Even-parity spin-triplet superconductivity in disordered electronic systems

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A purely electronic mechanism for superconductivity in disordered Fermi systems is given. It is based on long-time-tail effects in electronic correlations, is strongest in two-dimensional systems, and leads to spin-triplet even-parity pairing. Estimates are given for the mean-field transition temperature to the superconducting state. The superconducting state is discussed in detail. In particular, the superconducting gap function, the tunneling density of states in the superconducting phase, and the low-temperature specific heat are computed. We compare and contrast these results with those for conventional superconductors. We also discuss which classes of materials might provide an experimental realization of the superconducting state discussed here.

I. INTRODUCTION

In recent years there has been substantial interest in purely electronic mechanisms for superconductivity (SC) and in unconventional SC states. The reason is in large part due to high-temperature superconductivity where it is suspected that models closely related to the single-band repulsive Hubbard model contain the relevant physics.[1] For these systems it is currently unclear what the pairing mechanism for the SC is and whether they indeed represent a new type of SC with both a novel SC mechanism and state. Heavy-fermion systems are a better understood example of systems with unconventional SC. In some of these the pairing occurs in a spin-triplet state that is caused by an exchange of magnetic excitations.[2] Similarly, in ^{3}He the Cooper pairs are in a spin-triplet state and the effective attraction between helium atoms is, at least in part, of magnetic nature.[3]

In a recent work, we have presented an electronic mechanism for SC in a disordered electronic system.[4] To describe this work we first recall some general features of the effective interaction that leads to a SC instability in a BCS-like theory. Let us consider the particle-particle (p-p) vertex, which we denote by $K(12,34)$. The indices $1 \equiv (x_1, \omega_1)$, etc., comprise space and Matsubara frequency. According to the Pauli principle, the total effective interaction, including the spin degrees of freedom, must be antisymmetric under interchanges of two particles. The spin state of the pairs therefore puts restrictions on the symmetry of the vertex K. If the two quasiparticles (QP's) are in a spin singlet, the effective interaction $K(12,34)$ must be symmetric under interchanges of 1 and 2 or 3 and 4. If the QP's are in a spin-triplet state, then K must be antisymmetric. For singlet pairing, one can thus use a constant model potential. For triplet pairing, the Pauli principle is usually enforced by means of a frequency-independent interaction with an odd relative angular momentum.[5] The triplet superconductivity that results from such an odd-parity interaction is strongly suppressed by nonmagnetic impurities. This feature is in contrast to the case of conventional s-wave superconductivity, and it is believed to be one of the reasons why triplet superconductivity is not more prevalent. Alternatively, one can satisfy the antisymmetry requirement by considering a K which is even under interchange of $\mathbf{x}_1$ and $\mathbf{x}_2$, and odd under interchange of ω_1 and ω_2. However, since the interaction will then be an odd function of frequency, in a three-dimensional system this leads to extra powers of temperature in the T_c equation which also suppresses the triplet superconducting state.[6]

In the first part of the present paper we review and explain in detail our previous work[4] where we showed that the latter conclusion above can be avoided in the case of a two-dimensional (2D), disordered system with a strong QP interaction. In such a system we find a mechanism for triplet even-parity pairing which is of purely electronic origin. The relevant temperature scale should therefore be related to the Fermi temperature. A simple physical picture of our mechanism is as follows. Consider a Fermi liquid with a pointlike, static, repulsive interaction. We characterize the interaction with respect to the transferred momentum and the total spin. The triplet and singlet interaction amplitudes in the particle-hole channel we denote by K_t and K_s, respectively. The singlet amplitude in the p-p (Cooper) channel we denote by K_c. The symmetry arguments given above show that there is no triplet amplitude in the p-p channel due to the Pauli principle. A QP will charge polarize its environment by means of K_s (by repelling the like charges of the other QP's), and it will spin polarize it by means of K_t (by repelling unlike spins, the short-ranged nature of the interaction will not allow it to repel like spins because of the Pauli principle). In a clean Fermi liquid these polarization clouds are short lived. They decay exponentially with time and therefore essentially stay with the QP creating them. This gives rise to the renormalization of the effective mass, spin susceptibility, etc., in Fermi-liquid theory. In the presence of disorder, however, the

decay of the polarization clouds is much slower. The corresponding correlation functions display a long-time tail and go for large times like $t^{-D/2}$.[7] The polarization cloud will therefore still be present after the QP that caused it has diffused away. A second QP will then see the spin and charge fluctuations left behind by the first one. It will be attracted by the charge-polarization cloud regardless of its spin. It will be attracted or repelled by the spin-polarization cloud if the two QP's form a relative spin singlet or spin triplet, respectively. We conclude that, in the *p-p* channel, there should be an attractive triplet interaction amplitude which increases with increasing disorder, K_t and K_s. Notice that, in the triplet channel, there is no bare interaction, so the net effective interaction will always be attractive. The singlet amplitude will acquire a repulsive component due to K_t, an attractive one due to K_s, and, in addition, a renormalization of the bare repulsion K_c. Its net effect will therefore be repulsive. The attractive triplet interaction will be particularly effective in 2D. A $t^{-\alpha}$ long-time tail corresponds to an $\omega^{\alpha-1}$ singularity in frequency space. For $D=2$ we have $\alpha=1$, and the effect should extend all the way to zero frequency or temperature. (Actually, the asymptotic limit $T\to 0$ or $\omega\to 0$ in $D=2$ is presently not understood, but the preasymptotic behavior can be worked out. We will come back to this.) This is a manifestation of the strong fluctuations present in a low-dimensional system. We therefore expect a generic 2D, disordered Fermi liquid to be an even-parity spin-triplet superconductor at sufficiently low temperatures.

The argument given above for the SC instability suggests a superconducting state due to the peculiar frequency dependence and symmetry of the effective interaction. In the second part of this paper we discuss various features of this SC state in detail. The generic interesting feature of the even-parity spin-triplet SC state is that the zero-temperature gap function [and the single-particle density of states (DOS)] vanishes continuously as the frequency, ω, tends to zero. Among other things, this implies that the experimental tunneling DOS and optical conductivity are nonzero for $\omega\neq 0$. In this sense, the SC is "gapless," in contrast to conventional superconductors. In addition, the SC state will have thermal properties which vary as a power law rather than exponentially with temperature.

We stress that many aspects of the SC state discussed here are independent of the particular pairing potential which we will derive, and, consequently, many of our results can be expected to be generic features of an even-parity spin-triplet SC state.

The plan of this paper is as follows. In Sec. II we first review the field-theoretic description of disordered interacting electronic systems, and give the Gaussian propagators for the field theory. We then do a one-loop renormalization of the theory and show that the combination of disorder and electron-electron interaction induces an attractive interaction in the particle-particle (Cooper) triplet channel. As mentioned above, the induced attractive interaction naturally separates into a contribution due to charge fluctuations and into a contribution due to spin fluctuations. In Sec. III, the mean-field gap equation for the even-parity spin-triplet SC state is derived. In Sec. IV (V), the charge (spin) fluctuation contribution to the effective potential is considered. We first derive an expression for the critical temperature, T_c, for the mean-field SC phase transition. We then solve the zero-temperature gap equation and discuss the low-temperature behavior of the single-particle density of states and the specific heat. In Sec. VI, we discuss our results and some open problems related to them. We also discuss their experimental relevance and suggest some experiments to test our predictions. Let us finally mention that readers who find the physical argument for the existence of the triplet pairing given above convincing, and who are not interested in the technical derivation of the pairing potential, can skip Secs. I and III A and go directly to Sec. III B.

II. THE EFFECTIVE INTERACTION

In this section we first review the basic field-theoretic description of disordered interacting electronic systems, and give the Gaussian propagator of the field theory. We then do a one-loop renormalization of the theory and show that the renormalization process generates an attractive interaction in the particle-particle triplet channel. Note that, once the pairing potential has been established, the field-theoretic method is not needed and conventional SC theory can be used.

A. The model

For an arbitrary Fermi system, the partition function can be written as[8]

$$Z=\int D\bar{\psi}\,D\psi\exp(S)\ , \tag{2.1a}$$

where the functional integration measure is with respect to anticommuting Grassman fields $\bar{\psi}$ and ψ, and S is the action

$$S=\int_0^\beta d\tau\int d\mathbf{x}\,\bar{\psi}^i(\mathbf{x},\tau)\partial_\tau\psi^i(\mathbf{x},\tau)-\int_0^\beta d\tau\,H'(\tau)\ . \tag{2.1b}$$

Here $H'(\tau)$ is the Hamiltonian in imaginary-time representation, $\beta=T^{-1}$ is the inverse temperature, $i(=1,2)$ denotes spin labels, and summation over repeated spin indices is implied. Throughout the paper we use units such that $\hbar=k_B=1$, except where otherwise stated. Our basic model is an electron fluid moving in a static random potential, $V(\mathbf{x})$,

$$\begin{aligned}H'(\tau)=\int d\mathbf{x}\,\Big\{&\frac{1}{2m}\nabla\bar{\psi}^i(\mathbf{x},\tau)\cdot\nabla\psi^i(\mathbf{x},\tau)\\&+[V(\mathbf{x})-\mu]\bar{\psi}^i(\mathbf{x},\tau)\psi^i(\mathbf{x},\tau)\Big\}\\&+\frac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,u(\mathbf{x}-\mathbf{y})\bar{\psi}^i(\mathbf{x},\tau)\bar{\psi}^j(\mathbf{y},\tau)\\&\times\psi^j(\mathbf{y},\tau)\psi^i(\mathbf{x},\tau)\ .\end{aligned} \tag{2.2a}$$

Here m is the particle mass, μ is the chemical potential, and $u(\mathbf{x}-\mathbf{y})$ is the electron-electron interaction potential. We assume that $V(\mathbf{x})$ is δ correlated and obeys a

Gaussian distribution with second moment

$$\langle V(\mathbf{x})V(\mathbf{y})\rangle=\frac{1}{2\pi N_F\tau}\delta(\mathbf{x}-\mathbf{y}) , \tag{2.2b}$$

where the angular brackets denote the disorder average, N_F is the bare density of states per spin at the Fermi energy, and τ is the bare elastic mean free time.

All physical (thermodynamic and transport) quantities can be obtained from Eq. (2.1) by adding appropriate source terms to the action. The quenched disorder averages are conveniently performed by means of the replica trick.[9] One introduces N replicas of the system

$$Z^N=\int D\bar{\psi}D\psi\exp\left[\sum_{\alpha=1}^{N}S^\alpha\right] , \tag{2.3a}$$

with

$$S^\alpha=\sum_n\int d\mathbf{x}\,\bar{\psi}_n^{\alpha,i}(\mathbf{x})\left[ip_n+\frac{\nabla^2}{2m}+\mu-V(\mathbf{x})\right]\psi_n^{\alpha,i}(\mathbf{x})$$
$$-\frac{T}{2}\sum_{n_1n_2n_3}\int d\mathbf{x}\,d\mathbf{y}\,u(\mathbf{x}-\mathbf{y})\bar{\psi}_{n_1}^{\alpha,i}(\mathbf{x})\bar{\psi}_{n_2}^{\alpha,j}(\mathbf{y})$$
$$\times\psi_{n_3}^{\alpha,j}(\mathbf{y})\psi_{n_1+n_2-n_3}^{\alpha,i}(\mathbf{x}) . \tag{2.3b}$$

Here $p_n=\pi T(2n+1)$, $n=0,\pm1,\ldots$, is a fermionic Matsubara frequency. After calculations, the limit $N\to0$ is considered.

