Determination of currents in flat superconductors

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Explicit expressions are given for the current $I(y,z)$ in a flat superconductor when the magnetic field $H_x(y,z)$ perpendicular to its surface is measured, e.g., by the Faraday effect in a thin europium selenide layer evaporated on the specimen surface. The analytical results for a long flat strip are particularly instructive, showing what type of singularities occur in the current and field. The general solution for the current in flat conductors of arbitrary shape is given in form of an integral over $H_x(y,z)$ times an integral kernel that follows from an iteration.

The discovery of high-T_c superconductors has revived the interest in experimental methods that measure the critical current density J_c above which the Abrikosov vortices depin from material inhomogeneities and dissipation starts to occur. The persistent currents, which cause the magnetization in superconductors, depend on the applied magnetic field H_a and temperature T, and due to flux-line pinning also on the magnetic history and shape of the specimen. This shape dependence of the magnetic behavior is particularly complicated because of the nonlinearity and irreversibility of the magnetization, and in HTSC also because of their pronounced anisotropy and layered structure.

For long specimens in parallel field, i.e., when demagnetization effects may be disregarded, the nonlinearity of the magnetic response is well described by the Bean model[1–3] with appropriate critical current density $J_c(B,T)$. In this simple geometry, the flux lines are (macroscopically) parallel and the local current density J is determined by the *gradient* of the flux density (or induction) B, $J = (\partial H/\partial B)|\nabla B|$ where $B = |\mathbf{B}|$ and $H(B)$ is the reversible magnetic field. If H_a is not too small, $H_a \gg H_{c1}$, the reversible magnetization is negligibly small; then $B \approx \mu_0 H$ and $\partial H/\partial B \approx 1/\mu_0$. In many experiments, however, flat superconductors in the perpendicular or inclined field are used, e.g., thin films or disks. In this geometry demagnetization effects are crucial, and the current density is caused by the *curvature* of the flux lines if the thickness d of the specimen is much smaller than its width w.[4,5] This may be seen by writing $\mu_0\mathbf{J} = \nabla\times\mathbf{B} = \nabla B\times\hat{\mathbf{B}} + B\,\nabla\times\hat{\mathbf{B}}$ with $\hat{\mathbf{B}} = \mathbf{B}/B$. Here the first term originates from the gradient of B and is approximately proportional to $1/w$, and the second term comes from the curvature of the field lines and is approximately proportional to $1/d$. Recently, the currents inside a superconducting disk in the Bean critical state have been computed by various groups.[3,6–8]

A useful tool to investigate the spatial distribution of the current density J in flat superconductors are experiments using the Faraday effect in a thin layer of europium selenide evaporated onto the specimen surface.[8–12] Such experiments measure the magnetic field component $H_x(y,z)$ *perpendicular* to the specimen surface, which rotates the polarization angle of the reflected light. They allow time and space resolved observation of the surface field and thus of the circulating currents.

In this paper I show how the current density integrated over the specimen thickness

$$\mathbf{I}(y,z) = \int_{-d/2}^{d/2} \mathbf{J}(x,y,z)\,dx \tag{1}$$

can be calculated from the measured perpendicular surface field component $H_x(y,z)$. The solution will be given in terms of an integral over H_x times an integral kernel which inverts the Biot-Savart law. In the case of a long strip this kernel is given explicitly by an analytical function, and for flat specimens of arbitrary shape (circle, square, rectangle, etc.) the kernel follows from an iteration procedure which converges rapidly.

The presented results are very general. In principle, the thickness $d(y,z)$ may vary provided each surface $x(y,z)$ is "flat," i.e., its squared slope $(\partial x/\partial y)^2 + (\partial x/\partial z)^2 \ll 1$. The magnetostatic problem to be solved here is not specific for superconductors but describes the current-field relationship in any flat conductor. However, the general integral kernel will be obtained by considering the special problem of a flat superconductor in the ideal Meissner state, or equivalently, the hydrodynamic problem of laminar flow around a flat obstacle.

The detailed distribution of the current density over the specimen thickness is irrelevant here. If d is smaller than the penetration depth λ, then $J \approx$const over the thickness. For $d \gg \lambda$, J will flow in two surface layers of thickness λ if there are no flux lines or if the flux lines are pinned in straight positions (e.g., after field cooling). If the flux lines have constant curvature the current density will be constant over the thickness, $J(x,y,z) = I(y,z)/d$; this situation results when the pinning force is homogeneous or when the flux lines move and experience a constant viscous drag.

