Goldstone anomalies of dynamic susceptibilities and sound attenuation in the spherical-model limit

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Anomalies induced by Goldstone modes, in the phase of spontaneously broken continuous symmetry, are studied by means of a $1/n$ expansion of the time-dependent n-component Φ^4-model of a nonconserved order parameter. Obtaining the exact solution of the spherical model, we investigate the wave-number and frequency-dependent transverse and longitudinal response functions. The scaling function of the longitudinal susceptibility displays a crossover leading to divergent behavior in the hydrodynamic limit. An important, experimentally accessible quantity is the critical attenuation of ultrasound, whose asymptotic behavior in the critical and hydrodynamic regime is determined together with the scaling function connecting both limits. Contrary to an earlier prediction by Zeyher, the Goldstone modes do not alter the quadratic hydrodynamic frequency dependence of the coefficient of sound attenuation but their unequivocal signature is a cusp singularity of the scaling function. These results are relevant for solids with incommensurate phases.

I. INTRODUCTION

We study the influence of Goldstone modes on dynamic susceptibilities, the energy correlation function, and sound attenuation of systems that undergo a second-order transition to a low-temperature phase of broken continuous symmetry. Our investigation is based on the time-dependent Ginzburg-Landau model of a nonconserved order parameter with purely relaxational dynamics. The leading order of a $1/n$ expansion yields exact results of the spherical model.

The influence of Goldstone modes on critical behavior has attracted attention for a long time.[1-3] The interest in this topic is stimulated by the appearance of coexistence anomalies. At the phase transition itself, all modes of a n-component primary order parameter are massless and give rise to the well-known critical phenomena. Now if a continuous symmetry of the order parameter is spontaneously broken on passing through the transition point, the fluctuation spectrum of the order parameter displays the following distinctive feature. There is one massive longitudinal mode along the direction of the order parameter, and according to the Goldstone theorem,[4] there are $(n-1)$ transverse modes without an energy gap in their excitation spectrum. These Goldstone modes, being massless within the whole low-temperature phase, may lead to new kinds of anomalies, which are termed coexistence anomalies to stress their inherent origin.

There is a variety of systems with continuous symmetry. Ideally, it is realized by the gauge invariance of the Bose fluid liquid helium-4.[5] There the Goldstone mode is second sound. In the case of solids that undergo a transition to a phase of structurally incommensurate lattice modulation, the Goldstone modes are called phasons.[6] For all of these systems, it is of great importance to reveal the possible anomalies induced by the Goldstone modes.

The proper treatment of Goldstone modes is far from being straightforward, and several methods have been applied to handle the difficulties. Thereby, the $1/n$ expansion[1,7] proves to be a valuable tool for several reasons—especially in the spherical model limit $(n\rightarrow\infty)$.[8,9] The results obtained are explicit expressions and exact with respect to their dependence on space dimensionality. This facilitates the determination of possible coexistence anomalies resulting for example in nonanalytic behavior of scaling functions. Finally, the spherical model limit just sets focus on the transverse Goldstone modes, in whose peculiarities we are interested in.

Of course, it is desirable to put our theoretical conclusions to an experimental test. And even if the number of order-parameter components n is infinite in the spherical model, certain features of our results remain valid for finite n. Central to our investigation is the coefficient of sound attenuation $\alpha(\mathbf{k},\omega,|\tau|)$ for a sound wave with frequency ω propagating along $\mathbf{k}$ and $\tau=(T-T_c)/T$ measures the distance from the transition temperature T_c. The subsequent treatment introduces a Φ^2-correlation function as key quantity, which is determined in leading order of a $1/n$ expansion and provides valuable information.

The scaling behavior of the coefficient of sound attenuation can be inferred from the imaginary part of the Φ^2-correlation function. We analyze its asymptotic behavior in the critical $(T\rightarrow T_c)$ and hydrodynamic regime $(\omega\ll\omega_{\rm ch},\ k\ll\xi^{-1})$, where $\omega_{\rm ch}$ is the characteristic order-parameter rate and ξ is the correlation length. We will show that the Goldstone modes do not alter the hydrodynamic ω^2 law, in contradiction to the $\omega^{3/2}$ dependence predicted by Zeyher.[10] The actual coexistence anomalies are much more subtle. The scaling function describing the crossover from the critical to the hydrodynamic regime, a main result of our investigation, contains a cusp singularity at small scaling variables. This signature of the Goldstone modes, following from our theory, should be tested by experiments. This result is in accord with a forthcoming general renormalization-group theory of ultrasonic attenuation in the Φ^4-model.[11]

The real part of the Φ^2-correlation function is related to the energy correlation function. Again, we determine the universal scaling function within the spherical model and reveal a crossover in its wave-number dependence. The explicit expressions show that in the coexistence limit, the wave-number dependent corrections start with a $(k\xi)^2$-power law. A $(k\xi)^\epsilon$ term is present in the vicinity of the transition point itself.

For a computation of the Φ^2-correlation function, the wave-number- and frequency-dependent response functions of the transverse and longitudinal modes are required, where the latter one displays a well-known coexistence anomaly.[3] Our calculation of the longitudinal response function agrees with earlier findings[12] and, additionally, yields the pertinent scaling function.

This paper is organized as follows: In Sec. II, we present the model free energy together with stochastic equations of motion for the order parameter and a phonon variable. The perturbation theory for the transverse and longitudinal response functions, as well as the Φ^2-correlation function, is elaborated in Sec. III. In leading order of a $1/n$ expansion, the exact results of the spherical model are obtained. Section IV is devoted to the universal scaling behavior of these quantities. We characterize the asymptotic behavior in the hydrodynamic and critical limit by means of critical exponents. Moreover, the scaling functions connecting both limits are derived. Coexistence singularities are revealed and the consequences for the coefficient of sound attenuation and the energy correlation function are discussed. Our results are summarized in Sec. V. In the Appendixes, the scaling laws for the transverse and longitudinal susceptibilities, as well as for correlation functions of composite order-parameter fields, are derived.

II. ORDER-PARAMETER MODEL AND COUPLING TO SOUND WAVES

The model investigated is based on a n-component order parameter Φ with continuous symmetry in the order-parameter space. Thus, only $O(n)$-symmetric contributions appear in the model free energy. The acoustic sound waves are described by a phonon variable $\rho(\mathbf{x},t)$. The presence of critical sound anomalies depends on the coupling between these degrees of freedom. We will investigate an interaction, which is linear in the phonon variable and bilinear in the order parameter. This kind of coupling, relevant for a variety of systems is, of course, nonsymmetry breaking. The statics of our model are contained in the free energy functional

$$\mathcal{H}=\int d^d\mathbf{x}\left[\frac{1}{2}[r\Phi^2+(\nabla\Phi)^2]+\frac{\tilde{u}}{4!}\Phi^4+\frac{1}{2}\rho^2+\gamma\rho\Phi^2\right]$$

with (2.1)

$$\Phi^2=\sum_{j=1}^{n}\Phi_j^2,\quad (\nabla\Phi)^2=\sum_{j=1}^{n}(\nabla\Phi_j)^2,\quad \Phi^4=(\Phi^2)^2 .$$

The first three terms, later referred to as $\mathcal{H}_\Phi$, represent the well-known Φ^4-model, the fourth term is the purely harmonic elastic energy, and the last term is the phonon order parameter interaction with coupling strength γ. Such a Hamiltonian with $n=2$ can be derived from lattice dynamics for normal to incommensurate phase transitions of A_2BX_4 compounds.[6,13]