It has been shown how the field theory for $\langle Z^N\rangle$ can be mapped onto a nonlinear σ-model-like field theory.[10,11] The basic idea is to assume that all of the relevant physics can be expressed in terms of long-wavelength and low-frequency fluctuations of the number density, the spin density, and the single-particle spectral density. Technically this is achieved by repeatedly making long-wavelength approximations and by introducing composite variables that are related to the above fluctuations.

The Hamiltonian, $H[Q]$, for the composite variable-field theory is

$$H[Q]=\frac{\pi N_F}{4\tau}\mathrm{tr}Q^2-\mathrm{tr}\ln\hat{G}^{-1}+H_{\mathrm{int}}[Q] , \tag{2.4a}$$

where

$$\hat{G}^{-1}=i\hat{\Omega}+\left[\frac{\Delta}{2m}+\mu\right]\hat{I}+\frac{i}{2\tau}\hat{Q}\equiv\hat{G}^{(0)-1}+\frac{i}{2\tau}\hat{Q} . \tag{2.4b}$$

Here $\hat{I}$ is the unit matrix and a caret denotes a matrix quantity. In the long-wavelength limit, Eq. (2.4a) reduces to

$$H[Q]=\frac{1}{2G}\int d\mathbf{x}\{\mathrm{Tr}[\nabla Q(\mathbf{x})]^2-4HG\,\mathrm{Tr}[\Omega Q(\mathbf{x})]\}$$
$$+H_{\mathrm{int}}[Q] . \tag{2.5}$$

Here $G=4/\pi\sigma$, with σ the bare conductivity, is the disorder parameter and H is a frequency coupling parameter whose bare value is $\pi N_F/2$. Q in Eqs. (2.4) and (2.5) is an infinite matrix whose matrix elements $Q_{nm}^{\alpha\beta}$ are 4×4 matrices (spin quaternions) which depend on replica indices α,β and on Matsubara frequency indices n,m. Tr in Eq. (2.5) denotes a trace over all discrete degrees of freedom while tr in Eq. (2.4) denotes a trace over both discrete and continuous degrees of freedom. The matrix Q in Eq. (2.5) is subject to the constraints

$$Q^2=1,\quad \mathrm{Tr}Q=0,\quad Q^+=C^TQ^TC=Q . \tag{2.6}$$

The last condition is the hermiticity and charge-conjugation requirement. The matrix C has elements $\tau_1\otimes\sigma_2$. Here and in the following we denote by $\{\tau_r\}$ $(r=0,1,2,3)$ the quaternion basis and by $\{\sigma_i\}$ $(i=0,1,2,3)$ the Pauli matrices. $\Omega_{nm}^{\alpha\beta}=\delta_{nm}\delta_{\alpha\beta}\tau_0\otimes\sigma_0\omega_n$ with $\omega_n=2\pi Tn$ is a bosonic frequency matrix. If we expand Q in the spin-quaternion basis,

$$Q_{nm}^{\alpha\beta}=\sum_{r,i}{}_r^iQ_{mn}^{\alpha\beta}\tau_r\otimes\sigma_i , \tag{2.7}$$

we can use Eq. (2.6) to derive symmetry properties of ${}_r^iQ_{nm}^{\alpha\beta}$. In Eq. (2.7), $r=0,3$ and $r=1,2$ denotes the diffusion or particle-hole and the Cooperon or particle-particle degrees of freedom, respectively, while $i=0$ denotes spin singlet and $i=1,2,3$ denotes spin triplet. The diffusion degrees of freedom satisfy

$${}_r^iQ_{nm}^{\alpha\beta}=(-)^rS_i\,{}_r^iQ_{mn}^{\beta\alpha}\quad(r=0,3) , \tag{2.8a}$$

with $S_0=1$, $S_{1,2,3}=-1$. For the Cooperon degrees of freedom we have

$${}_r^iQ_{nm}^{\alpha\beta}=S_iQ_{mn}^{\beta\alpha}\quad(r=1,2) . \tag{2.8b}$$

These symmetry relations are a direct consequence of the commutation properties of the underlying Grassman fields and reflect the Pauli principle. We will come back to them later.

The interaction part of the Hamiltonian consist of three parts,[12]

$$H_{\mathrm{int}}[Q]=\sum_\nu\left[\frac{\pi TK_\nu}{2}\right]\int d\mathbf{x}[Q(\mathbf{x})\cdot Q(\mathbf{x})]_\nu$$
$$(\nu=s,t,c) , \tag{2.9}$$

where the K_ν are the three bare interaction amplitudes discussed in Sec. I. The "products" $Q\cdot Q$ can be written

$$[Q\cdot Q]_s=\sum_{\substack{n_1n_2\\n_3n_4}}\delta_{n_1+n_3,n_2+n_4}$$
$$\times\sum_\alpha[\mathrm{Tr}(\tau_+\otimes\sigma_0Q_{n_1n_2}^{\alpha\alpha})$$
$$\times\mathrm{Tr}(\tau_+\otimes\sigma_0Q_{n_3n_4}^{\alpha\alpha})+(\tau_+\to\tau_-)] , \tag{2.10a}$$

$$[Q\cdot Q]_t=\sum_{\substack{n_1n_2\\n_3n_4}}\delta_{n_1+n_3,n_2+n_4}$$
$$\times\sum_\alpha\sum_{i=1}^{3}[\mathrm{Tr}(\tau_+\otimes\sigma_iQ_{n_1n_2}^{\alpha\alpha})$$
$$\times\mathrm{Tr}(\tau_+\otimes\sigma_iQ_{n_3n_4}^{\alpha\alpha})$$
$$+(\tau_+\to\tau_-)] , \tag{2.10b}$$

$$[Q\cdot Q]_c=\sum_{\substack{n_1 n_2\\ n_3 n_4}}\delta_{n_1+n_2,n_3+n_4}\times\sum_{\alpha}[\mathrm{Tr}(\tau_+\otimes\sigma_0 Q^{\alpha\alpha}_{n_1 n_2}\cdot\tau_-\otimes\sigma_0 Q^{\alpha\alpha}_{n_3 n_4})+(\tau_+\leftrightarrow\tau_-)]\ , \tag{2.10c}$$

where $\tau_\pm=(\tau_0\pm i\tau_3)/2$. From Eq. (2.10c) we see explicitly that the symmetry given by Eq. (2.8b) (i.e., the Pauli principle) does not allow for a bare p-p triplet interaction amplitude (if the bare interaction potential is frequency independent).

In order to develop a loop expansion, we use the same parametrization of the matrix Q as before,[13]

$$Q_{nm}=\begin{array}{c|cc} & m\geq 0 & m<0\\ \hline \left.\begin{matrix}(1-qq^+)^{1/2}\\ q^+\end{matrix}\right. & \begin{matrix}q\\ -(1-q^+q)^{1/2}\end{matrix} & \left.\begin{matrix}\\ \end{matrix}\right|_{n<0}\end{array}\Bigg|^{n\geq 0}\ . \tag{2.11}$$

This parametrization eliminates the constraints of the σ model given by Eq. (2.6). For future reference we note, however, that the expansion coefficients ${}^r_i q^{\alpha\beta}_{nm}$ of the matrix q do not obey obvious symmetry relations in contrast to ${}^r_i Q^{\alpha\beta}_{nm}$.

B. The Gaussian theory

To obtain the Gaussian theory, we expand Eq. (2.5) in powers of q using Eq. (2.11),

$$H[Q]=H^{(2)}+H^{(3)}+H^{(4)}+\cdots\ , \tag{2.12}$$

where $H^{(k)}$ is of order q^k. We first concentrate on the Gaussian part of the effective Hamiltonian:

$$H^{(2)}[q]=\frac{4}{G}\int_{\mathrm{p}}\sum_{r,i}\sum_{\substack{1,2\\3,4}}{}^r_i q_{12}(\mathbf{p})\,{}^r_i M_{12,34}(p)\,{}^r_i q_{34}(-\mathbf{p})\ , \tag{2.13a}$$

where the basis of Eq. (2.7) has been used for the small q's also. In Eq. (2.13a), $\int_{\mathrm{p}}=\int d\mathbf{p}/(2\pi)^d$ and $1=(n_1,\alpha_1)$, etc. The matrix M in Eq. (2.13a) is

$${}^{0,3}_{i}M_{12,34}(p)=S_{\nu_i}\delta_{1-2,3-4}\{\delta_{13}\delta_{24}[p^2+GH(\omega_{n_1}-\omega_{n_2})]+\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}2\pi TGK_{\nu_i}\}\ , \tag{2.13b}$$

with $\nu_0=s$, $\nu_{1,2,3}=t$, $S_s=1$, and $S_t=-1$. For the Cooperon degrees of freedom,

$${}^{1,2}_{i}M_{12,34}(p)=-S_{\nu_i}\delta_{1+2,3+4}\{\delta_{13}\delta_{24}[p^2+GH(\omega_{n_1}-\omega_{n_2})]+\delta_{i0}\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}2\pi TGK_c\}\ . \tag{2.13c}$$

The matrix M is easily inverted. The inverse, which determines the Gaussian propagators, reads

$${}^{0,3}_{i}M^{-1}_{12,34}(p)=S_{\nu_i}\delta_{1-2,3-4}\left\{\delta_{13}\delta_{24}D_{n_1-n_2}(p)+\frac{\delta_{\alpha_1\alpha_2}}{n_1-n_2}\Delta D^{\nu_i}_{n_1-n_2}(p)\right\} \tag{2.14a}$$

and

$${}^{1,2}_{i}M^{-1}_{12,34}(p)=-S_{\nu_i}\delta_{1+2,3+4}\left\{\delta_{13}\delta_{24}D_{n_1-n_2}(p)-\delta_{i0}D_{n_1-n_2}(p)D_{n_3-n_4}(p)\times\frac{k_c}{1+k_c f_{n_1+n_2}(p)}\right\}\ . \tag{2.14b}$$

Here we have introduced the diffusion propagator

$$D_n(p)=[p^2+GH\omega_n]^{-1}\ , \tag{2.15a}$$

and the singlet and triplet propagators

$$D^{s,t}_n(p)=[p^2+G(H+K_{s,t})\omega_n]^{-1}\ . \tag{2.15b}$$

In Eqs. (2.14), $\Delta D^{s,t}_n=D^{s,t}_n-D_n$, $k_c=G2\pi TK_c$, and

$$f_n(p)=\sum_{n_1 n_2}\delta_{n_1+n_2,n}\theta(n_1-n_2)D_{n_1-n_2}(p) \tag{2.15c}$$

with $\theta(x)$ the step function. We note that, with unrestricted frequency sums, $f_n(p)$ diverges logarithmically in the ultraviolet (UV) for all p and n and in the infrared (IR) for $p,n\to 0$. The UV divergence is due to our unphysical treatment of the high-frequency behavior and should be cut off by a large cutoff, N, while the low-frequency singularity is the usual divergence one obtains in the p-p channel [for $K_c<0$, Eq. (2.14b) displays the usual Cooper instability]. These singularities are independent of dimensionality and any loop expansion and should not be renormalized. In addition, we are, in general, interested only in logarithmic singularities that lead to infinities and not zero. Consequently, at the end of our calculations we let $f_n(p)\to\infty$ and thereby neglect terms that are logarithmically small. We must not, however, neglect the second term in Eq. (2.14b) immediately, since the calculation will produce additional factors of $f_n(p)$ that appear in the numerator. This renormalization procedure is different from that used in the previous literature.[12]

C. One-loop renormalization

Our strategy is now as follows. We use Eq. (2.12) to obtain the contribution to $H[Q]$ of order q^3, q^4, etc.