First, I consider a long flat strip with thickness $d = 2\epsilon \ll$ width $w = 2a \ll$ length, and with cross section $|x| \le \epsilon$, $|y| \le a$. The current $I(y)$ flowing along this strip (along z) causes a magnetic field outside the strip which is obtained by linear superposition of the fields of straight wires carrying currents $I(y)dy$. The field around each wire is circular with strength $I(y)dy/2\pi r$ where r is the distance from this wire. For the field at the specimen surface $x = \pm\epsilon$ this gives the applied field plus

$$H_x(y) = \frac{1}{2\pi}\int_{-a}^{a} \frac{I(u)\,du}{y-u}, \tag{2}$$

$$H_y(y) = \tfrac{1}{2} I(y)\ \mathrm{sgn}\, x \tag{3}$$

($\mathrm{sgn}\, x = x/|x|$, $|y| \leq \infty$). This means that on the considered length scale $> d$ the parallel field component H_y is completely determined by the closest current path (local relationship), but the perpendicular component H_x is generated by all current paths (nonlocal relationship). Equations (2) and (3) apply also away from the specimen for $|y| > a$ in the plane $x = 0$; note that $I(y) = 0$ for $|y| > a$. For numerical purposes one may replace in the integral (2) the diverging function $1/y$ by $y/(y^2 + \epsilon^2)$ with $\epsilon \ll a$.

If H_y is measured, one immediately knows the current $I(y) = 2H_y(y)$. If H_x is measured, the linear relationship (2) has to be inverted to obtain I and H_y. This means one has to find the current distribution which at the specimen surface generates a perpendicular field H_x which vanishes everywhere except at a given position $y = u$. This problem is identical to finding the shielding currents which are caused in an ideal superconducting strip by two magnetic poles of linear shape and opposite sign at the position $y = u$ on its two surfaces, Fig. 1. The resulting field lines coincide with the stream lines of the laminar flow around a strip driven by a fan positioned in a slit at $y = u$. This two-dimensional problem of the theory of potentials can be solved by the method of complex functions.[13] One finds for $|y| \leq a$

$$I(y) = \frac{2}{\pi} \int_{-a}^{a} \frac{H_x(u)}{u - y} \left(\frac{a^2 - u^2}{a^2 - y^2} \right)^{1/2} du. \tag{4}$$

In particular for $H_x(y)$=const=$-H_a$ one obtains the current which shields an applied perpendicular constant field H_a, or which flows in a normal conducting strip immediately after H_a is applied, i.e., before H_a diffuses into the conductor

$$I(y) = 2yH_a/(a^2 - y^2)^{1/2}. \tag{5}$$

Inserting (4) and (5) back into (2) and putting $a = 1$ one obtains useful relationships valid for all $|y| \leq 1$, $|u| \leq 1$:

$$\int_{-1}^{1} \frac{dv}{(v - y)(v - u)} \left(\frac{1 - u^2}{1 - v^2} \right)^{1/2} = \pi \delta(y - u), \tag{6}$$

$$\int_{-1}^{1} \frac{(1 - u^2)^{1/2}}{y - u}\, du = \pi y, \tag{7}$$

$$\int_{-1}^{1} \frac{u\ du}{(u - y)(1 - u^2)^{1/2}} = \pi. \tag{8}$$

The magnetic field caused in the Bean critical state in a strip with partial penetration of the current, $I(y) = J_c d\, \mathrm{sgn}\, y$ for $0 \leq \rho < |y| < a$ and $I(y) = 0$ for $|y| < \rho$, Fig. 2, follows from (1) ($x = 0$, $|y| < \infty$)

$$H_x(y) = \frac{J_c d}{2\pi} \ln \left| \frac{a^2 - y^2}{y^2 - \rho^2} \right|. \tag{9}$$

For full penetration ($\rho = 0$) $H_x(y) \propto \ln |a^2/y^2 - 1|$. A current $I(y) = y$ would yield $H_x(y) = \pi^{-1}(y\, \mathrm{Arth}\, y - 1)$.