The dynamics of our model are expressed by stochastic equations of motion of the Langevin type,[14]

$$\dot{\Phi}_j=-\lambda\frac{\delta\mathcal{H}}{\delta\Phi_j}+r_j(\mathbf{x},t) ,$$

(2.2a)

$$\langle r_i(\mathbf{x},t)r_j(\mathbf{x}',t')\rangle=2\lambda\delta_{i,j}\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') ,$$

and

$$M\ddot{\rho}=\nabla^2\frac{\delta\mathcal{H}}{\delta\rho}+DM\nabla^2\dot{\rho}+R(\mathbf{x},t) ,$$

(2.2b)

$$\langle R(\mathbf{x},t)R(\mathbf{x}',t')\rangle=2DM\nabla^4\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') .$$

The nonconserved order-parameter components Φ_j $(j=1,\ldots,n)$ follow purely relaxational dynamics according to (2.2a). The equation of motion (2.2b) for the phonon variable contains propagation, with the bare sound velocity $1/\sqrt{M}$, and the bare damping is D. The stochastic forces r_j and R produce a Gaussian white noise, with vanishing mean value $\langle r_j(\mathbf{x},t)\rangle=\langle R(\mathbf{x},t)\rangle=0$, and the variances given in Eqs. (2.2) obey the Einstein relations.

From these equations of motion, the critical behavior of the sound mode can be determined. The complete treatment, presented elsewhere,[15] eliminates the phonon variable. The final result for the coefficient of critical sound attenuation $\alpha(\mathbf{k},\omega,|\tau|)$ can be expressed by means of a Φ^2-correlation function

$$\alpha(\mathbf{k},\omega,|\tau|)=\frac{\omega\mathrm{Im}\{4n\gamma^2\Pi(\mathbf{k},\omega,|\tau|)\}}{2Mc^3|1+4n\gamma^2\Pi(\mathbf{k},\omega,|\tau|)|^2} . \quad (2.3)$$

Here c is the sound velocity and the Φ^2-correlation function explicitly reads

$$2n\,\Pi(\mathbf{k},\omega,|\tau|)=-\frac{1}{2}\int_0^\infty dt\, e^{i\omega t}\frac{d}{dt}\langle\Phi^2(\mathbf{k},t)\Phi^2(-\mathbf{k},0)\rangle_{\mathcal{H}_\Phi}.$$

(2.4)

The crucial point is that the thermal average necessary for a computation of the Φ^2-correlation function requires only the part $\mathcal{H}_\Phi$ of the energy functional (2.1), which solely depends on the order parameter. It is thus a quantity of pure critical dynamics. Solely the coupling constant is shifted to a new value ($\tilde{u}\to u=\tilde{u}-12\gamma^2$) through elimination of the phonon variable. Equation (2.3) is closely related to, but not identical with an earlier phenomenological approach,[16] which is confirmed in significant limiting cases.

In the static limit, the Φ^2-correlation function of (2.4) gives the energy correlation function $C(\mathbf{k},|\tau|)$, which introduces another quantity of interest,

$$C(\mathbf{k},|\tau|)=4n\lim_{\omega\to 0}\Pi(\mathbf{k},\omega,|\tau|)$$
$$=[\langle\Phi^2(\mathbf{k})\Phi^2(-\mathbf{k})\rangle-\langle\Phi^2(\mathbf{k})\rangle^2] . \quad (2.5)$$

The actual computation of time-dependent expectation values will be performed with the help of the stochastic functional $\mathcal{J}$ of the path integral,[17,18] equivalent to the simplified model described by $\mathcal{H}_\Phi$ with coupling constant u and a single equation of motion (2.2a). An explicit representation is offered in the next section. Introduction of auxiliary fields $\tilde{\Phi}$ (Ref. 19) reduces the nonlinearity of the stochastic functional, which appears as weight factor of expectation values of any combination of the fields $A(\Phi,\tilde{\Phi})$,

$$\langle A(\Phi,\tilde{\Phi})\rangle=\frac{1}{N}\int\mathcal{D}[\Phi]\mathcal{D}[i\tilde{\Phi}]A(\Phi,\tilde{\Phi})\exp[\mathcal{J}(\Phi,\tilde{\Phi})] \; . \tag{2.6}$$

If the order parameter is coupled to an external field h_j in the energy functional (2.1), an additional contribution $(\lambda\sum h_j\tilde{\Phi}_j)$ appears in the stochastic functional. Then from (2.6) one obtains

$$\langle \Phi_i(\mathbf{x},t)\lambda\tilde{\Phi}_j(\mathbf{x}',t')\rangle=\frac{\delta\langle\Phi_i(\mathbf{x},t)\rangle}{\delta h_j(\mathbf{x}',t')} \; . \tag{2.7}$$

The expression on the right-hand side of (2.7) is the dynamic susceptibility or order-parameter response function. Generally, the relation between response and correlation functions is the content of fluctuation-dissipation theorems.[18] In this way, the correlation function of (2.4) is related to a composite field response function

$$-\frac{1}{2}\Theta(t-t')\frac{d}{dt}\langle\Phi^2(\mathbf{k},t)\Phi^2(\mathbf{k}',t')\rangle =\langle\Phi^2(\mathbf{k},t)(\Phi\lambda\tilde{\Phi})(\mathbf{k}',t')\rangle \; . \tag{2.8}$$

Recognizing the physical meaning of the dynamic expectation values of (2.7) and (2.8), we now turn to their explicit computation in the phase of broken symmetry.

III. PERTURBATION THEORY IN THE LOW-TEMPERATURE PHASE

In the low-temperature phase, the expectation value of the order parameter is finite. Without loss of generality, we take

$$\langle\Phi_i\rangle=\sqrt{3/u}\,m\,\delta_{i,n} \; . \tag{3.1}$$

We introduce new wave-number and frequency-dependent fields

$$\Phi(\mathbf{k},\omega)\to\begin{bmatrix}\pi^\alpha(\mathbf{k},\omega)\\ \sqrt{3/u}\,m\,\delta(\mathbf{k})\delta(\omega)+\sigma(\mathbf{k},\omega)\end{bmatrix}\quad(\alpha=1,\ldots,n-1)\;, \tag{3.2}$$

where both the $(n-1)$ transverse fields π^α and the longitudinal field σ have zero expectation value, and $\delta(\mathbf{k})=(2\pi)^d\delta^d(\mathbf{k})$.

After substitution of (3.2), the dynamic functional is decomposed into a harmonic part $\mathcal{J}_{\mathrm{harm}}$ yielding the zeroth-order propagators of our model and the interacting part $\mathcal{J}_{\mathrm{int}}$ containing higher-order vertices. The following shorthand notation will be used

$$\int_{\mathbf{k}_1,\ldots,\mathbf{k}_p}\int_{\omega_1,\ldots,\omega_p}\Psi^{i_1}\ldots\Psi^{i_p} =\int\frac{d^d\mathbf{k}_1}{(2\pi)^d}\cdots\int\frac{d^d\mathbf{k}_p}{(2\pi)^d}\int\frac{d\omega_1}{2\pi}\cdots\int\frac{d\omega_p}{2\pi}\Psi^{i_1}(\mathbf{k}_1,\omega_1)\ldots\Psi^{i_p}(\mathbf{k}_p,\omega_p)\delta\left[\sum_{l=1}^{p}\mathbf{k}_l\right]\delta\left[\sum_{l=1}^{p}\omega_l\right]\;, \tag{3.3}$$

where each Ψ^i stands for one of the fields π^α, σ, $\tilde{\pi}^\alpha$, or $\tilde{\sigma}$. The harmonic part of the dynamic functional then reads

$$\mathcal{J}_{\mathrm{harm}}[\{\pi^\alpha\},\sigma,\{\tilde{\pi}^\alpha\},\tilde{\sigma}]=\int_{\mathbf{k}_1,\mathbf{k}_2}\int_{\omega_1,\omega_2}\left[\sum_\alpha\lambda\tilde{\pi}^\alpha\tilde{\pi}^\alpha+\lambda\tilde{\sigma}\tilde{\sigma}-\sum_\alpha\tilde{\pi}^\alpha[i\omega_1+\lambda(r+m^2/2+k_1^2)]\pi^\alpha -\tilde{\sigma}[i\omega_1+\lambda(r+3m^2/2+k_1^2)]\sigma\right]\;. \tag{3.4}$$