This leads to a loop or disorder expansion. We can then use standard techniques to obtain renormalizations of the coupling constants G, H, $K_{s,t,c}$. The calculation proceeds along the lines of our earlier work[14] and is straightforward. The result shows that, in addition to providing renormalizations of the bare coupling constants, the perturbation theory generates a new term in the triplet Cooperon channel. It is the purpose of this paper to investigate this new disorder-generated term. For completeness, and because we will want to use the results elsewhere, we also give the one-loop renormalization of K_c. We obtain,

$$ {}^{1,2}_{i}M^{\text{1-loop}}_{12,34}(p)=-S_{\nu_i}\delta_{1+2,3+4}\{\delta_{13}\delta_{24}[p^2(1+GR_G)+GH(\omega_{n_1}-\omega_{n_2})(1+GR_H)]+\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}2\pi TGK^{c,\nu_i}_{n_1n_2,n_3n_4}\} , \tag{2.16a} $$

where R_G and R_H determine the renormalizations of G and H which have been given elsewhere,[14] and

$$ K^{c,s}_{n_1n_2,n_3n_4}=K_c-\frac{G}{8}\int_q\{K_sD^s_{n_3-n_2}(q)-3K_tD^t_{n_3-n_2}(q)+K_sD_{n_3-n_2}(q)[1-GK_s|\omega_{n_2}-\omega_{n_4}|D^s_{|n_2-n_4|}(q)] $$
$$ -3K_tD_{n_3-n_2}(q)[1-GK_t|\omega_{n_2}-\omega_{n_4}|D^t_{|n_2-n_4|}(q)]\} \tag{2.16b} $$

and

$$ K^{c,t}_{n_1n_2,n_3n_4}=\frac{G^2}{8}\int_q\{-(\omega_{n_3}-\omega_{n_2})D_{n_3-n_2}(q)[(K_s)^2D^s_{n_3-n_2}(q)+(K_t)^2D^t_{n_3-n_2}(q)] $$
$$ +|\omega_{n_2}-\omega_{n_4}|D_{n_3-n_2}(q)[(K_s)^2D^s_{|n_2-n_4|}(q)+(K_t)^2D^t_{|n_2-n_4|}(q)]\} . \tag{2.16c} $$

In giving these results we have set the external momentum $\mathbf{p}$ equal to zero. Power counting shows that the integrals in Eqs. (2.16b) and (2.16c) go as $n^{(D-2)/2}$, where n is some frequency index. This is the long-time-tail behavior mentioned in Sec. I. Finally, we mention that we have derived identical results using a different parametrization of $Q^{\alpha\beta}_{n_1n_2}$.[15]

The frequency dependence of $K^{c,s,t}$ deserves some comment. $K^{c,t}$ ($K^{c,s}$) as given by Eqs. (2.16) is not antisymmetric (symmetric) under interchange of n_1 and n_2 or n_3 and n_4. This does *not* violate the Pauli principle since Eqs. (2.16) have been derived as couplings in the q formulation of the model only. In the Q formulation, the proper symmetry can be built in by symmetrization (cf., below).

As already mentioned, we focus here on $K^{c,t}$ in $D=2$. Integration gives

$$ K_{n_1n_2n_3n_4}=\frac{-\bar{G}}{16}\left\{K_sg_{\gamma_s}\left[\left|\frac{n_2-n_4}{n_3-n_2}\right|\right]+K_tg_{\gamma_t}\left[\left|\frac{n_2-n_4}{n_3-n_2}\right|\right]\right\} $$
$$ ={}^sK^{c,t}_{n_1n_2,n_3n_4}+{}^tK^{c,t}_{n_1n_2,n_3n_4} , \tag{2.17a} $$

where $\gamma_{s,t}=K_{s,t}/H$, $\bar{G}=GS_d/(2\pi)^d$ with S_d the surface area of a d-dimensional unit sphere, and,

$$ g_\gamma(x)=\frac{1}{x(1+\gamma)-1}[(x-1)\ln(1+\gamma)-\gamma x\ln x] . \tag{2.17b} $$

The second equality in Eq. (2.17a) gives $K^{c,t}$ as a sum of a charge or singlet fluctuation contribution and a spin or triplet fluctuation contribution.

For real electronic systems interacting through screened Coulomb interactions, a compressibility sum rule fixes $\gamma_s=-1$ and, consequently, g_{γ_s} in Eq. (2.17) does not exist. To correctly evaluate Eq. (2.16c) for the long-range case, one should (in D^s) replace $H+K_s$ by[16]

$$ H+K_s\to\frac{H}{1+F_0^s}\frac{(q/\kappa)^{D-1}}{1+(q/\kappa)^{D-1}} , \tag{2.18} $$

where F_0^s is a Fermi-liquid parameter and $\kappa^{d-1}=c_de^2(\partial n/\partial\mu)$ gives the inverse screening length, κ, with $c_2=2\pi$, $c_3=4\pi$, and $(\partial n/\partial\mu)$ is the thermodynamic compressibility. For this case, the singlet contribution to $K^{c,t}$ in two dimensions to leading logarithmic accuracy is

$$ {}^sK^{c,t}_{n_1n_2,n_3n_4}=\frac{\bar{G}H}{16}\left\{F\left[\frac{GH}{\bar{\kappa}^2}|\omega_{n_3}-\omega_{n_2}|\right]-\frac{|\omega_{n_2}-\omega_{n_4}|}{|\omega_{n_3}-\omega_{n_2}|}\times F\left[\frac{GH}{\bar{\kappa}^2}\frac{|\omega_{n_2}-\omega_{n_4}|^2}{|\omega_{n_3}-\omega_{n_2}|}\right]\right\} , \tag{2.19a} $$

where

$$ F(x)=\frac{1}{1+x}[\ln x-\pi x^{1/2}] \tag{2.19b} $$

and

$$ \bar{\kappa}=\kappa(1+F_0^s) . \tag{2.19c} $$

III. THE GAP EQUATION

In the first part of this section we discuss some general features of the saddle-point structure of the field theory

presented in Sec. II. We point out some dubious points in Finkelshtein's treatment[11] and we discuss how to properly solve the problem. We then derive the imaginary axis gap equation for the triplet even-parity SC state. The gap equation is then transformed to the real frequency axis and we comment on the causality and symmetry properties of the new SC state.

A. Saddle-point solutions of the field theory

Let us first discuss the saddle point (SP) solution of the field theory given by Eq. (2.4) with Eqs. (2.9) and (2.10). The SP condition is

$$\frac{\delta H[Q]}{\delta {}^r_i Q^{\alpha\beta}_{nm}(\mathbf{x})}=0 \ . \tag{3.1}$$

It is obvious that the functional derivatives of the first and the third terms on the right-hand side of Eq. (2.4a) are proportional to Q. Differentiation of the second term yields

$$\frac{\delta}{\delta {}^r_i Q^{\alpha\beta}_{nm}(\mathbf{x})}\mathrm{tr}\ln\hat{G}^{-1}=\frac{-i}{2\tau}\,{}^r_i G^{\beta\alpha}_{mn}(\mathbf{x})$$

$$=\frac{-i}{2\tau}\delta_{r0}\delta_{i0}G^{(0)\beta\alpha}_{mn}(\mathbf{x})+O(Q) \ . \tag{3.2a}$$

Here $G^{(0)}$ is the bare Green's function [cf., Eq. (2.4b)] and we have used the expansion

$$G^{\alpha\beta}_{nm}=\sum_{r,i}{}^r_i G^{\alpha\beta}_{nm}\tau_r\otimes\sigma_i \ . \tag{3.2b}$$

We see that the derivative of $\mathrm{tr}\ln\hat{G}^{-1}$ is the only term in the SP equation which can provide an inhomogeneity (viz., the bare Green's function). Consequently, if either $r\neq 0$ or $i\neq 0$, then the self-energy, which is given by $iQ/2\tau$, can be nonzero only if there is a broken symmetry. In that case, ${}^{0}_{1,2,3}Q\neq 0$ describes magnetism, and ${}^{1,2}_{0}Q\neq 0$ describes singlet superconductivity. Assuming the latter, one indeed recovers BCS-Gorkov theory[17] from Eq. (3.1). In the absence of a broken symmetry, only ${}^0_0Q^{\alpha\beta}_{nm}$ is relevant, and $[Q\cdot Q]_t$ and $[Q\cdot Q]_c$ in Eq. (2.9) can be neglected. In this case, Eq. (3.1) yields

$$\frac{\pi N_F}{2\tau}\,{}^0_0Q^{\alpha\beta}_{nm}$$

$$=\frac{i}{2\tau}\,{}^0_0G^{\alpha\beta}_{nm}(\mathbf{x},\mathbf{x})-\pi T K_s\delta_{\alpha\beta}\sum_{n_3n_4}\delta_{m+n_3,n+n_4}\,{}^0_0Q^{\alpha\alpha}_{n_3n_4} \ . \tag{3.3a}$$

In Finkelshtein's treatment of the SP problem, he neglects the last or interaction term in Eq. (3.3a). The Q's are composite variables for a product of annihilation and creation operators and the SP value is an expectation value of these operators. For the Fermi-liquid phase, this suggests an ansatz

$${}^r_i Q^{\alpha\alpha}_{nm}(\mathbf{x})=\delta_{r0}\delta_{i0}\delta_{\alpha\beta}\delta_{p_n,p_m}Q(p_n) \ . \tag{3.3b}$$

Equation (3.3b) in Eq. (3.3a) gives

$$Q(p_n)=\frac{i}{\pi N_F}\int_{\mathrm{p}}\left[ip_n+\xi_{\mathrm{p}}+\frac{i}{2\tau}Q(p_n)\right]^{-1}$$

$$-\frac{2\tau T}{N_F}K_s\sum_m Q(p_m) \tag{3.3c}$$

with $\xi_{\mathrm{p}}=\mu-p^2/2m$. To illustrate a point, we solve Eq. (3.3b) by summing over n using $\sum_n=2N$ with N as high-frequency cutoff. We obtain

$$Q(p_n)=\frac{i}{\pi N_F}\int_{\mathrm{p}}\left[ip_n+\xi_{\mathrm{p}}+\frac{i}{2\tau}Q(p_n)\right]^{-1}$$

$$-\frac{2\pi\tau T\bar{K}_s i}{N_F^2}\sum_m\int_{\mathrm{p}}\frac{1}{[ip_n+\xi_{\mathrm{p}}+(i/2\tau)Q(p_m)]} \ , \tag{3.4a}$$

where

$$\bar{K}_s=\frac{K_s}{[1+4\tau T(NK_s/N_F)]} \ . \tag{3.4b}$$

We next comment on the structure of Eqs. (3.4). First, the renormalization of K_s by Eq. (3.4b) is somewhat unphysical (although it is of no real consequence as we will see). To understand the meaning of this renormalization, we note that in the derivation of Eq. (2.4) similar terms proportional to $T\tau N$ have been neglected. As a consequence, only terms linear in K_ν were kept in the action. Finkelshtein argued that these terms are negligible in the limit $T\tau\ll 1$. To be consistent, one should then neglect the correction in the denominator of Eq. (3.4b) as well. In order to justify this, one must make the cutoff sufficiently small that $\tau TN\ll 1$. In general, this can be done within a long-wavelength theory designed to describe critical behavior because the precise values of the bare coupling constants are irrelevant and because cutoffs can be chosen arbitrarily small without changing the relevant physics. If one is not satisfied with this argument, one can go back to the original derivation and retain the neglected terms. We have performed this calculation and found that the disordered SP value of Q is then given by Eq. (3.4a) with $\bar{K}_s$ replaced by K_s. We conclude that Finkelshtein's approximation is unnecessary, and that the problem discussed above is an artifact. With $\bar{K}_s\to K_s$, one obtains the expected result for the SP in a disordered Fermi liquid: Eq. (3.4a) has the structure of a disordered Hartree-Fock theory for the electronic self-energy.

