appendix we establish the equivalence of our set of BAE (4.3) with the set obtained by Sutherland (3.73). We use a method developed by Bares *et al.* in Ref. [25] (see Appendix C of their paper) to show the equivalence of the solutions of Sutherland and Lai (see also Ref. 26).

We start by expressing the second set of our BAE,

$$\left[\frac{\tilde{\lambda}_l+i/2}{\tilde{\lambda}_l-i/2}\right]^L=\prod_{j=1}^{N_\downarrow}\frac{\lambda_j^{(1)}-\tilde{\lambda}_j-i/2}{\lambda_j^{(1)}-\tilde{\lambda}_l+i/2}\ ,\quad l=1,\ldots,N_h+N_\downarrow\ ,$$

$$1=\prod_{j=1}^{N_h+N_\downarrow}\frac{\lambda_k^{(1)}-\tilde{\lambda}_j-i/2}{\lambda_k^{(1)}-\tilde{\lambda}_j+i/2}\ ,\quad k=1,\ldots,N_\downarrow\ , \tag{C1}$$

as a polynomial equation of degree $N_h+N_\downarrow$:

$$p(w)=\prod_{j=1}^{N_h+N_\downarrow}\left[w-\tilde{\lambda}_j-\frac{i}{2}\right]-\prod_{j=1}^{N_h+N_\downarrow}\left[w-\tilde{\lambda}_j+\frac{i}{2}\right]=0\ . \tag{C2}$$

Among the roots w_j, $j=1\cdots N_h+N_\downarrow$ of (C2), we consider $N_\downarrow$ roots $w_1,\ldots,w_{N_\downarrow}$, which we identify with $\lambda_1^{(1)},\ldots,\lambda_{N\downarrow}^{(1)}$. The N_h other roots of (C2) we denote by w_j'. Using the residue theorem we can derive the following equality:

$$\sum_{j=1}^{N_\downarrow}-i\ln\left[\frac{\tilde{\lambda}_l-w_j+i/2}{\tilde{\lambda}_l-w_j-i/2}\right]=\sum_{j=1}^{N_\downarrow}\frac{1}{2\pi i}\oint_{C_j}dz(-i)\ln\left[\frac{\tilde{\lambda}_l-z+i/2}{\tilde{\lambda}_l-z-i/2}\right]\frac{d}{dz}\ln[p(z)]\ , \tag{C3}$$

where C_j is a small contour around w_j. The branch cut of the logarithm extends from $z_n=\tilde{\lambda}_l+i/2$ to $z_p=\tilde{\lambda}_l-i/2$. By deforming the contours on the rhs of (C3), we arrive at the following equality:

$$\sum_{j=1}^{N_\downarrow}-i\ln\left[\frac{\tilde{\lambda}_l-w_j+i/2}{\tilde{\lambda}_l-w_j-i/2}\right]=-\sum_{j=1}^{N_h}-i\ln\left[\frac{\tilde{\lambda}_l-w_j'+i/2}{\tilde{\lambda}_l-w_j'-i/2}\right]-i\ln\left[\frac{p(z_n)}{p(z_p)}\right]\ , \tag{C4}$$

where the last term on the rhs comes from integration around the branch cut. The form of the polynomial p now implies that

$$p(z_n)=-\prod_{m=1}^{N_h+N_\downarrow}(\tilde{\lambda}_l-\tilde{\lambda}_m+i)\ , \tag{C5}$$

$$p(z_p)=\prod_{m=1}^{N_h+N_\downarrow}(\tilde{\lambda}_l-\tilde{\lambda}_m-i)\ .$$

Inserting (C5) into (C4) and exponentiating the result, we obtain the identity

$$\prod_{j=1}^{N_\downarrow}\frac{\tilde{\lambda}_l-w_j+i/2}{\tilde{\lambda}_l-w_j-i/2}=\prod_{k=1}^{N_h}\frac{\tilde{\lambda}_l-w_k'-i/2}{\tilde{\lambda}_l-w_k'+i/2}\prod_{\substack{m=1\\m\neq l}}^{N_h+N_\downarrow}\frac{\tilde{\lambda}_l-\tilde{\lambda}_m+i}{\tilde{\lambda}_l-\tilde{\lambda}_m-i}\ . \tag{C6}$$

Now we use (C6) in the first set of the BAE in (C1) with the result

$$\left[\frac{\tilde{\lambda}_l+i/2}{\tilde{\lambda}_l-i/2}\right]^L=\prod_{\substack{m=1\\m\neq l}}^{N_h+N_\downarrow}\frac{\tilde{\lambda}_l-\tilde{\lambda}_m+i}{\tilde{\lambda}_l-\tilde{\lambda}_m-i}\prod_{k=1}^{N_h}\frac{\tilde{\lambda}_l-w_k'-i/2}{\tilde{\lambda}_l-w_k'+i/2},\quad l=1,\ldots,N_h+N_\downarrow\ . \tag{C7}$$

This is precisely the first set of the BAE in the Sutherland form (3.73), if we make the identification $w_k'=\tilde{\lambda}_k^{(1)}$. The second set of the BAE (3.73) is also fulfilled by the spectral parameters $\tilde{\lambda}_k^{(1)}$ because they are roots of the equivalent polynomial equation (C2) (this is because the BAE of the nesting for the Sutherland solution and our solution are identical). Thus, we have established the equivalence of the BAE (4.3) and (3.73).

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