The interacting part

$$\mathcal{J}_{\mathrm{int}}[\{\pi^\alpha\},\sigma,\{\tilde{\pi}^\alpha\},\tilde{\sigma}]=-\lambda\frac{\sqrt{3u}}{6}m\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\int_{\omega_1,\omega_2,\omega_3}\left[\sum_\alpha 2\tilde{\pi}^\alpha\pi^\alpha\sigma+\sum_\alpha\tilde{\sigma}\pi^\alpha\pi^\alpha+3\tilde{\sigma}\sigma\sigma\right] -\frac{\lambda u}{6}\int_{\mathbf{k}_1,\ldots,\mathbf{k}_4}\int_{\omega_1,\ldots,\omega_4}\left[\sum_{\alpha,\beta}\tilde{\pi}^\alpha\pi^\alpha\pi^\beta\pi^\beta+\sum_\alpha\tilde{\pi}^\alpha\pi^\alpha\sigma\sigma+\sum_\alpha\tilde{\sigma}\sigma\pi^\alpha\pi^\alpha+\tilde{\sigma}\sigma\sigma\sigma\right] \tag{3.5}$$

contains third- and fourth-order interactions. There is a further vertex with one longitudinal response field

$$\mathcal{J}[\tilde{\sigma}]=-\lambda\sqrt{3/u}\,m(r+m^2/2)\int_{\mathbf{k}}\int_\omega\tilde{\sigma}(-\mathbf{k},-\omega)\;,$$

which we take into account in the derivation of the equation of state later on. Finally, the contribution of a functional determinant

$$\mathcal{J}[\Phi,\tilde{\Phi}]=\int d^d\mathbf{x}\int dt\sum_j\frac{\lambda}{2}\frac{\delta}{\delta\Phi_j}\frac{\delta\mathcal{H}_\Phi}{\delta\Phi_j}$$

guarantees to cancel acausal terms in the perturbation theory. To make our treatment more transparent, we

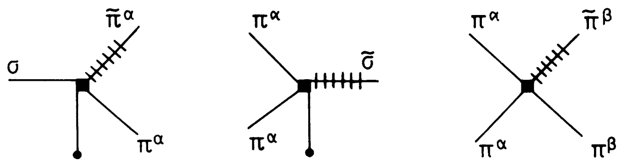

FIG. 1. Anharmonic vertices of the Φ^4 theory in the low-temperature phase with at least two transverse fields.

will use a diagrammatic representation of perturbation theory. The vertices contributing in the spherical model limit are displayed in Fig. 1.

As a consequence of the symmetry breaking (3.1), we have to distinguish between transverse and longitudinal susceptibilities

$$\langle \pi^\alpha(\mathbf{k},\omega)\tilde{\pi}^\beta(\mathbf{k}',\omega')\rangle = g_\perp(\mathbf{k},\omega)\delta_{\alpha\beta}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') , \tag{3.6a}$$

$$\langle \sigma(\mathbf{k},\omega)\tilde{\sigma}(\mathbf{k}',\omega')\rangle = g_\parallel(\mathbf{k},\omega)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega') . \tag{3.6b}$$

The response functions, given in harmonic approximation through $\mathcal{J}_{\rm harm}$, are supplemented by self-energies due to the interactions of $\mathcal{J}_{\rm int}$

$$g_\perp(\mathbf{k},\omega) = [\lambda(r+m^2/2+k^2) - i\omega + \Sigma_\perp(\mathbf{k},\omega)]^{-1} , \tag{3.7a}$$

$$g_\parallel(\mathbf{k},\omega) = [\lambda(r+3m^2/2+k^2) - i\omega + \Sigma_\parallel(\mathbf{k},\omega)]^{-1} . \tag{3.7b}$$

The self-energies $\Sigma_\perp(\mathbf{k},\omega)$ and $\Sigma_\parallel(\mathbf{k},\omega)$ are determined perturbatively in a $1/n$ expansion. To decide which diagrams contribute, one has to take into account the n dependence of the coupling constant $u=O(1/n)$ and $m=O(1)$.[7] On the other hand, contraction of internal transverse lines within diagrams yields combinatoric factors $O(n)$. Now, one easily convinces oneself that to leading order the diagrams of Fig. 2 appear. Thereby, the longitudinal self-energy $\Sigma_\parallel(\mathbf{k},\omega)$ already requires a geometric series of the transverse order-parameter bubble $\pi(\mathbf{k},\omega)$, shown in Fig. 3. All diagrams are of the same order in $1/n$ and the analytic expression of the series reads

$$\frac{2n\,\pi(\mathbf{k},\omega)}{1+(\lambda u n/3)\pi(\mathbf{k},\omega)} + O(1/n) . \tag{3.8}$$

Equation (3.7a), for the transverse response function, is an implicit equation because the self-energy $\Sigma_\perp(\mathbf{k},\omega)$ already has to be calculated with the propagator $g_\perp(\mathbf{k},\omega)$ marked by a full line in Fig. 2. The Hartree bubble of

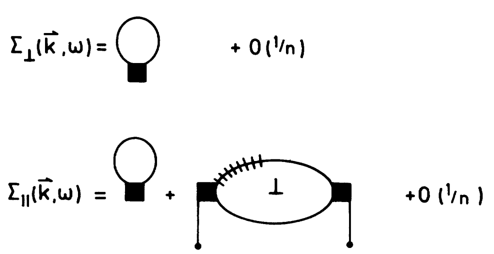

FIG. 2. Transverse and longitudinal self-energies of the order-parameter susceptibilities in the spherical model.

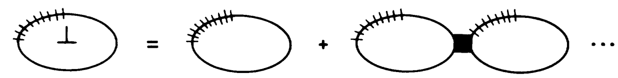

FIG. 3. The dressed bubble representing a geometric series of transverse order-parameter bubbles.