Now the SP problem for the triplet even-parity SC state is considered. It leads to the BCS-Gorkov theory for this SC state. We note that our results could be derived in a number of ways given the pairing potential. We begin by neglecting normal-state self-energies for simplicity and we consider the triplet even-parity SC state by replacing $H_{\mathrm{int}}[Q]$ in Eq. (2.4a) by

$$H_{\mathrm{int}}[Q]=\frac{\pi T}{2}\int d\mathbf{x}\sum_{\substack{n_1n_2\\ n_3n_4}}\tilde{K}^{c,t}_{n_1n_2,n_3n_4}\delta_{n_1+n_2,n_3+n_4}\sum_{i=1}^{3}\sum_{\alpha}[\mathrm{Tr}(\tau_+\otimes\sigma_i Q^{\alpha\alpha}_{n_1n_2}\cdot\tau_-\otimes\sigma_i Q^{\alpha\alpha}_{n_3n_4})+(\tau_+\leftrightarrow\tau_-)] \ , \tag{3.5a}$$

with $\tilde{K}^{c,t}$ given by

$$\tilde{K}^{c,t}_{n_1n_2,n_3n_4}=\tfrac{1}{4}[K^{c,t}_{n_1n_2,n_3n_4}-K^{c,t}_{n_2n_1,n_3n_4}-K^{c,t}_{n_1n_2,n_4n_3}+K^{c,t}_{n_2n_1,n_4n_3}] . \quad (3.5b)$$

Here we have antisymmetrized the potential for use in the Q formulation, and $K^{c,t}$ are given by Eqs. (2.16c), (2.17), and (2.19).

Assuming a SP diagonal in the replica indices the SP condition yields

$${}^0_0Q_{n_1n_2}=\frac{i}{\pi N_F}{}^0_0G_{n_1n_2} , \quad (3.6a)$$

and ($r=1,2$; $i=1,2,3$),

$${}^r_iQ_{n_1n_2}=\frac{i}{\pi N_F}{}^r_iG_{n_1n_2}+\frac{2\pi T\tau}{N_F}\sum_{n_3n_4}\tilde{K}^{c,t}_{n_1n_2,n_3n_4}\delta_{n_1+n_2,n_3+n_4}{}^r_iQ_{n_3n_4} . \quad (3.6b)$$

Using that ${}^0Q$ is related to a product of an annihilation and creation operator and that ${}^rQ$ ($r=1,2$) is related to a product of two annihilation or two creation operators, one looks for solutions of the form

$${}^0_0G_{n_1n_2}=\delta_{p_{n_1},p_{n_2}}G^0(p_{n_1}) , \quad (3.7a)$$

$${}^0_0Q_{n_1n_2}=\delta_{p_{n_1},p_{n_2}}Q^0(p_{n_1}) , \quad (3.7b)$$

$${}^r_iG_{n_1n_2}=\delta_{p_{n_1},-p_{n_2}}G^r_i(p_{n_1}) \quad (r=1,2) , \quad (3.7c)$$

$${}^r_iQ_{n_1,n_2}=\delta_{p_{n_1},-p_{n_2}}Q^r_i(p_{n_1}) \quad (r=1,2) . \quad (3.7d)$$

Equations (3.7) in Eqs. (3.6) give

$$Q^0(p_{n_1})=\frac{i}{\pi N_F}G^0(p_{n_1}) , \quad (3.8a)$$

$$Q^r_i(p_{n_1})=\frac{i}{\pi N_F}G^r_i(p_{n_1})+\frac{2\pi T\tau}{N_F}\sum_{n_3}K^{(t)}(p_{n_1},p_{n_3})Q^r_i(p_{n_3}) , \quad (3.8b)$$

where

$${}^r_iG_{nm}=\frac{1}{4}\left[\mathrm{tr}\tau^+_r\otimes\sigma^+_i\left\langle n\mathbf{x}\left|\left[i\hat{\Omega}+\mu+\frac{\Delta}{2m}+\frac{i}{2\tau}\hat{Q}\right]^{-1}\right|m\mathbf{x}\right\rangle\right] \quad (3.8c)$$

and

$$K^{(t)}(p_{n_1},p_{n_3})=\tilde{K}^{c,t}_{n_1,-n_1-1;n_3,-n_3-1} . \quad (3.8d)$$

Following the same arguments as given below Eq. (3.4), we can replace Eq. (3.8b) by

$$Q^r_i(p_{n_1})=\frac{i}{\pi N_F}G^r_i(p_{n_1})+\frac{2\pi T\tau}{N_F}\sum_{n_3}K^{(t)}(p_{n_1},p_{n_3})G^r_i(p_{n_3}) . \quad (3.8e)$$

In this paper we restrict ourselves to a triplet even-parity SC state where

$$Q^1_1\neq 0, \quad Q^1_2=Q^1_3=0 \quad (3.9a)$$

and

$$Q^2_i=0 . \quad (3.9b)$$

More complicated states are possible, but here we consider the above case for simplicity. Defining a gap function $\bar{\Delta}$ by

$$Q^1_1=\frac{2\tau}{i}\bar{\Delta} , \quad (3.10)$$

Eqs. (3.8) and (3.9) give

$$Q^0(p_{n_1})=\frac{1}{\pi N_F}\int_{\mathrm{p}}\frac{[p_{n_1}-i\xi_{\mathrm{p}}+(1/2\tau)Q^0(p_{n_1})]}{\{[p_{n_1}+(1/2\tau)Q^0(p_{n_1})]^2+\xi_{\mathrm{p}^2}+\bar{\Delta}^2(p_{n_1})\}} \quad (3.11a)$$

and

$$\bar{\Delta}(p_{n_1})=\frac{1}{2\pi N_F\tau}\int_{\mathrm{p}}\frac{\bar{\Delta}(p_{n_1})}{\{[p_{n_1}+(1/2\tau)Q^0(p_{n_1})]^2+\xi_{\mathrm{p}^2}+\bar{\Delta}^2(p_{n_1})\}}+\frac{\pi T}{N_F}\sum_{n_2}K(p_{n_1},p_{n_2})\int_{\mathrm{p}}\frac{\bar{\Delta}(p_{n_2})}{\{[p_{n_2}+(1/2\tau)Q^0(p_{n_2})]^2+\xi_{\mathrm{p}^2}+\bar{\Delta}^2(p_{n_2})\}} . \quad (3.11b)$$

In deriving Eqs. (3.11), we have used the symmetry conditions

$$Q^0(-p_{n_1})=-Q^0(p_{n_1}) , \tag{3.12a}$$

$$\bar{\Delta}(-p_{n_1})=-\bar{\Delta}(p_{n_1}) , \tag{3.12b}$$

and we have used that Q^0 can be taken to be purely real. The symmetry given by Eq. (3.12b) is just Eq. (2.8b) and is a required symmetry in the even-parity triplet SC state.

Finally, the disorder dependence of the gap equation can be eliminated in the same way as in BCS-Gorkov theory for even-parity singlet SC (Ref. 17) (where the result is known as Anderson's theorem[18]). We define a gap function, Δ, by

$$\bar{\Delta}(p_{n_1})=\Delta(p_{n_1})+\mathcal{F}(p_{n_1}) \tag{3.13a}$$

with

$$\mathcal{F}(p_{n_1})=\frac{1}{2\pi N_F\tau}\int_{\mathbf{p}}\frac{\bar{\Delta}(p_{n_1})}{\{[p_{n_1}+(1/2\tau)Q^0(p_{n_1})]^2+\xi_{\mathbf{p}^2}+\bar{\Delta}^2(p_{n_1})\}} . \tag{3.13b}$$

It is now easily shown that Δ satisfies the gap equation

$$\Delta(p_{n_1})=\pi T\sum_{n_2}K^{(t)}(p_{n_1},p_{n_2})\int d\xi\frac{\Delta(p_{n_2})}{p_{n_2}^2+\xi^2+\Delta^2(p_{n_1})} . \tag{3.14}$$

Equation (3.14) has the structure of a clean SC gap equation in the presence of the pairing potential $K^{(t)}(p_{n_1},p_{n_2})$. For later reference, we note that, for $p_{n_1},p_{n_2}>0$, we have

$$K^{(t)}(p_{n_1},p_{n_2})=K_c^{(t)}(p_{n_1},p_{n_2})+K_s^{(t)}(p_{n_1},p_{n_2}) , \tag{3.15a}$$

where

$$K_c^{(t)}(p_{n_1},p_{n_2})=\frac{\bar{G}H}{16}\left\{F\left[\frac{GH}{\bar{\kappa}^2}|p_{n_1}+p_{n_2}|\right]-\left|\frac{p_{n_1}-p_{n_2}}{p_{n_1}+p_{n_2}}\right|\times F\left[\frac{GH}{\bar{\kappa}^2}\frac{|p_{n_1}-p_{n_2}|^2}{|p_{n_1}+p_{n_2}|}\right]\right\} \tag{3.15b}$$

with F given by Eq. (2.19b),

$$K_s^{(t)}(p_{n_1},p_{n_2})=-\frac{\bar{G}}{16}g_{\gamma_t}\left[\left|\frac{p_{n_1}+p_{n_2}}{p_{n_1}-p_{n_2}}\right|\right] , \tag{3.15c}$$

and g_γ given by Eq. (2.17b).

In Secs. IV and V, we will use Eqs. (3.14) and (3.15) to determine the SC transition temperature. Identical results can be obtained by examining the stability of the normal state to the existence of $K^{(t)}$. This further corroborates the validity of replacing Eq. (3.8b) by Eq. (3.8e).