From the above results one recognizes the following general features, which apply on the surface of flat conductors of any shape: (a) $H_\parallel - H_{a\parallel}$ always equals $\frac{1}{2} I$. (b) If I is finite at the specimen edge then $H_\perp$ has a logarithmic singularity at this edge and $H_\perp - H_{a\perp}$ changes sign at some distance from the edge. (c) $H_\perp$ has a symmetric logarithmic singularity where I exhibits an abrupt jump. (d) I and $H_\parallel - H_{a\parallel}$ have $1/\sqrt{y}$ singularities where $H_\perp$ exhibits a jump, and at the edges if there $H_\perp \neq H_{a\perp}$.

If an applied perpendicular field $H_{a\perp}$ is switched on or suddenly changes at $t = 0$, then at the very first moment I is infinite at the edges of a flat specimen, and the total $H_\perp$ is zero inside, and ∞ just outside, the edge. At $t > 0$, magnetic flux starts to penetrate, either by linear diffusion if the specimen is a normal conductor or if the flux lines in the superconductor move viscously in the flux flow or thermally assisted flux-flow states,[5] or nonlinearly if flux pinning is dominant. As a consequence,

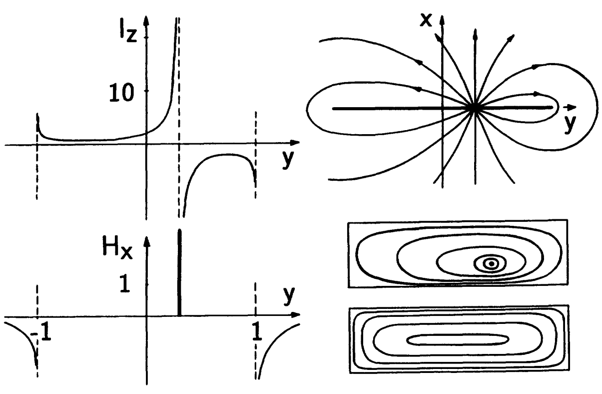

FIG. 1. Right: the current density $I_z(y)$ which generates a perpendicular surface field $H_x(y) = \delta(y - u)$ in an ideally shielding superconducting strip of half-width $a = 1$ for $u = 0.3$, cf. the integrand of Eq. (4). Left, top: the magnetic field lines around this strip. Left, bottom: stream lines of surface current $\mathbf{I}(y, z)$ [lines $g(y, z)$ =const with g from (16)] for $f(y, z) = \delta(y - y')\delta(z - z')$ and for $f(y, z)$ =const ($f = H_x$ is the perpendicular field caused at the surface by the currents).

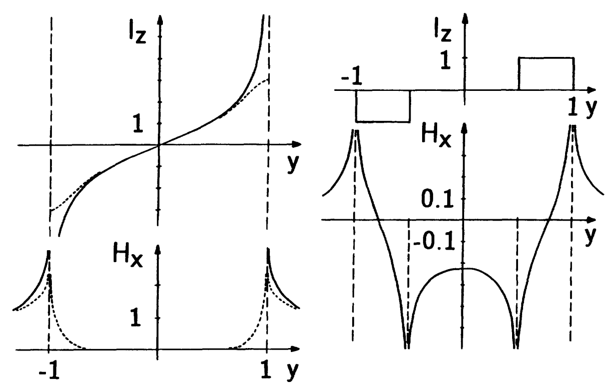

FIG. 2. Left: shielding current $I_z(y)$ (5) and perpendicular magnetic field $H_x(y) + H_a$ at the surface of an ideally shielding infinite superconducting strip. Half-width $a = 1$, shielded field $H_a = 1$. The dotted lines indicate the situation where part of the field has penetrated diffusively. Right: the perpendicular field component $H_x(y)$ (9) generated by a constant current density $|I_z(y)| = J_c d = 1$ which has penetrated half-way into a superconducting strip ($a = 1$, $\rho = \frac{1}{2}$).

the current density I becomes finite everywhere, and the perpendicular field generated by this current changes sign close to the edge. This means an overshooting $H_\perp > H_a$ inside the edge. Outside the edge, such logarithmic overshooting always occurs if $I \neq 0$ at the edge.