Fig. 2 is independent of external wave number and frequency. Therefore, the response function takes the form

$$g_\perp(\mathbf{k},\omega) = [\lambda(r_\perp + k^2) - i\omega]^{-1} , \tag{3.9}$$

where the self-energy

$$\Sigma_\perp = \frac{\lambda u n}{6}\int_q \frac{1}{r_\perp + q^2} + O(1/n) \tag{3.10}$$

only contributes to the transverse mass

$$r_\perp = r + \frac{m^2}{2} + \frac{un}{6}\int_q \frac{1}{r_\perp+q^2} . \tag{3.11}$$

Thus, the implicit propagator Eq. (3.7a) has been turned into an implicit equation (3.11) for the transverse mass $r_\perp$, which considerably simplifies the problem. At this point, the equation of state is required. It can be determined from the requirement that according to definition (3.2) the longitudinal field is purely fluctuating: $\langle \sigma \rangle = 0$. Besides the vertex with a single longitudinal response field $\mathcal{J}[\tilde{\sigma}]$ mentioned after (3.5), again the Hartree bubble of the transverse self-energy $\Sigma_\perp$ appears in the diagrams to leading order. This yields the equation

$$r + \frac{m^2}{2} + \frac{un}{6}\int_q \frac{1}{r_\perp+q^2} = 0 , \tag{3.12}$$

from which the following results can be obtained. (i) As a consequence of the interaction of order-parameter fluctuations, the transition temperature is shifted. This shift is determined from the vanishing of the spontaneous magnetization at the transition point, $m^2(r_c)=0$, i.e.,

$$r_c = -\frac{un}{6}\int_q \frac{1}{r_\perp+q^2} . \tag{3.13}$$

Utilizing the temperature variable $\tau = r - r_c$, which is a linear measure of the distance from the true transition temperature, the order parameter m of (3.1) obeys

$$m^2 = |2\tau|^{2\beta} , \tag{3.14}$$

yielding the critical exponent $\beta = 1/2$ for the spherical model. (ii) Combining Eqs. (3.11) and (3.12), the transverse mass vanishes in absence of an external field

$$r_\perp = 0 . \tag{3.15}$$

Thereby, we confirm the Goldstone theorem[4] within the spherical model. Next, we investigate the longitudinal response function. The analytic expression for the longitudinal self-energy shown in Fig. 2 reads

$$\Sigma_\parallel(\mathbf{k},\omega) = \frac{\lambda u n}{6}\int_q \frac{1}{r_\perp+q^2} - \frac{\lambda u n}{3}\,\frac{\lambda m^2 \pi(\mathbf{k},\omega)}{1+(\lambda u n/3)\pi(\mathbf{k},\omega)} + O(1/n) . \tag{3.16}$$

Inserting (3.16) into $g_{\|}(\mathbf{k},\omega)$ of (3.7b) and using $r_\perp$ from (3.11) yields

$$g_{\|}(\mathbf{k},\omega)=\left[\lambda(r_\perp+k^2)-i\omega+\frac{\lambda m^2}{1+(\lambda un/3)\pi(\mathbf{k},\omega)}\right]^{-1} \tag{3.17}$$

Finally, if we take into account the vanishing of the transverse mass (3.15) and switch to the more familiar temperature variable τ, the response functions (3.9) and (3.17) read

$$g_\perp(\mathbf{k},\omega)=[\lambda k^2-i\omega]^{-1}\ , \tag{3.18a}$$

$$g_{\|}(\mathbf{k},\omega)=\left[\lambda k^2-i\omega+\frac{\lambda|2\tau|}{1+(\lambda un/3)\pi(\mathbf{k},\omega)}\right]^{-1} . \tag{3.18b}$$

We now turn to a computation of the Φ^2-correlation function (2.4), which by means of the fluctuation-dissipation-theorem (2.8), is equivalently expressed as a composite field response function

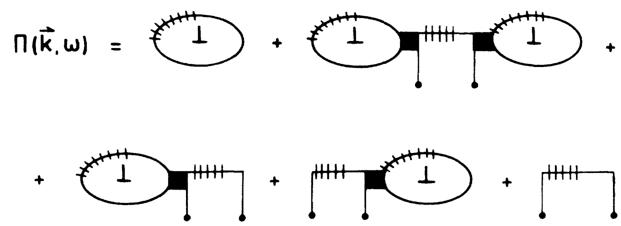

FIG. 4. The Φ^2-correlation function in the spherical model. All diagrams can be summed with the dressed bubble of Fig. 3, and the longitudinal response function $g_{\|}$.

$$\Pi(\mathbf{k},\omega,|\tau|)=\frac{\lambda}{2n}\langle \Phi^2(\mathbf{k},\omega)(\Phi\tilde{\Phi})(-\mathbf{k},-\omega)\rangle\ . \tag{3.19}$$

In terms of transverse and longitudinal fields combined with the n dependence of u and m, the following contributions are left in leading order

$$\Pi(\mathbf{k},\omega,|\tau|)=\frac{\lambda}{2n}[\langle \pi^2(\pi\tilde{\pi})\rangle_{\mathbf{k},\omega}+2\sqrt{3/u}\ \langle (m\sigma)(\pi\tilde{\pi})\rangle_{\mathbf{k},\omega}+\sqrt{3/u}\ \langle \pi^2(m\tilde{\sigma})\rangle_{\mathbf{k},\omega}+\frac{6}{u}\langle (m\sigma)(m\tilde{\sigma})\rangle_{\mathbf{k},\omega}]+O(1/n)\ , \tag{3.20}$$

where $\overset{(\sim)}{\pi}=(\overset{(\sim)}{\pi}{}^1,\dots,\overset{(\sim)}{\pi}{}^{n-1})$. In a diagrammatic representation, again diagrams of arbitrary loop number have to be summed. With the help of the geometric series of transverse bubbles shown in Fig. 3 and the longitudinal response function $g_{\|}(\mathbf{k},\omega)$, all diagrams for the Φ^2-correlation function in the spherical limit are displayed in Fig. 4. From this representation, the analytical result immediately is obtained

$$\Pi(\mathbf{k},\omega,|\tau|)=\frac{3\lambda|2\tau|}{un}\frac{g_{\|}(\mathbf{k},\omega)}{[1+(\lambda un/3)\pi(\mathbf{k},\omega)]^2}+\frac{\lambda\pi(\mathbf{k},\omega)}{1+(\lambda un/3)\pi(\mathbf{k},\omega)}+O(1/n)\ . \tag{3.21}$$

This expression is of a certain generality, because $g_{\|}(\mathbf{k},\omega)$ and $\pi(\mathbf{k},\omega)$ might stem from an order-parameter dynamics different from (2.2a), as long as no mode couplings are introduced additionally.

We still have to compute the transverse order-parameter bubble $\pi(\mathbf{k},\omega)$. Inserting the transverse response function (3.18a), we are left with an integral that can be performed analytically. For space dimensions $2<d<4$, we obtain

$$\begin{aligned}\pi(\mathbf{k},\omega)&\equiv\int_{\mathbf{k}_1,\omega_1}\frac{1}{\lambda(\mathbf{k}+\mathbf{k}_1)^2-i(\omega+\omega_1)}\frac{2\lambda}{(\lambda k_1^2-i\omega_1)(\lambda k_1^2+i\omega_1)}\\&=\frac{K_d}{2\lambda}\frac{\Gamma(\epsilon/2)\Gamma(2-\epsilon/2)}{2}\left[\frac{\lambda k^2-i\omega}{2\lambda}\right]^{-\epsilon/2}\int_0^1 dx\,x^{-\epsilon/2}\left[1-x\frac{1}{2}\frac{\lambda k^2}{\lambda k^2-i\omega}\right]^{-\epsilon/2}\\&=\frac{K_d}{2\lambda}\frac{\pi/2}{\sin(\pi\epsilon/2)}\left[\frac{\lambda k^2-i\omega}{2\lambda}\right]^{-\epsilon/2}F\left[1-\epsilon/2,\epsilon/2,2-\epsilon/2,\frac{1}{2}\frac{\lambda k^2}{\lambda k^2-i\omega}\right]\ .\end{aligned} \tag{3.22}$$

Here $K_d=2^{-d+1}\pi^{-d/2}/\Gamma(d/2)$ and as usual $\epsilon=4-d$. $F(a,b,c,z)$ is the hypergeometric function and $\Gamma(z)$ is the gamma function.[20]