B. Real frequency gap equation and causality properties

Using standard techniques, the imaginary frequency gap equation given by Eq. (3.14) can be transformed to real frequencies. We will use this equation only at zero temperature where it reads, for $\omega\geq 0$,

$$\Delta(\omega)=\frac{1}{i\pi}\int_0^\infty d\Omega\,\hat{K}^{(t)}(\omega,\Omega)F_r(\Omega) . \tag{3.16}$$

Here $\hat{K}^{(t)}(\omega,\Omega)$ is the real frequency pairing potential which can be written

$$\hat{K}^{(t)}(\omega,\Omega)=\hat{K}_c^{(t)}(\omega,\Omega)+\hat{K}_s^{(t)}(\omega,\Omega) , \tag{3.17a}$$

where, for $\omega,\Omega\geq 0$,

$$\hat{K}_c^{(t)}(\omega,\Omega)=K_c^{(t)}(-i\omega,-i\Omega) , \tag{3.17b}$$

$$\hat{K}_s^{(t)}(\omega,\Omega)=K_s^{(t)}(-i\omega,-i\Omega) , \tag{3.17c}$$

with $-i=\exp[-i\pi/2]$.[19] In Eq. (3.16), F_r is a retarded function given by

$$F_r(\Omega)=F(z=\Omega+i\delta),\quad(\delta\to 0^+) , \tag{3.18a}$$

with $F(z)$ the Gorkov function,

$$F(z)=\int_{-\infty}^{+\infty}d\xi\frac{\Delta(z)}{\xi^2+\Delta^2(z)-z^2} . \tag{3.18b}$$

For later use we note that Eq. (3.12b) implies

$$\Delta(z)=-\Delta(-z) . \tag{3.19}$$

We next examine the symmetry and causality properties of the triplet even-parity SC state. In general, $F(z)$ will have a spectral representation[20]

$$F(z)=\int_{-\infty}^{+\infty}\frac{d\omega}{\pi}\frac{F''(\omega)}{\omega-z} \tag{3.20a}$$

with spectrum

$$F''(\omega)=\frac{[F(\omega+i\delta)-F(\omega-i\delta)]}{2i} . \tag{3.20b}$$

Causality of the diffusive propagators that lead to $\hat{K}^{(t)}$ in Eq. (3.16) should also imply that $\Delta(\omega)$ is a causal function

with spectral representation

$$\Delta(z)=\int_{-\infty}^{+\infty}\frac{d\omega}{\pi}\frac{\Delta''(\omega)}{\omega-z}\ . \tag{3.20c}$$

For the kernel $\hat{K}_s^{(t)}$ in the limit $\gamma_t\to\infty$, we have checked the causality of Δ by direct calculation. Equations (3.19) and (3.20) imply

$$\begin{aligned}&F''(\omega)=F''(-\omega)\ ,\\ &\Delta''(\omega)=\Delta''(-\omega)\ ,\\ &F'(\omega)=-F'(-\omega)\ ,\\ &\Delta'(\omega)=-\Delta'(-\omega)\ ,\end{aligned} \tag{3.21a}$$

where F' and Δ' are determined by

$$F'(\omega)=\frac{F(\omega+i\delta)+F(\omega-i\delta)}{2} \tag{3.21b}$$

and an analogous expression for Δ'. Finally, using that the imaginary axis quantities, $F(ip_n)$ and $\Delta(ip_n)$, are real, Eqs. (3.20) imply that F', F'', Δ', and Δ'' are all purely imaginary in the triplet even-parity SC state. We also note that Δ (and F) satisfies the Kramers-Kronig relations

$$\Delta''(\omega)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega'\Delta'(\omega')P\frac{1}{\omega'-\omega}\ , \tag{3.22a}$$

$$\Delta'(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}d\omega'\Delta''(\omega')P\frac{1}{\omega'-\omega}\ , \tag{3.22b}$$

where P denotes the principal part. The symmetry properties given above also imply that the Gorkov function for this SC state is an odd function of real or imaginary time.

IV. THE SINGLET OR CHARGE FLUCTUATION MECHANISM

In this section we investigate the SC state induced by the charge fluctuation contribution $K_c^{(t)}(p_{n_1},p_{n_2})$ to $K^{(t)}(p_{n_1},p_{n_2})$. In principle, one should treat $K_c^{(t)}$ and $K_s^{(t)}$ together to compute physical quantities like T_c, the tunneling DOS, and the specific-heat coefficient. However, we will eventually find that the singlet or charge fluctuation contribution to $K^{(t)}$ is numerically insignificant for typical parameter values and can be neglected whenever spin fluctuations are present. Nevertheless, we investigate its effects in detail for several reasons. First, we will see that it is easier to derive definite numerical values for observables from this contribution than from the triplet or spin fluctuation contribution. This is partly due to the fact that this contribution does not depend on the Fermi-liquid parameter F_0^a, and that there is an ultraviolet length scale given by the screening length, κ^{-1}. Finally, the triplet mechanism involves a renormalization process whose validity is not clear (cf., Sec. V). The corresponding renormalization process for the singlet mechanism is neglected here for simplicity.

A. T_c equation

Here we use the linearized gap equation to determine T_c from K_c. With the definition

$$\bar{T}_c=\frac{GH2\pi T_c}{\kappa^2(1+F_0^s)^2}\ , \tag{4.1}$$

the linearized gap equation for T_c is given by Eq. (3.14) with $K^{(t)}\to K_c^{(t)}$ and can be written

$$\begin{aligned}\Delta(n)=\frac{G}{32\pi}\sum_{m=0}^{\infty}\frac{\Delta(m)}{2m+1}\Bigg\{&\frac{1}{1+\bar{T}_c(n+m+1)}\{\pi[\bar{T}_c(n+m+1)]^{1/2}-\ln\bar{T}_c(n+m+1)\}\\ &-\frac{|n-m|}{(n+m+1)[1+\bar{T}_c|n-m|^2/(n+m+1)]}\\ &\times\left[\pi\left[\frac{\bar{T}_c|n-m|^2}{n+m+1}\right]^{1/2}-\ln[\bar{T}_c|n-m|^2/(n+m+1)]\right]\Bigg\}\ .\end{aligned} \tag{4.2}$$

We compute T_c to leading logarithmic accuracy, for $T_c\to 0$, by replacing Eq. (4.2) with

$$\Delta(n)\simeq-\frac{G}{32\pi}(\ln\bar{T}_c)\sum_{m=0}^{\infty}\frac{\Delta(m)}{2m+1}\left\{1-\frac{|n-m|}{n+m+1}\right\}\ . \tag{4.3}$$

In the limit $T_c\to 0$, we can further replace the sum in Eq. (4.3) by an integral according to $2\pi T\sum_{m=0}^{\infty}=\int_0^{\infty}dp_m$. Then Δ is independent of frequency for positive frequencies. The resulting T_c equation is

$$\begin{aligned}1&\simeq-\frac{G}{64\pi}(\ln\bar{T}_c)\int_0^{\infty}\frac{dx}{x}\left[1-\frac{|1-x|}{1+x}\right]\\ &=-\frac{G}{16\pi}(\ln 2)(\ln\bar{T}_c)\end{aligned} \tag{4.4a}$$

or

$$T_c=\frac{\kappa^2(1+F_0^s)^2}{2\pi GH}\exp\left\{-\frac{16\pi}{G\ln 2}\right\}\ . \tag{4.4b}$$

The prefactor in Eq. (4.4b) is given by the diffusive energy

scale $D\bar{\kappa}^2/2\pi$ with $D=1/GH$ the thermal diffusion constant. In a two-dimensional system, conductivity and conductance are the same, and we have

$$G=\left[\frac{4}{\pi}\right]\hat{R}_\square \ , \tag{4.5a}$$

where

$$\hat{R}_\square=\frac{R_\square}{\hbar/e^2}=\frac{R_\square}{4108\Omega}$$

is the dimensionless resistance per square. The Einstein relation allows us to write

$$D\bar{\kappa}^2=\frac{2\pi}{\hat{R}_\square}\left[\frac{\bar{\kappa}}{k_F}\right]^2 T_F \ , \tag{4.5b}$$

with k_F the Femi wave number and T_F the Fermi temperature. The T_c formula can then be written as

$$T_c=T_F\frac{(\bar{\kappa}/k_F)^2}{\hat{R}_\square}\exp\left[-\frac{4\pi^2/\ln 2}{\hat{R}_\square}\right] . \tag{4.6}$$

B. The gap equation at $T=0$

Scaling the frequency and gap with the diffusive energy scale, $D\bar{\kappa}^2$,

$$\Omega\rightarrow\Omega D\bar{\kappa}^2 \ , \tag{4.7a}$$

$$\Delta\rightarrow\Delta D\bar{\kappa}^2 \ , \tag{4.7b}$$

leads to a zero-temperature gap equation given by

$$\Delta(\omega)=\int_0^\infty d\Omega\frac{\Delta(\Omega)}{\sqrt{\Omega^2-\Delta^2(\Omega)}}\tilde{K}_c^{(t)}(\omega,\Omega) \ . \tag{4.8a}$$

Here $\tilde{K}_c^{(t)}(\omega,\Omega)$ is the effective scaled potential due to charge fluctuations,

$$\tilde{K}_c^{(t)}(\omega,\Omega)=-\lambda\left[f(\Omega+\omega)-\frac{|\Omega-\omega|}{\Omega+\omega}f\left[\frac{|\Omega-\omega|^2}{\Omega+\omega}\right]\right] , \tag{4.8b}$$

where

$$f(x)=\frac{1}{1-ix}\left[\ln x-\frac{i\pi}{2}+\pi(1-i)\left[\frac{x}{2}\right]^{1/2}\right] \tag{4.8c}$$

and

$$\lambda=\frac{G}{4\pi}=\frac{\hat{R}_\square}{\pi^2} \ . \tag{4.8d}$$

We next examine the structure of Eqs. (4.8) and point out some of its important features. First there are two frequency scales. The first one is

$$\omega_1^*=D\bar{\kappa}^2 \ , \tag{4.9a}$$

and it is analogous to the Debye frequency in conventional SC. ω_1^* sets the scale for the frequency and the gap, and it has been scaled out of the equations by means of Eq. (4.7). In conventional SC there is a second frequency scale related to the critical temperature or the gap at zero frequency. A second scale, ω_2^*, also exists in Eq. (4.8). As in conventional SC, it separates high and low frequencies and is given by the equation

$$\omega_2^*=\Delta(i\omega_2^*) \ . \tag{4.9b}$$

For weak coupling, $\lambda\ll 1$, we can determine ω_2^* analytically. From the imaginary-axis gap equation, Eq. (3.14) with $K_c^{(t)}$ as a kernel, one finds, by asymptotic analysis,

$$\Delta(i\omega\rightarrow 0)=\lambda\omega\ln\omega\ln(\omega e^3)+O(\omega) \ , \tag{4.9c}$$

restoring the frequency scale ω_1^*, we have

$$\omega_2^*=\omega_1^* e^{-1/\lambda^{1/2}} \ . \tag{4.9d}$$

From Eq. (4.9c), the asymptotic solution on the real axis is obtained by analytic continuation

$$\Delta(\omega\rightarrow 0)=-i\lambda\omega(\ln\omega)^2-\lambda\omega(3i+\pi)\ln\omega+O(\omega) \ . \tag{4.10a}$$

We have also determined the asymptotic solution of Eqs. (4.8) directly and found the same result. Asymptotic analysis also yields the high-frequency solution

$$\Delta(\omega\rightarrow\infty)=c\frac{\lambda}{\omega^2}\left[\ln\omega-i\frac{\pi}{2}-2\right]+O(\omega^{-5/2}) \ , \tag{4.10b}$$

where c is an undetermined constant.