Next I consider the current $\mathbf{I}(y,z)$ and surface field $\mathbf{H}(\pm\epsilon, y, z)$ of a flat conductor of arbitrary shape, with area A and extension $\gg$ thickness $d = 2\epsilon$. The current density $\mathbf{J}$ causes a magnetization $\mathbf{M}$ with $\mathbf{J} = \mathrm{curl}\mathbf{M}$. Integrated over the thickness d this gives $\mathbf{I}(y,z) = \mathrm{curl}[g(y,z)\hat{\mathbf{x}}]$ with $g(y,z) = \hat{\mathbf{x}} \int M(x,y,z)\,dx$, cf. (1). The field component parallel to the specimen surface is

$$H_y = \tfrac{1}{2} I_z \,\mathrm{sgn}\, x, \quad H_z = -\tfrac{1}{2} I_y \,\mathrm{sgn}\, x, \tag{10}$$

or $\mathbf{H}_\parallel(x,y,z) = -\frac{1}{2}\nabla g(y,z)\,\mathrm{sgn}\,x$ for $|x| = \epsilon \to 0$. Since outside the specimen $\mathrm{curl}\mathbf{H} = \mathbf{0}$, one may write $\mathbf{H}(x,y,z) = -\nabla\phi(x,y,z)$. Writing $\mathbf{r} = (0,y,z)$ and $\mathbf{s} = (0,u,v)$ one obtains for the scalar potential

$$\phi(x,\mathbf{r}) = \int_A \frac{x\, g(\mathbf{s})\, d^2s}{[x^2 + (\mathbf{r}-\mathbf{s})^2]^{3/2}}\, \frac{d^2s}{4\pi}. \tag{11}$$

For the perpendicular surface field this gives

$$\begin{aligned} H_x(\pm\epsilon, \mathbf{r}) &= -\partial\phi(x,\mathbf{r})/\partial x \\ &= f(\mathbf{r}) \\ &= f(y,z) \\ &= \int_A \frac{2\epsilon^2 - (\mathbf{r}-\mathbf{s})^2}{[\epsilon^2 + (\mathbf{r}-\mathbf{s})^2]^{3/2}}\, g(\mathbf{s})\, \frac{d^2s}{4\pi}. \end{aligned} \tag{12}$$

Equation (12) may be transcribed into a form which allows us to determine $g(y,z)$ for a given $f(y,z)$. Using the representations for the two-dimensional delta function $(\epsilon/2\pi)/(\epsilon^2 + r^2)^{3/2} = \delta_2(\mathbf{r}) = (3\epsilon^3/2\pi)/(\epsilon^2+r^2)^{5/2}$ $(\epsilon \to +0)$ one may write (12) as

$$f(\mathbf{r}) = g(\mathbf{r})C(\mathbf{r}) - \int_A \frac{g(\mathbf{s}) - g(\mathbf{r})}{|\mathbf{r}-\mathbf{s}|^3}\, \frac{d^2s}{4\pi} \tag{13}$$

with

$$C(\mathbf{r}) = \frac{1}{4\pi} \int_{\infty - A} \frac{d^2\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}. \tag{14}$$

The integration in (14) is over the infinite area outside the specimen. For a rectangular specimen with $|y| \le a$ and $|z| \le b$, $C(y,z)$ has four terms

$$C(y,z) = \frac{1}{4\pi} \sum_{p,q} [(a - py)^{-2} + (b - qz)^{-2}]^{1/2} \tag{15}$$

with $p,q = \pm 1$. For the long strip above ($b \gg a$, $b \gg z$), (15) gives $C(y,z) = C(y) = (a/2\pi)/(a^2 - y^2)^{1/2}$, and (5) gives $g(y) = 2H_a(a^2 - y^2)^{1/2}$. From (13) one gets

$$g(\mathbf{r}) = \frac{1}{C(\mathbf{r})} \left(f(\mathbf{r}) - \int_A \frac{g(\mathbf{s}) - g(\mathbf{r})}{|\mathbf{r}-\mathbf{s}|^3}\, \frac{d^2s}{4\pi} \right). \tag{16}$$

The integral equation (16) is the central result of this paper. The current $\mathbf{I}(y,z) = \mathrm{curl}[g(y,z)\hat{\mathbf{x}}] = (0; \partial g/\partial z; -\partial g/\partial y)$ which causes a given perpendicular field $H_x = f(y,z)$ is obtained by iterating Eq. (16) with $f(y,z)$ inserted, starting with $g = 0$. This iteration may be stabilized by replacing its original form $g_{n+1} = F(g_n)$ by $g_{n+1} = pF(g_n) + (1-p)g_n$ with $p < 1$. Substitutions like $\tilde{y}(y) = \sin(\pi y/2a)$ are useful since at the edges $g \propto (a - |y|)^{1/2} \propto 1 - |\tilde{y}|$, etc. In general, $g(y,z)$ vanishes along the edges since $C(y,z) = \infty$ there. The current flows along the lines $g(y,z) = \mathrm{const}$ since $I = |\nabla g|$. $\int g(y,z)\,dy\,dz$ is the total magnetic moment which determines the torque and force on the specimen.