IV. SCALING LAWS FOR THE COEFFICIENT OF SOUND ATTENUATION AND ENERGY CORRELATION FUNCTION

It will be the purpose of this section to reveal the scaling properties of the results obtained in the preceding section. For systems with continuous symmetry, we have to distinguish between two different situations. In the vicinity of the phase-transition point ($T\to T_c$), the correlation length ξ of the longitudinal modes diverges. All n components of the order parameter are critical modes and lead to the well-known critical point anomalies characterized by critical exponents, amplitude ratios, and universal scaling functions.[21]

A different type of critical behavior is realized in the low-temperature phase of broken continuous symmetry along the whole coexistence curve ($T<T_c,\omega\to 0,k\to 0$). In the absence of an external field conjugate to the order-parameter fields, there are $(n-1)$ transverse Goldstone modes without energy gap. As a consequence of the presence of these massless modes, we expect to observe scaling behavior along the whole coexistence line. Our spe-

cial interest will be directed towards the anomalies induced by the Goldstone modes, either through alteration of asymptotic behavior or through the presence of singularities within the scaling functions. In this section, we demonstrate the capability of the $1/n$ expansion to produce results with scaling form and to reveal critical point anomalies as well as coexistence anomalies. In leading order, equivalent to the spherical model, no recourse to the renormalization-group machinery is necessary.[21]

It is convenient to represent our results in terms of auxiliary functions

$$R(x,y) \equiv R_1(x,y)+iR_2(x,y) = (x^2-iy)^{-\epsilon/2} F\left[1-\epsilon/2,\epsilon/2,2-\epsilon/2,\frac{1}{2}\frac{x^2}{x^2-iy}\right], \quad (4.1)$$

which obviously are homogeneous:

$$R(x,y)=(y)^{-\epsilon/2}\hat{R}\left[\frac{x^2}{y}\right]=(x^2)^{-\epsilon/2}\tilde{R}\left[\frac{y}{x^2}\right]. \quad (4.2)$$

Together with the constant

$$a=\frac{\pi/2}{\sin(\pi\epsilon/2)}2^{\epsilon/2}\frac{unK_d}{6}, \quad (4.3)$$

the transverse order-parameter bubble $\pi(\mathbf{k},\omega)$ (3.22) can be written as

$$\frac{\lambda un}{3}\pi(\mathbf{k},\omega)=aR(k,\omega/\lambda). \quad (4.4)$$

First, we examine the critical behavior of the Φ^2-correlation function at $T=T_c$, i.e., $\tau=0$ in Eq. (3.21). The resulting expression is indeed a homogeneous function of frequency and wave number at small values of these arguments. An expansion with respect to frequency yields

$$\Pi(\mathbf{k},\omega,0)=\frac{3}{un}\left[1-\frac{1}{a}\left[\frac{\omega}{\lambda}\right]^{\epsilon/2}\frac{1}{\hat{R}(\lambda k^2/\omega)}\right]. \quad (4.5)$$

After extraction of an overall power of the frequency, the function only depends on the ratio $(\lambda k^2/\omega)$. In the real part a constant appears as leading term.

Next, we investigate the Φ^2-correlation function at an arbitrary temperature below the transition point $(T<T_c)$. To this end, the complete expression (3.21) is split into real and imaginary part. For the purpose of offering explicit expressions, we present

$$\Pi(\mathbf{k},\omega,|\tau|)=\frac{3}{un}\frac{N(k,\omega,|\tau|)}{D(k,\omega,|\tau|)}, \quad (4.6)$$

in terms of the real denominator

$$D(k,\omega,|\tau|)=1+2\frac{\omega}{\lambda|2\tau|}aR_2(k,\omega/\lambda)+2\frac{k^2}{|2\tau|}[1+aR_1(k,\omega/\lambda)] + \left[\left[\frac{\omega}{\lambda|2\tau|}\right]^2+\left[\frac{k^2}{|2\tau|}\right]^2\right]\{[1+aR_1(k,\omega/\lambda)]^2+a^2R_2^2(k,\omega/\lambda)\}, \quad (4.7a)$$

and the numerator

$$N(k,\omega,|\tau|)=D(k,\omega,|\tau|)-\frac{k^2}{|2\tau|}+i\frac{\omega}{\lambda|2\tau|}-\left[\left[\frac{\omega}{\lambda|2\tau|}\right]^2+\left[\frac{k^2}{|2\tau|}\right]^2\right][1+aR^*(k,\omega/\lambda)]. \quad (4.7b)$$

To analyze the coexistence behavior of these expressions, we have to investigate the limit of small frequency and wave number. Thereby, the infrared divergent bubble $\pi(\mathbf{k},\omega)$ (3.22) is of central importance and three different realizations have to be considered. (i) At $\mathbf{k}\equiv 0$, we have $\pi(\mathbf{k}=0,\omega)\propto(i\omega)^{-\epsilon/2}$ resulting in a power-law divergence at small frequency. (ii) Analogously for $\omega\equiv 0$, we find $\pi(\mathbf{k},\omega=0)\propto(k^2)^{-\epsilon/2}$ being again divergent at low wave number. (iii) Finally, ω and $\mathbf{k}$ are both nonzero. If one of the variables is sent to zero at a finite value of the other one, no divergence occurs at all. This is only realized if both tend to zero simultaneously. If k^2 and ω vanish at different rate, one of cases (i) or (ii) effectively holds. If both vanish at finite ratio $i\omega/(\lambda k^2)=$const, the transverse bubble

$$\pi(\mathbf{k},\omega)\propto(i\omega)^{-\epsilon/2}\propto(k^2)^{-\epsilon/2}$$

is likewise divergent.

First, we recognize that in the strict coexistence limit $\mathbf{k}=\omega=0$ the right-hand side of Eq. (4.6) remains finite

$$\Pi(0,0,|\tau|)=\frac{3}{un}. \quad (4.8)$$

This is remarkable because $\Pi(\mathbf{k},\omega,|\tau|)$ contains multiple contributions of the infrared divergent bubble $\pi(\mathbf{k},\omega)$. Nontrivial cancellations, typical for $O(n)$-symmetric functions, are responsible for this observation.[22]

We now turn to the limit of small k and ω. In this limit, the functions (4.7) are consistently expanded with respect to these arguments, i.e., a factor $[1+aR_1(k,\omega/\lambda)]$ gives $aR_1(k,\omega/\lambda)$. This expansion, appropriate for the coexistence limit, is valid at any temperature in the low-temperature phase. We now approach the transition point $(T\rightarrow T_c)$ in the vicinity of the coexistence line. Therefore, we set $\tau=0$ after applying the coexistence expansion to the Φ^2-correlation function. In doing so, we observe that our expressions, derived for a proper description of the coexistence region are still capable to produce the correct critical point behavior (4.5) at low wave number and frequency. Based on this observation, it is justified to utilize the correlation length $\xi \propto |\tau|^{-\nu}$, and the order-parameter rate $\omega_{\text{ch}} \propto |\tau|^{z\nu}$ to parametrize the temperature dependence of our results. The critical exponents $\nu=1/(2-\epsilon)$ and $z=2$ acquire their spherical model values and we set

$$\xi=\left[\frac{a}{|2\tau|}\right]^{\nu} \tag{4.9}$$

and

$$\omega_{\text{ch}}=\lambda\left[\frac{|2\tau|}{a}\right]^{z\nu} . \tag{4.10}$$

With the help of the scaling variables $x=k\xi$ and $y=\omega/\omega_{\text{ch}}$, the scaling law of the Φ^2-correlation function can be written in a compact form

$$\Pi(\mathbf{k},\omega,|\tau|)=\frac{3}{un}\left[1+\frac{|2\tau|^{-\alpha}}{a^{2/(2-\epsilon)}}P(k\xi,\omega/\omega_{\text{ch}})\right] . \tag{4.11}$$