We have used Eqs. (4.10) to construct trial solutions which serve as input for an iterative numerical solution of Eqs. (4.8). For small λ ($\lambda\leq 0.2$), we found good convergence for almost any trial solution. For λ larger than a critical value $\lambda_c^+\simeq 0.2$, the iteration failed to converge. We have found that this is not due to a failure of the iteration procedure, but rather signalizes the absence of causal solutions of the gap equation. The mathematical reason for this is as follows. From Eq. (4.10a) we see that the reactive and dissipative parts of the gap, Δ' and Δ'', respectively, at small frequencies read

$$\Delta'(\omega\rightarrow 0)=-i\lambda\omega(\ln\omega)^2 \ , \tag{4.11a}$$

$$\Delta''(\omega\rightarrow 0)=i\pi\lambda\omega\ln\omega \ . \tag{4.11b}$$

One easily checks that Eqs. (4.11a) and (4.11b) are indeed related by Kramers-Kronig relations, Eqs. (3.22). The argument of the square root in Eq. (4.8a) can be written

$$f(\omega)=\omega^2-\Delta^2(\omega)=\omega^2+[i\Delta'(\omega)]^2-[i\Delta''(\omega)]^2-2i\Delta'(\omega)\Delta''(\omega) \ . \tag{4.12}$$

For sufficiently small frequencies $|i\Delta'(\omega)|\gg|i\Delta''(\omega)|$, and the real part of $f(\omega)$ is positive. With increasing frequency, the Kramers-Kronig relations will force $\Delta'(\omega)$ to decrease and go through zero (there must be anomalous dispersion in a region where Δ'' is substantially nonzero). Since $|\Delta''(\omega\rightarrow 0)|\gg\omega$, this zero will occur in the region where $|\Delta''(\omega)|>\omega$ if only λ is large enough. Therefore, one will cross the cut of the square root in Eq. (4.8a) if λ is larger than some λ_c^+. We have been unable to find a physical solution in the region $\lambda>\lambda_c^+$.

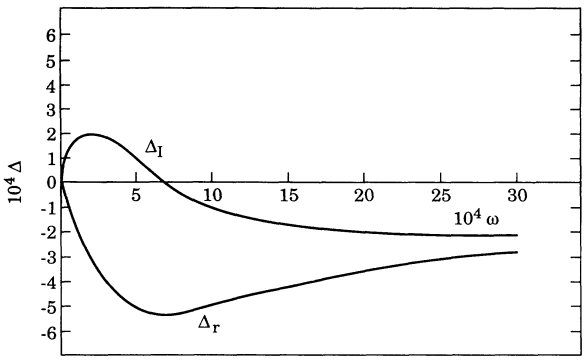

FIG. 1. Real and imaginary parts of the gap function Δ vs frequency ω for the charge fluctuation mechanism induced by $K_c^{(t)}$. Δ and ω are measured in units of ω_1^*, and the coupling constant is $\lambda=0.1$.

In Sec. VI, we will come back to possible physical implications of this. For now we ignore this problem and consider only the case $\lambda<\lambda_c^+$. The choice $\lambda=0.1$ produces a representative result. Figure 1 shows the real and imaginary parts of Δ for this coupling. For $\omega\to 0$ and $\omega\to\infty$, the numerical solution reproduces the analytical results given in Eqs. (4.10). We will further discuss this result in Sec. VI.

V. THE TRIPLET OR SPIN FLUCTUATION MECHANISM

In this section we investigate the SC state induced by the triplet or spin fluctuation contribution to $K^{(t)}(p_{n_1},p_{n_2})$, which we denoted by $K_s^t(p_{n_1},p_{n_2})$. As for the singlet contribution to $K^{(t)}$, we first use the linearized gap equation to compute T_c from $K_s^{(t)}$. We expect $K_s^{(t)}$ to, in general, lead to a much higher T_c than $K_s^{(t)}$. The reason is that renormalization-group (RG) calculations in $D=2$ suggest that the renormalized triplet interaction amplitude, k_t, becomes large as $T\to 0$.[11,12,14] Because the spin fluctuation contribution to $K_s^{(t)}$ is proportional to k_t, this, in principle, can lead to a high T_c. We then discuss the SC state induced by $K_s^{(t)}$, breaking up the discussion into three steps. First we treat the problem by ignoring the renormalization of k_t. Then the model with k_t renormalized perturbatively will be considered. Physically this should be appropriate for moderate temperatures. Finally, we examine the SC state assuming that k_t or the magnetic susceptibility diverges as a power law as $\omega\to 0$. The motivation for this ansatz will be discussed.

A. T_c equation

For simplicity we consider the linearized gap equation in the $\gamma_t\to\infty$ limit. We will self-consistently verify that the renormalized γ_t is indeed large near T_c. In this limit, Eqs. (3.14) and (3.15) give

$$\Delta(n)=\frac{y_0}{16}\sum_{m=0}^{\infty}\frac{\Delta(m)}{2m+1}\ln\left|\frac{n+m+1}{n-m}\right| \tag{5.1a}$$

with

$$y_0=\frac{\bar{G}K_t}{H}=\frac{2}{\pi^2}\gamma_{t0}\hat{R}_\square\ , \tag{5.1b}$$

where $\gamma_{t0}=K_t/H$. The logarithmic singularity in Eq. (5.1a) at $n=m$ is an artifact of the $\gamma_t\to\infty$ limit and $\ln\infty$ for $n=m$ should be replaced by $\ln(1+\gamma_t)$. This singularity will be irrelevant for our low-T estimate of T_c.

If we replace the sum in Eq. (5.1) by an integral, we see that Δ is again independent of frequency for positive frequencies. The resulting condition for criticality is

$$1\simeq\frac{y_0}{32}\int_0^\infty\frac{dx}{x}\ln\left|\frac{x+1}{x-1}\right|=\frac{y_0\pi^2}{64}\ . \tag{5.2}$$

Equation (5.2) seems to imply a threshold value which the coupling strength y_0 has to exceed in order for the system to become superconducting. We have to remember, however, that so far we have been working strictly in perturbation theory, so the bare coupling constant y_0 appears in Eq. (5.2). Since we know that the coupling constants become strongly scale dependent due to disorder renormalization, we should replace y_0 in Eq. (5.2) by its scale-dependent counterpart, $y(T)$. This will determine the critical temperature. In Ref. 21 we have considered the RG flow of y in $d=2$. We showed that, for sufficiently small RG length rescaling factors b, the disorder is not renormalized, $g(b)\equiv g_0=2\hat{R}_\square/\pi^2$, where $g_0=y_0/\gamma_{t0}$. The regime where this holds can be divided into two regions, the boundary between which can be written as $g_0\gamma_t\ln\gamma_t=1$, where γ_t is the scale-dependent or renormalized counterpart of γ_{t0}. For $g_0\gamma_t\ln\gamma_t\gg 1$, the frequency or temperature rescaling factor H is not renormalized, and b is related to the temperature by $b^2=\hat{T}^{-1}$, where $\hat{T}=T/T_0$ with T_0 a microscopic temperature scale on the order of the Fermi temperature T_F. With the help of the explicit flow equation for γ_t,[14,21] one can express the condition for the two regions and the y flow in them as[21]

$$\frac{dy}{dx}=\frac{y^2}{4}\ , \tag{5.3a}$$

which holds if

$$\frac{y_0}{1-y_0x/4}\ln\left[\frac{y_{t0}}{1-y_0x/4}\right]\ll 1\ , \tag{5.3b}$$

and

$$\frac{dy}{dx}=y^2\ , \tag{5.3c}$$

which holds if

$$\frac{y_0}{1-y_0x}\ln\left[\frac{\gamma_{t0}}{1-y_0x}\right]\gg 1\ . \tag{5.3d}$$

In Eqs. (5.3), $x=\ln b$, and we have used the explicit flow equations to rewrite the conditions $g_0\gamma_t\ln\gamma_t\ll 1$ ($\gg 1$). In the regime intermediate between Eqs. (5.3b) and (5.3d), the beta function interpolates between Eqs. (5.3a) and (5.3c).

At this point, let us mention two important unsolved problems. (1) The RG flow equations given by Eqs. (5.3) break down for $T\to 0$, or at $T=0$, for $b\to\infty$. Equations (5.3) show that there are two intermediate length or temperature scales, while the behavior at still larger scales is unknown. Indeed, the problem of determining the correct asymptotic flow is very complicated. We call this the 2D ground-state problem (in the absence of SC), and we will come back to it later. (2) It has not been proven that the field theory, Eq. (2.5), describing disordered electronic systems is renormalizable. In fact, we have shown here that the disorder generates a new interaction amplitude, $K^{c,t}$, which was not in the bare theory. However, our use of RG ideas has been very limited. Equations (3.5) only use the well-established fact that, on intermediate length scales in $d=2$, the dominant effect is a rapid increase of γ_t with increasing b.[11,12,14] On larger scales the renormalization of the disorder becomes important, the relationship between b^2 and $\hat{T}$ becomes more complicated, and the question of renormalizability must be addressed. This means that we cannot trust our results if the scale b corresponding to our critical temperature is too close to $\exp(1/y_0)$.

With $b^2=\hat{T}^{-1}$, we obtain, from Eq. (5.3c),

$$y(T)=\frac{y_0}{1-(y_0/2)\ln\hat{T}^{-1}} . \tag{5.4}$$

With $y_0\to y(T)$ in Eq. (5.2), we obtain an equation for T_c:

$$1=\frac{y_0}{y_c[1-(y_0/2)\ln(T_0/T_c)]} \tag{5.5a}$$

or

$$T_c=T_0\exp\left[-\frac{2}{y_0}\left[1-\frac{y_0}{y_c}\right]\right] , \tag{5.5b}$$

where $y_c=64/\pi^2$. With this result the condition given after Eq. (5.3d) is

$$\hat{R}_\square \ll \frac{\pi^2}{2}y_c e^{-1/y_c} . \tag{5.6}$$

The value on the rhs of the inequality (5.6) is quite large. However, one should keep in mind that the exact flow in 2D is not known and our estimates are rather crude.

If Eq. (5.6) is not satisfied, then Eq. (5.3a) must be used. In addition, the relationship between $x=\ln b$ and $\hat{T}$ is more complicated and is determined by $\hat{T}=[h_0/h(b)b^2]$ with[21]

$$\frac{dh}{dx}=\frac{3hy}{4} . \tag{5.7}$$

Equations (5.3a) and (5.7) give

$$T_c=T_0\left[1-\frac{y_0}{4}x\right]^3\exp\left[-\frac{8}{y_0}\left[1-\frac{y_0}{y_c}\right]\right] \tag{5.8a}$$

with $x=x(T_c)$ given by

$$x(T_c)=\tfrac{1}{2}\ln\frac{T_0}{T_c}+\tfrac{3}{2}\ln\left[1-\frac{y_0}{4}x(T_c)\right] . \tag{5.8b}$$

Comparing Eqs. (5.8) and (5.5), it is clear that T_c given by Eq. (5.8a) is much less than that given by Eq. (5.5b). A discussion of numerical results will be given in Sec. VI.