If one has to evaluate a set of experimental data $H_{xi} = f_i(y,z)$ with $i = 1,2,\ldots$, corresponding, e.g., to different times, one can speed up the determination of $g_i(y,z)$ if one calculates first an integral kernel K by iteration of (16) with $f(y,z) = \delta(y-y')\delta(z-z')$ inserted. This yields $K(y,z;y',z') = g(y,z)$ and finally

$$g_i(\mathbf{r}) = \int_A K(\mathbf{r};\mathbf{r}')\, f_i(\mathbf{r}')\, d^2\mathbf{r}' \tag{17}$$

from which $\mathbf{I}_i(y,z)$ is obtained. It should be remarked that in general it is *not* allowed to take the derivative under the integral sign since $K(\mathbf{r};\mathbf{r}')$ may have poles, cf. Eqs. (6)–(8). The integrand of (16) is, however, well behaved at the point $\mathbf{r} = \mathbf{s}$.

The kernel $K(\mathbf{r};\mathbf{r}')$ is required in the theory of superconducting vibrating reeds,[14–19] cf. Eq. (1) of Ref. 15. The magnetic energy (stray field energy) U of the currents $\mathbf{I}(y,z)$ in a flat conductor is

$$U = \mu_0 \int H_x(\mathbf{r})\, K(\mathbf{r};\mathbf{r}')\, H_x(\mathbf{r}')\, d^2\mathbf{r}\, d^2\mathbf{r}'. \tag{18}$$

Equation (18) gives also the magnetic energy of an arbitrarily curved flat superconductor with $d \gg \lambda$ in the ideal Meissner state if $H_x(\mathbf{r}) = H_a\phi(\mathbf{r})$ is inserted where ϕ is the (small) local tilt angle with respect to the applied longitudinal magnetic field H_a. In general, Eq. (18) applies to flat type-II superconductors if their reversible magnetization is small and $\phi(y,z)$ is the tilt angle of the flux lines at the surface with respect to H_a. If the flux lines are strongly pinned and sufficiently long, then ϕ is the tilt angle of the specimen.[14] For flat, long, narrow strips or reeds, (18) may be simplified since now the kernel depends on z and z' only, $\overline{K}(z;z') = \int K(y,z;y',z')\,dy\,dy'$, cf. Fig. 5 of Ref. 17. From (5) one can show that for a long strip one has $\int K(y,z;y',z')\,dy'\,dz' = (a^2 - y^2)^{1/2}$. As stated above, the infinities of the current $I \propto 1/\sqrt{\delta}$ at the edges require that $K \propto \sqrt{\delta}$ at all edges (δ is the distance from the edge). Finite thickness $d = 2\epsilon$ smears the singularities, $1/I \propto K \propto (\delta^2 + \epsilon^2)^{1/2}$ at the edges.

In conclusion, it was shown how the currents $\mathbf{I}(y,z)$ in a thin flat conductor or superconductor of arbitrary shape can be calculated from the perpendicular field component $H_x(y,z) = f(y,z)$ measured at the surface, e.g., by the Faraday effect. For a long strip the result is Eq. (4); else, Eq. (16) has to be solved by iteration. The resulting magnetization (or dipolar density) $g(y,z)$ yields the currents by taking derivatives, and the magnetic field $\mathbf{H} = -\nabla\phi$ outside the specimen by integration according to (11). The general integral kernel K which follows by iterating (16) with $f(y,z) = \delta(y-y')\delta(z-z')$ inserted, yields also the magnetic energy. With $f(y,z) =$

$-H_a$ =const inserted Eq. (16) applies to ideal screening. The resulting $g(y,z)$ then yields the surface screening currents of a flat ideal superconductor in a perpendicular field H_a, or of a superconductor tilted in a longitudinal field. The presented method allows also the exact calculation of finite-size corrections to the resonance frequency of vibrating thin superconductors, which were estimated in Refs. 15 and 17 for long narrow reeds. Equation (16) applies to flat conductors of arbitrary shape. The above equations will also be useful for treating the dynamics of linear and nonlinear flux penetrations into flat superconductors put into an oblique magnetic field; this topic has recently become of new interest after the observation of double peaks in the temperature dependent dissipation of vibrating superconductors.[19]

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