The specific-heat exponent $\alpha=-\epsilon/(2-\epsilon)$ and the scaling function describing the crossover from the coexistence to the critical point limit, at small and large scaling variables, respectively, is of the form

$$P(x,y)=\frac{-x^2+iy-[x^4+y^2]R^*(x,y)}{1+2x^2R_1(x,y)+2yR_2(x,y)+[x^4+y^2]|R(x,y)|^2} . \tag{4.12}$$

We now discuss applications of the results (4.11) and (4.12). First, we investigate the static energy correlation function. At zero frequency, the imaginary part of the Φ^2-correlation function vanishes, $R_2(x,0)=0$ and $R_1(x,0)=(x^2)^{-\epsilon/2}F_\epsilon(1/2)$ with the abbreviation $F_\epsilon(z)=F(1-\epsilon/2,\epsilon/2,2-\epsilon/2,z)$. The scaling law for the energy correlation function then reads

$$C(\mathbf{k},|\tau|)=\frac{12}{u}\left[1-\frac{|2\tau|^{-\alpha}}{a^{2/(2-\epsilon)}}(k\xi)^{\epsilon}\frac{(k\xi)^{2-\epsilon}}{1+F_\epsilon(1/2)(k\xi)^{2-\epsilon}}\right] . \tag{4.13}$$

The striking feature of this expression is the crossover in its $(k\xi)$ dependence. The following power laws are found in the critical and hydrodynamic regime:

$$[C(0,|\tau|)-C(\mathbf{k},|\tau|)] \propto |2\tau|^{-\alpha}\begin{cases}(k\xi)^{\epsilon} & \text{for } (k\xi)\gg 1,\\ (k\xi)^2 & \text{for } (k\xi)\ll 1 ,\end{cases} \tag{4.14}$$

implying a k^ϵ dependence at the critical point and a k^2 law in the coexistence limit. These results have been obtained under the assumption of small wave number. The complete wave-number dependent energy correlation function at the transition point is given by

$$C(\mathbf{k},0)=\frac{12}{u}\left[1+\frac{1}{aF_\epsilon(1/2)}k^{\epsilon}\right]^{-1} . \tag{4.15}$$

Equation (4.15) is in accord with the first line of (4.14) and in the homogeneous case $(k=0)$ leads to the finite value $C(0,0)=12/u$, which is the well-known spherical model limit of the specific heat. Our result for the energy correlation function agrees with that of Nicoll[23] in the limit $(n\to\infty)$ who, however, did not determine the scaling function explicitly, nor the limiting wave number dependence. Another study, by Lawrie,[22] is based on renormalization-group methods at arbitrary component number n. Applying the spherical model limit $(n\to\infty)$ to his final result (4.36), only the constant $C_{n\to\infty}=12/u$ is left.

Our primary goal is the coefficient of sound attenuation given in Eq. (2.3). Its frequency and temperature dependence is dominated by the factor

$$\tilde{\alpha}(\mathbf{k},\omega,|\tau|)\equiv\omega\,\text{Im}\{\Pi(\mathbf{k},\omega,|\tau|)\} . \tag{4.16}$$

The denominator of Eq. (2.3), $c^3|1+4n\gamma^2\Pi|^2$, depends on the nonuniversal coupling constant γ, and obeys a simple scaling law only if $4n\gamma^2\Pi \propto |\tau|^{-\alpha}$ dominates, which is asymptotically the case for a positive specific-heat exponent. Because $\alpha<0$ in the spherical model, this denominator does not alter the asymptotic power laws as given by (4.16), which is also obvious from Eqs. (4.5) and (4.8). Therefore, we concentrate on (4.16).

From renormalization-group theory[24] one finds near T_c, the general scaling law

$$\tilde{\alpha}(\mathbf{k},\omega,|\tau|)=\text{const}\,\omega^2|2\tau|^{-\rho}g(k\xi,\omega/\omega_{\text{ch}}) , \tag{4.17}$$

with the critical exponent $\rho=\alpha+z\nu$. To verify the scaling form (4.17) and to evaluate $g(k\xi,\omega/\omega_{\text{ch}})$, we insert (4.11) into (4.16),

$$\tilde{\alpha}(\mathbf{k},\omega,|\tau|)=\frac{3}{un\lambda}\omega^2|2\tau|^{-(\alpha+z\nu)}\frac{\text{Im}\{P(k\xi,\omega/\omega_{\text{ch}})\}}{\omega/\omega_{\text{ch}}} . \tag{4.18}$$

The exponent ρ acquires its spherical limit value

$$\rho=\alpha+z\nu=1 , \tag{4.19}$$

and the scaling function g is given by

$$g(k\xi,\omega/\omega_{\text{ch}})=\frac{\omega_{\text{ch}}}{\omega}\text{Im}\{P(k\xi,\omega/\omega_{\text{ch}})\} . \tag{4.20}$$

The scaling property of g is evident, and using for example the homogeneity property Eq. (4.2), one easily convinces oneself that $g(0,0)$ is finite. We arrive at the important conclusion that the hydrodynamic character of sound attenuation $[\alpha(\omega\to 0)\propto\omega^2]$ is not altered by the Goldstone modes. However, the scaling function g displays singular behavior at small scaling arguments, which will be proofed by the following analysis.

In ultrasonic experiments, the wave length of the sound wave is much larger than the correlation length, which allows one to set $k\xi=0$. Although our result (4.18) incorporates the $(k\xi)$ dependence, we may drop it in the following for the sake of transparency. At vanishing wave number (4.1) reduces to

$$R(0,y)=y^{-\epsilon/2}\exp\left[i\frac{\pi\epsilon}{4}\right] \tag{4.21}$$

and the scaling function reads

$$g(0,\omega/\omega_{ch}) = \frac{1+\sin(\pi\epsilon/4)(\omega/\omega_{ch})^{1-\epsilon/2}}{1+2\sin(\pi\epsilon/4)(\omega/\omega_{ch})^{1-\epsilon/2}+(\omega/\omega_{ch})^{2-\epsilon}} . \tag{4.22}$$

It is obvious that this function is finite at vanishing frequency, however, it has a cusp singularity

$$g(0,\omega/\omega_{ch}) = 1-\sin\left[\frac{\pi\epsilon}{4}\right](\omega/\omega_{ch})^{1-\epsilon/2} \quad \text{for } (\omega/\omega_{ch})\to 0 . \tag{4.23}$$

As shown in Fig. 5, this coexistence singularity is fairly pronounced for $d=3(\epsilon=1)$. This anomaly is the unique signature of the Goldstone modes revealed by means of the $1/n$ expansion for the first time. Instead of the hydrodynamic ω-square law and the cusp singularity, Zeyher[10] predicted a $\omega^{3/2}$ power law. The error in Zehyer's work can be traced back to a violation of causality requirements in dynamic perturbation theory.