B. The gap equation at $T=0$

In the $\gamma_t\to\infty$ limit discussed in the previous subsection, the zero-temperature gap equation is

$$\Delta(\omega)=\lambda\int_0^\infty d\Omega\frac{\Delta(\Omega)}{\sqrt{\Omega^2-\Delta^2(\Omega)}}\ln\left|\frac{\omega+\Omega}{\omega-\Omega}\right| \tag{5.9a}$$

with

$$\lambda=\frac{y_0}{32} . \tag{5.9b}$$

λ (or y_0) should actually be renormalized just as in the previous subsection. It is useful, however, to first ignore this effect and to study Eq. (5.9a) for a constant λ.

We first note that Eq. (5.9a) does not have an ultraviolet scale analogous to the one given in Eq. (4.9a) for the singlet contribution to $K^{(t)}$. One expects a scale related to T_0 of the previous subsection to appear. Indeed, this scale can be built into the theory by imposing an UV cutoff on the order of the Fermi wave number on the momentum integrals in Eqs. (2.16) that leads to $\tilde{K}^{(t)}$. We ignore this effect here because it only modifies the high-frequency behavior of the gap function which is physically not very relevant and depends on details of the model. Nevertheless, it should be remembered that effects or structure at the frequency scale

$$\omega_1^*\sim T_0 \tag{5.10a}$$

is being neglected. This scale would also appear if a disorder renormalized λ was used. As in the previous section there is, in addition, a second frequency scale, ω_2^*, that separates high- and low-frequency behavior and is given by

$$\omega_2^*=\Delta(i\omega_2^*) . \tag{5.10b}$$

We now proceed as in the previous section. Asymptotic analysis for the imaginary axis gap equation yields

$$\Delta(i\omega\to 0)=-2\lambda\omega\ln\omega+O(\omega) . \tag{5.10c}$$

From this we find

$$\omega_2^*=\omega_1^* e^{-1/2\lambda} . \tag{5.10d}$$

The small-frequency solution on the real axis we can again obtain either by analytically continuing Eq. (5.10c) or by direct solution of the real-axis gap equation. Either way we find

$$\Delta(\omega\to 0)=2i\lambda\omega\ln\omega+O(\omega) . \tag{5.11a}$$

For large frequencies, we find

$$\Delta(\omega\to\infty)=C_1\sin(s\ln\omega)+C_2\cos(s\ln\omega)O(\omega^{-1}) , \tag{5.11b}$$

where C_1 and C_2 are undetermined constants. s in Eq. (5.11b) is determined by the transcendental equation

$$s=\pi\lambda\tanh\left[\frac{\pi s}{2}\right] . \tag{5.11c}$$

We note several features of Eqs. (5.9) and (5.11). First, the problem discussed below Eq. (4.10) for Eq. (4.8) also holds for Eq. (5.9). This implies the existence of a critical value of λ which we denote by λ_c^+. We consider only the case $\lambda<\lambda_c^+(\simeq 0.5)$. Second, Eq. (5.11c) leads to a complex s for $\lambda<2/\pi^2=\lambda_c^-$ (or $y_0<y_c=64/\pi^2$). From Eq. (5.2) we see that this was to be expected: Because we have not used a renormalized λ, the SC state exists only if $\lambda>2/\pi^2$. We further note that, even in the physical region, $\lambda_c^-<\lambda<\lambda_c^+$, the gap function does not decay for $\omega\to\infty$. This unphysical result can be eliminated by using an UV cutoff on the momentum integrals that lead to $K_t^{(t)}$. As already mentioned, we ignore this problem here.

We have attempted to solve Eq. (5.9) by the same iterative technique we used in the previous section and failed to find convergence. We believe that this reflects a failure of the solution technique and that Eq. (5.9a) does have a physical solution for $\lambda_c^-<\lambda<\lambda_c^+$. To verify or argue this we have solved Eq. (5.9a) with the logarithmic kernel replaced by a simpler kernel which has the same high- and low-frequency limits:

$$\ln\left|\frac{\omega+\Omega}{\omega-\Omega}\right|\to 2\frac{\omega}{\Omega}\Theta(\Omega-\omega)+2\frac{\Omega}{\omega}\Theta(\omega-\Omega) . \tag{5.12}$$

With Eq. (5.12) in Eq. (5.9a), the integral equation is equivalent to the differential equation

$$\frac{d^2\Delta(\omega)}{d\omega^2}+\frac{1}{\omega}\frac{d\Delta(\omega)}{d\omega}-\frac{\Delta(\omega)}{\omega^2}+\frac{4\lambda\Delta(\omega)}{\omega\sqrt{\omega^2-\Delta^2(\omega)}}=0 . \tag{5.13}$$

Equation (5.13) can be easily solved. Analytically the high- and low-frequency solutions are given by Eqs. (5.11), i.e., the differential equation with the model kernel is equivalent to the integral equation with the logarithmic kernel in these limits. For other frequencies we have solved Eq. (5.13) numerically using a Runge-Kutta method. For boundary conditions we require Eq. (5.11a)

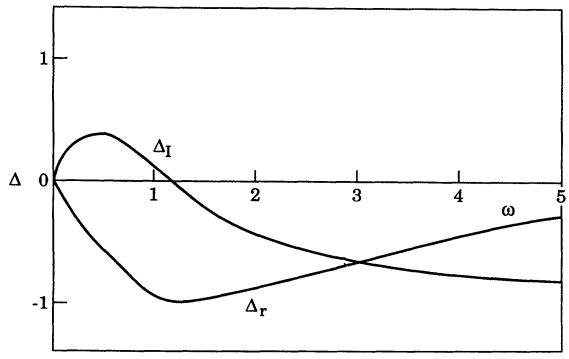

FIG. 2. Real and imaginary parts of the gap function Δ vs frequency ω for the spin fluctuation mechanism induced by $K_s^{(t)}$. Δ and ω are measured in units of ω_1^*, and the coupling constant is $\lambda=0.375$.

and the Kramers-Kronig relations given by Eq. (3.22) to be satisfied in the $\omega\to 0$ limit.

In Fig. 2, we show the real and imaginary parts of Δ for a typical value of $\lambda=0.375$. For $\omega\to 0$ and $\omega\to\infty$, the exact analytic results given by Eqs. (5.11) are recovered in the numerical solution given in Fig. 2. We note that, for intermediate frequencies, the approximation given by Eqs. (5.12) violates Eqs. (3.22). We have performed a Kramers-Kronig check and found that the solution of Eq. (5.13) qualitatively still has the behavior of a causal function. We will further discuss this result in Sec. VI.

So far we have ignored the fact that λ in Eq. (5.9) is dramatically renormalized by disorder as $\omega\to 0$ (or as $T\to 0$ at $\omega=0$). Neglecting this effect has two consequences. First, a lower critical disorder parameter, λ_c^-, has appeared. This implies that a critical amount of disorder is needed to induce the SC at $T=0$. As we have shown in the previous subsection this is not correct if renormalization effects in λ are taken into account, i.e., $\lambda_c^-=0$. Second, low-frequency singularities in λ modify the low-frequency or low-temperature result given by Eq. (5.11a). Indeed it is the logarithmic singularity in $K_c^{(t)}$ that leads to the extra logarithmic factors in Eq. (4.10a) as compared with Eq. (5.11a).

To correctly build in the frequency dependence of λ is difficult. We shall see, however, that scaling or power-law dependencies are insensitive to the precise way in which λ is renormalized. We first note that the RG equations given in the previous subsection apply to the $T=0$, $\omega\neq 0$ case if, in these equations, we replace T by ω. For example, Eq. (5.4) gives

$$y(\omega)=\frac{y_0}{1-(y_0/2)\ln(T_0/\omega)} . \tag{5.14}$$

This result, however, breaks down when $[1-(y_0/2)\ln T_0/\omega]\to 0$ and, consequently, cannot be used as $\omega\to 0$. The precise behavior of $y(\omega\to 0)$ is not known and we previously referred to this as the 2D ground-state problem. However, all work suggests that $y(\omega\to 0)$ is divergent in the absence of SC. It is therefore of interest to examine the properties of an even-parity spin-triplet SC state with ($\lambda\sim y$),

$$y(\omega\to 0)\simeq\frac{c}{\omega^{\mu}} , \tag{5.15}$$

where c is a constant and μ is an unknown positive exponent. The picture that emerges for λ or y is as follows. For high frequencies ($\omega\gg T_0$)$\lambda(\omega)$ is not appreciably modified by disorder and is given (for small disorder) by Eq. (5.9b): with decreasing frequency Eq. (5.16) implies, for $\omega\leq T_0$,

$$\lambda(\omega\gg T_0)\simeq\lambda_0=\frac{y_0}{32} , \tag{5.16a}$$

$$\lambda(\omega\leq T_0)\simeq\frac{\lambda_0}{1-(y_0/2)\ln(T_0/\omega)}\simeq\lambda_0\left\{1+\frac{y_0}{2}\ln\frac{T_0}{\omega}\right\} . \tag{5.16b}$$

For very low frequencies, $\omega \ll T_0$, Eq. (5.15) gives

$$\lambda(\omega \ll T_0) \simeq \frac{c}{\omega^{\mu}} . \tag{5.16c}$$

Note that λ in Eq. (5.9a) should depend on both external, ω, and internal, Ω, frequencies: $\lambda \rightarrow \lambda(\Omega,\omega)$. For our considerations this point is irrelevant. For intermediate frequencies, Eq. (5.16b) implies that, for decreasing frequencies, the gap function, Δ, will resemble the result discussed in Sec. IV due to the extra logarithmic singularity in λ. Alternatively, one expects at low but finite temperatures that $K_c^{(t)}$ and $K_s^{(t)}$ will lead to similar SC states. At zero temperature, Eq. (5.16c) can be used to determine the low-frequency scaling properties of Δ, ρ, and C. One finds

$$\Delta(\omega \rightarrow 0) \sim \omega^{1-\mu} . \tag{5.17}$$

VI. DISCUSSION

In this paper we have presented a superconducting state of 2D systems which is characterized by spin-triplet pairing and a gap function which has even parity and is an odd function of complex frequency. In this last section we further discuss our results and estimate numerical values for various observables.