We complete our discussion of sound attenuation with a closer look at the critical regime $(T\to T_c)$. In this limit the scaling function behaves as

$$g(0,\omega/\omega_{ch})\propto\sin\left[\frac{\pi\epsilon}{4}\right](\omega/\omega_{ch})^{-1+\epsilon/2}$$

according to (4.22), where the power is just the spherical model value of $-1-\alpha/z\nu$. Accordingly we introduce an alternative scaling function,

$$G(k\xi,\omega/\omega_{ch})=(\omega/\omega_{ch})^{1+\alpha/(z\nu)}g(k\xi,\omega/\omega_{ch}) , \tag{4.24}$$

in terms of which the scaling law (4.17) reads

$$\tilde{\alpha}(\mathbf{k},\omega,|\tau|)\propto\omega^{1-\alpha/(z\nu)}G(k\xi,\omega/\omega_{ch}) . \tag{4.25}$$

The scaling function G is finite in the critical limit $(T\to T_c)$, hence, the attenuation has a finite, temperature-independent value on approaching the transition point. The frequency dependence is now described by the universal law in (4.25). Critical sound attenuation at the transition point is due to order-parameter fluctuations. It is absent in the mean-field Landau-Khalatnikov theory,[25] where fluctuations are totally neglected.

The scaling function $G(0,\omega/\omega_{ch})$ is shown in Fig. 6 for space dimension $d=3$. For $2<d\le 3$, this function is monotonically increasing with the scaling argument. For $3<d<4$, $G(0,\omega/\omega_{ch})$ has a maximum at a finite value of the scaling variable. In real systems, with a finite number of order-parameter components $(n=2,3)$, there is a maximum of the attenuation for $T<T_c$ even for $d=3$.[26] This is also revealed in a general renormalization-group theory.[11] Nevertheless, it is gratifying that within the spherical model limit the asymptotics in the critical and hydrodynamic limit are properly described by universal critical exponents. Moreover, the scaling functions connecting both limits can be calculated even in the low-temperature phase.

We close this section by mentioning that the same procedure can be applied to the transverse and longitudinal response functions [Eqs. (3.18a) and (3.18b)]. The derivation and discussion of their scaling behavior are deferred to Appendix A.

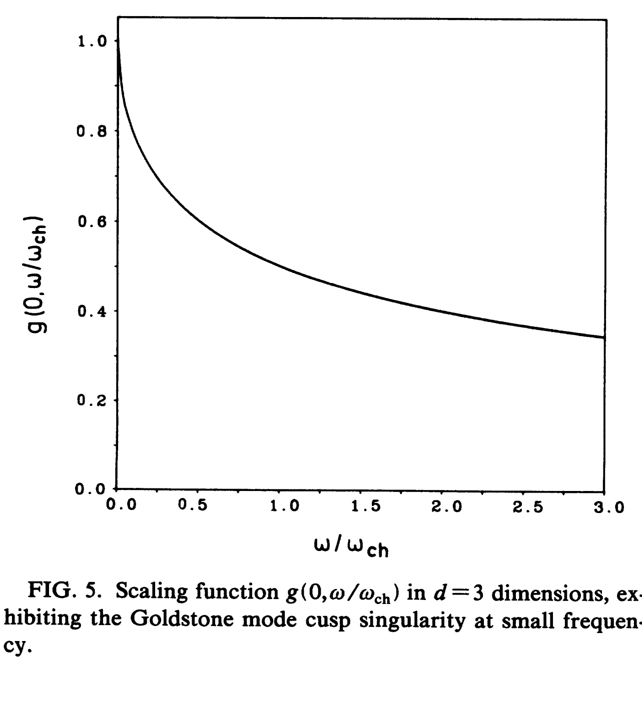

FIG. 5. Scaling function $g(0,\omega/\omega_{ch})$ in $d=3$ dimensions, exhibiting the Goldstone mode cusp singularity at small frequency.

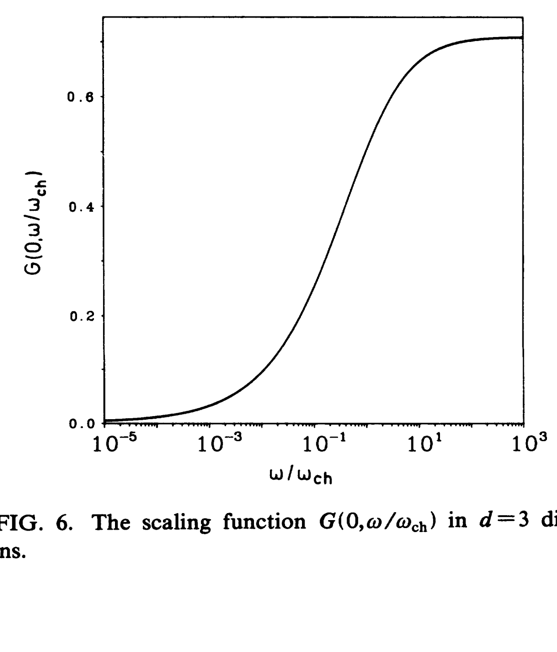

FIG. 6. The scaling function $G(0,\omega/\omega_{ch})$ in $d=3$ dimensions.

V. DISCUSSION

In this work, we have derived the dynamic susceptibilities, the energy correlation function, and the coefficient of sound attenuation in the spherical model limit. Special emphasis has been put on their scaling properties. Thereby, valuable insight into the peculiar nature of the massless Goldstone modes could be obtained. Besides the asymptotic behavior in the critical and hydrodynamic regimes described by universal critical exponents, the scaling functions connecting these limits have been obtained and coexistence anomalies were found. In the case of sound attenuation, our investigation revealed that the coefficient of sound attenuation follows the hydrodynamic ω^2 dependence but has a subtle cusplike coexistence anomaly.

In experiments below incommensurate phase transitions,[26] one actually finds for longitudinal acoustic modes at small frequency the hydrodynamic ω^2 law. In view of the $[\omega^{3/2}=\omega^2(1/\omega^{1/2})]$ prediction by Zeyher, experimental investigators have been led to introduce a phason mass in order to eliminate the $1/\omega^{1/2}$ divergence of the Onsager coefficient. While it is conceivable that pinning effects lead to a phason mass, certainly acoustic attenuation does not require the introduction of such a mass. The complete theory based either on the proper $1/n$ expansion of this paper or a forthcoming renormalization-group analysis[11] gives ω^2 with cusp corrections. The coupling $\gamma\rho\mathbf{\Phi}^2$ assumed in Eq. (2.1) is appropriate for incommensurate solids as K_2SeO_4 and Rb_2ZnCl_4 for longitudinal acoustic phonons propagating along high-symmetry directions perpendicular to the incommensurate modulation axis. For other directions and for transverse phonons there is also a coupling to the bilinear field $\Phi_1\Phi_2$. However for incommensurate transitions the pertinent coupling coefficient g_{12} is proportional to the wave number. This leads to an extra factor of ω^2 in the attenuation, which compensates the Goldstone anomaly of the $\Phi_1\Phi_2$-correlation function $\Pi^{(5)}$ given in Appendix B together with other correlation functions of bilinear fields. We note that the extra wave number dependence of g_{12} insures that elimination of the acoustic phonons leads to a static free-energy functional, the relevant terms of which remain rotationally invariant.

In systems with a finite number n of order-parameter components, of course, critical exponents will change and the shape of the scaling functions might differ quantitatively from their $(n=\infty)$-limit. The qualitative behavior of the transverse Goldstone modes, especially with respect to the ensuing coexistence anomalies, is not expected to change.

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APPENDIX A: SCALING FORM OF ORDER-PARAMETER RESPONSE FUNCTIONS

The scaling law of the transverse and longitudinal response function is commonly represented in the form

$$g_{\perp,\parallel}(\mathbf{k},\omega,|\tau|)=\frac{1}{\lambda}\frac{1}{k^{2-\eta}}G_{\perp,\parallel}(k\xi,\omega/\omega_{\mathrm{ch}}) . \tag{A1}$$

Here, η is the exponent describing the decay of the static correlation function at T_c. Exactly at the critical point ($|\tau|=0$) both response functions [(3.18a) and (3.18b)] coincide because the symmetry is not yet broken:

$$g_{\perp}(\mathbf{k},\omega,0)=g_{\parallel}(\mathbf{k},\omega,0)=\frac{1}{\lambda k^2}\left[1-\frac{i\omega}{\lambda k^2}\right]^{-1} . \tag{A2}$$

This is a homogeneous function of wave number and frequency implying the spherical model value of the exponent $\eta=0$.

In the coexistence limit, the longitudinal response function (3.18b) yields at small wave number and frequency,

$$g_{\parallel}(\mathbf{k}\to 0,\omega\to 0,|\tau|)=\left[\lambda k^2-i\omega+\frac{3|2\tau|}{un\pi(\mathbf{k},\omega)}\right]^{-1} . \tag{A3}$$

If we now apply the critical point limit ($|\tau|\to 0$) in the vicinity of the coexistence line, we recover the previously determined expression (A2). Based on this observation, the scaling functions are derived from (3.18a) and (A3)

$$G_{\perp}(k\xi,\omega/\omega_{\mathrm{ch}})=\left[1-\frac{i\omega/\omega_{\mathrm{ch}}}{(k\xi)^2}\right]^{-1} , \tag{A4a}$$

$$G_{\parallel}(k\xi,\omega/\omega_{\mathrm{ch}})=\left[1-\frac{i\omega/\omega_{\mathrm{ch}}}{(k\xi)^2}+\frac{(k\xi)^{-2+\epsilon+\eta}[1-(i\omega/\omega_{\mathrm{ch}})/(k\xi)^2]^{\epsilon/2}}{F(1-\epsilon/2,\epsilon/2,2-\epsilon/2,\frac{1}{2}\{(k\xi)^2/[(k\xi)^2-i\omega/\omega_{\mathrm{ch}}]\})}\right]^{-1} . \tag{A4b}$$

The transverse scaling function (A4a) of the Goldstone modes maintains its form for all values of the scaling variables and has a pole at

$$(k\xi)^2=i\omega/\omega_{\mathrm{ch}} . \tag{A5}$$

The longitudinal function (A4b) displays a crossover behavior from the critical regime to the hydrodynamic regime

$$G_{\parallel}[k\xi,\omega/\omega_{\mathrm{ch}}\propto(k\xi)^2]\propto\begin{cases}[k\xi]^0 & \text{for } k\gg\xi^{-1}\\ [k\xi]^{2-\epsilon-\eta} & \text{for } k\ll\xi^{-1}\end{cases} . \tag{A6}$$

TABLE I. Elements of the scaling laws for $\Pi^{(i)}$.

i	A_i	α_i	$P^{(i)}(x,y)$	$\lim_{y\to 0}\mathrm{Im}P^{(i)}(0,y)/y \propto$
1	$\frac{3}{un}a^{-2/(2-\epsilon)}$	α	$\frac{-(x^2-iy)}{1+(x^2-iy)R(x,y)}$	y^0
2	$\frac{3}{un}$	$\frac{\alpha+\hat{\alpha}}{2}$	$\frac{1}{1+(x^2-iy)R(x,y)}$	$y^{-\epsilon/2}$
3	$\frac{3}{un}a^{2/(2-\epsilon)}$	$\hat{\alpha}$	$\frac{R(x,y)}{1+(x^2-iy)R(x,y)}$	$y^{-1-\epsilon/2}$
4	$\frac{3}{un}a^{2/(2-\epsilon)}$	$\hat{\alpha}$	$R(x,y)$	$y^{-1-\epsilon/2}$
5	$\frac{3}{u}a^{2/(2-\epsilon)}$	$\hat{\alpha}$	$\frac{1}{x^2-iy}$	y^{-2}

Combining (A1) and (A6), we retrieve the well-known coexistence singularity of the longitudinal response function[27] $g_\parallel \propto k^{-\epsilon}$. The absence of such a divergence in case of the Φ^2-correlation function again illustrates the special coexistence behavior of $O(n)$-symmetric functions.

APPENDIX B: ANISOTROPIC CONTRIBUTIONS TO SOUND ATTENUATION

For a multicomponent order parameter, besides the Φ^2-correlation function, also correlation functions of anisotropic bilinear operators exist. These can be classified by group theory leading to irreducible functions of well defined scaling behavior. There are response functions built from the $O(n-1)$ scalar $\Phi^2-(n/n-1)\pi^2$ and from composite operators of two transverse fields $\pi_1\pi_2$ or the longitudinal field times a transverse field $\Phi_n\pi$, namely

$$\Pi^{(2)}(\mathbf{k},\omega,|\tau|)\equiv\frac{\lambda}{2n}\left\langle\Phi^2(\mathbf{k},\omega)\left[\Phi\tilde{\Phi}-\frac{n}{n-1}\pi\tilde{\pi}\right](-\mathbf{k},-\omega)\right\rangle , \tag{B1a}$$

$$\Pi^{(3)}(\mathbf{k},\omega,|\tau|)\equiv\frac{\lambda}{2n}\left\langle\left[\Phi^2-\frac{n}{n-1}\pi^2\right](\mathbf{k},\omega)\left[\Phi\tilde{\Phi}-\frac{n}{n-1}\pi\tilde{\pi}\right](-\mathbf{k},-\omega)\right\rangle , \tag{B1b}$$

$$\Pi^{(4)}(\mathbf{k},\omega,|\tau|)\equiv\lambda\langle(\pi_1\pi_2)(\mathbf{k},\omega)(\pi_1\tilde{\pi}_2)(-\mathbf{k},-\omega)\rangle , \tag{B1c}$$

$$\Pi^{(5)}(\mathbf{k},\omega,|\tau|)\equiv\frac{\lambda}{n}\langle[(\sqrt{3/u}\,m+\sigma)\pi](\mathbf{k},\omega)[\pi\tilde{\sigma}+(\sqrt{3/u}\,m+\sigma)\tilde{\pi}](-\mathbf{k},-\omega)\rangle . \tag{B1d}$$

In the spherical model, the scaling laws for these functions read

$$\Pi^{(i)}(\mathbf{k},\omega,|\tau|)=\frac{3}{un}\delta_{i,1}+A_i|2\tau|^{-\alpha_i}P^{(i)}(k\xi,\omega/\omega_{\mathrm{ch}}) , \tag{B2}$$

with amplitudes A_i, temperature exponents α_i, and scaling functions $P^{(i)}(x,y)$ for $i=1,\ldots,5$ given in Table I. We include the Φ^2-correlation function, denoted as $\Pi^{(1)}(\mathbf{k},\omega,|\tau|)$. Hereby, $\hat{\alpha}=\alpha+2(\phi-1)=\epsilon/(2-\epsilon)$ is related to the crossover exponent ϕ. The last column of Table I displays more pronounced coexistence singularities for $i=2,\ldots,5$ than the cusp singularity of the Φ^2-correlation function. For incommensurate transitions, the elastic deformations can couple to $\Phi_1\Phi_2$, however, the coupling coefficient is proportional to the wave number, which gives an additional factor of ω^2 in the attenuation, which compensates the Goldstone anomaly of $\Pi^{(5)}$.

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