A. Possible values for T_c

In Eq. (4.6) we have given T_c due to the charge fluctuation mechanism for a truly 2D system. In a real quasi-2D system, say a thin metal film of thickness d, the DOS per spin at the Fermi level is $N_F = k_F dm/2\pi^2$, the diffusion constant is $D = k_F^2 \tau/3m^2$, and the sheet resistance is $R_{\square} = \rho/d = 1/2N_F D$, when ρ is the bulk resistivity. The T_c equation for such a system reads

$$T_c = T_F \frac{(\bar{\kappa}/k_F)^2}{\hat{R}_{\square}} \frac{\pi}{k_F d} \exp\left[\frac{-4\pi^2/\ln 2}{\hat{R}_{\square}}\right] . \tag{6.1}$$

For thin metal films, typical numbers are $\hat{R}_{\square} \leq 1$, $T_F \simeq 5 \times 10^3$ K, $\pi/k_F d \simeq 1$, and $\bar{\kappa}/k_F \simeq 1$. The resulting T_c is unobservably small. The problem is that the achievable resistance values in thin films are not high enough. This is different in the inversion layer in a Si metal-oxide-semiconductor field-effect transistor (MOSFET) which can be driven through a metal-insulator transition.[22] These systems are truly two dimensional, so Eq. (4.6) applies. If we take our result at face value, T_c/T_F will reach a maximum of $6 \times 10^{-3} (\bar{\kappa}/k_F)^2$ for $\hat{R}_{\square} = 4\pi^2/\ln 2 = 57 = 9R_M/(e^2/\hbar)$, where $R_M = 2\pi e^2/\hbar = 25.8 k\Omega$ is the Mott resistance. Several caveats are in order, however. First, we have restricted ourselves to a one-loop approximation. Starting at two-loop order, one would expect to find T_c-degradation effects of the kind that occur in conventional superconductors.[23] These effects will bring down the maximum T_c/T_F, and shift it to smaller values of $\hat{R}_{\square}$. Second, the experimentally controllable parameter in a MOSFET is the carrier density N_s. $\hat{R}_{\square}$ increases with decreasing N_s, while T_F decreases, making the absolute T_c quite small where T_c/T_F is maximal. Finally, one should keep in mind that estimating T_c from many-body theory is notoriously difficult, with the history of the theory of ^{3}He providing an infamous example. If we proceed nevertheless, we should be careful not to assign too high a value to $\hat{R}_{\square}$, $\hat{R}_{\square} \simeq \pi - 2\pi$ (i.e., $R_{\square}$ about one-half to one times the Mott number) seems the largest reasonable choice. A typical carrier density for this resistance is [22] $N_s = 5 \times 10^{11}$ cm^{-2}, which yields $T_F = 35$ K, $\kappa = 1.25 \times 10^7$ cm^{-1}, and $k_F = 1.25 \times 10^6$ cm^{-1}. For this carrier concentration, a reasonable value for the Fermi-liquid parameter F_0^s is -0.9.[16] Then we obtain $T_c = 15$–640 μK for $\hat{R}_{\square} = \pi - 2\pi$. An observation of superconductivity in this temperature regime does not appear absolutely hopeless.

We now turn to the spin fluctuation mechanism. From Eq. (5.5b) we obtain

$$T_c = T_0 e^{-1/\lambda_{\text{eff}}} \tag{6.2a}$$

with

$$\lambda_{\text{eff}} = \frac{2}{\pi} \hat{R}_{\square} \gamma_{t0} \left[1 - \frac{\pi}{16} \hat{R}_{\square} \gamma_{t0}\right]^{-1} . \tag{6.2b}$$

It seems that T_c given by Eqs. (2.6) can be an appreciable fraction to T_0. We must keep in mind, however, that all the caveats mentioned above in connection with the charge fluctuation mechanism apply here as well. Also, if Eq. (5.6) is not satisfied we should apply Eqs. (5.8) which yield a much lower T_c. We conclude that T_c due to the spin fluctuation mechanism is most likely a very rapidly varying function of both disorder and γ_{t0} (or F_0^a), and might be observable only in a narrow region in parameter space. Moderate values of disorder and large values of γ_{t0} should be most favorable.

B. The gap function, the tunneling density of states, and the specific heat

We now turn to the solutions of the gap equation which we found in Secs. IV B and V B. The solution, $\Delta(\omega)$, directly determines the single-particle or tunneling DOS, $\rho(\omega)$, via

$$\rho(\omega) = N_F \text{Re}\left\{\frac{\omega}{[\omega^2 - \Delta^2(\omega)]^{1/2}}\right\} . \tag{6.3}$$

Let us first discuss the implications of the upper critical coupling strength λ_c^+ which we found in Secs. IV and V. As discussed after Eq. (4.12) for $\lambda > \lambda_c^+$, there is a frequency $\omega > 0$ where $\text{Im}\Delta(\omega) = 0$, while $\text{Re}\Delta(\omega) > \omega$. This implies that $\rho(\omega)$ vanishes at this frequency. Furthermore, with the choice of Riemann sheets that guarantees a causal Gorkov function,[24] $\rho(\omega)$ will become negative for larger ω. We have also considered the possibility that $\rho(\omega) = 0$ for a range of frequencies, but again were unable to construct a causal solution with this property. It seems likely that the present theory has indeed no causal solution for $\lambda > \lambda_c^+$. We offer two suggestions for what physically happens at stronger coupling. (1) The triplet SC state may be unstable against some other collective state still to be identified, or (2) building the SC self-

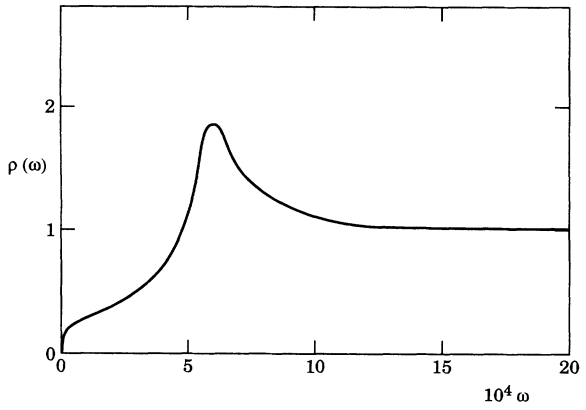

FIG. 3. Single-particle density of states, ρ, vs frequency ω for the gap function shown in Fig. 1. ρ is measured in units of N_F, and ω in units of ω_1^*.

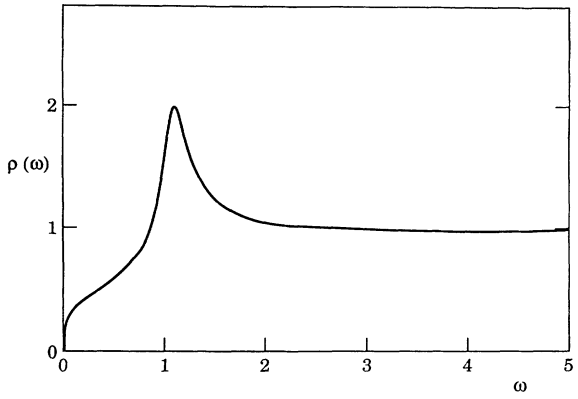

FIG. 5. Single-particle density of states, ρ, vs frequency ω for the gap function shown in Fig. 2. ρ is measured in units of N_F, and ω in units of ω_1^*.

consistently into the pairing potential may effectively prevent λ from ever being larger than λ_c^+. In this context we note that our pairing potential has been constructed from correlation functions in the normal state, while any theory of a purely electronic SC mechanism should, in principle, self-consistently consider the effects of SC on the pairing mechanism.[25] This remains to be investigated in the future.

For $\lambda<\lambda_c^+$, our solutions for $\Delta(\omega)$ determine the tunneling DOS via Eq. (6.3). Using Eq. (4.10a) in Eq. (6.3), we see that the charge fluctuation mechanism leads to a DOS which vanishes for small frequencies as

$$\rho(\omega)=\frac{N_F}{\lambda\{\ln[D(\bar{\kappa}^2/\omega)]\}^2} . \tag{6.4}$$

The numerical solution for all frequencies and $\lambda=0.1$ is shown in Fig. 3. Compared to conventional SC, there are several interesting features to note in Fig. 3. First, there is no gap, and $\rho(\omega)$ vanishes only very slowly as $\omega\to 0$. We note that experimentally it is common practice to *define* a "gap energy" as the position of the first peak in $\rho(\omega)$. Adopting this convention here one would find

$$\Delta_{\rm gap}\simeq(6\times 10^{-4})D\bar{\kappa}^2 . \tag{6.5}$$

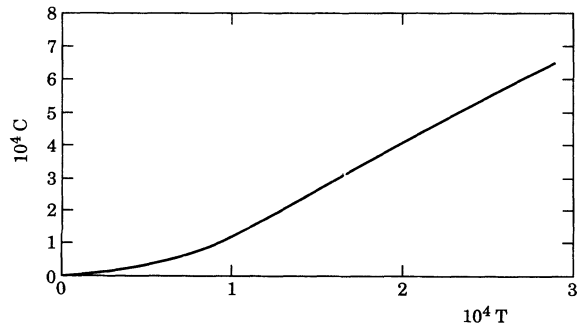

FIG. 4. Specific heat, C, vs temperature T for the density of states shown in Fig. 3. C is measured in units of $N_F\omega_1^*$, and T in units of ω_1^*.

Second, the curvature of the DOS in Fig. 3 is very different from what one is used to in conventional superconductors. In particular, we point out the pronounced shoulder of the DOS in Fig. 3. The only experimental tunneling data we are aware of which bear some resemblance to Fig. 3 are those by Gurvitch *et al.* on YBCO.[26] We do not know whether this is a coincidence.

In Fig. 4 we show the specific heat as a function of temperature for $\lambda=0.1$. We have used the formula, valid at low temperatures,

$$C(T)=\frac{1}{T^2}\int_0^\infty d\omega\,\rho(\omega)\omega^2\exp[-\beta\omega] . \tag{6.6}$$

Analytically, Eqs. (6.4) and (6.6) give

$$C(T\to 0)=\frac{2N_F}{\lambda}\frac{T}{[\ln(D\bar{\kappa}^2/T)]^2} . \tag{6.7}$$

For the spin fluctuation mechanism, Eq. (5.10c) in Eq. (6.4) yields, for small frequencies,

$$\rho(\omega)=\frac{N_F}{2\lambda\ln(T_0/\omega)} . \tag{6.8}$$

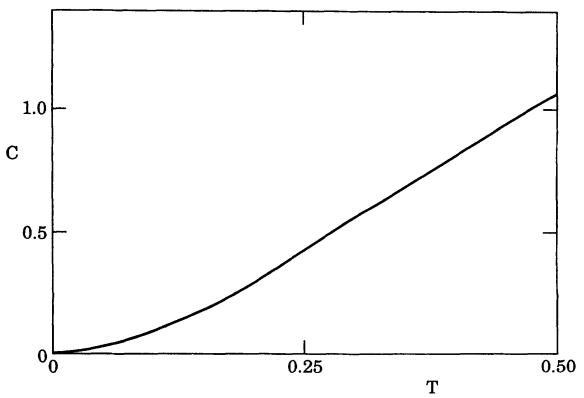

FIG. 6. Specific heat, C, vs temperature T for the density of states shown in Fig. 5. C is measured in units of $N_F\omega_1^*$, and T in units of ω_1^*.

Figure 5 shows the DOS for all frequencies at $\lambda=0.375$. Notice that the result for the spin fluctuation mechanism is qualitatively similar to that for the charge fluctuation mechanism, Fig. 3. In Fig. 6, we also show the specific heat for the spin fluctuation mechanism. Analytically, we have from Eqs. (6.8) and (6.6) for low temperatures

$$C(T\to 0)=\frac{N_F}{\lambda}\frac{T}{\ln(T_0/T)} . \tag{6.9}$$

Finally, we have to remember that the above results for the spin fluctuation mechanism do not apply for asymptotically low frequencies or temperatures because of the 2D ground-state problem. If we adopt the scaling conjecture expressed in Eq. (5.17), we obtain asymptotically

$$\rho(\omega\to 0)\propto\omega^{\mu} , \tag{6.10}$$

$$C(T\to 0)\propto T^{1+\mu} . \tag{6.11}$$

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