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Vortex motion and the Hall effect in type-II superconductors: A time-dependent Ginzburg-Landau theory approach

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Vortex motion in type-II superconductors is studied starting from a variant of the time-dependent Ginzburg-Landau equations, in which the order-parameter relaxation time is taken to be complex. Using a method due to Gor'kov and Kopnin, we derive an equation of motion for a single vortex ($B \ll H_{c2}$) in the presence of an applied transport current. The imaginary part of the relaxation time and the normal-state Hall effect both break "particle-hole symmetry," and produce a component of the vortex velocity parallel to the transport current, and consequently a Hall field due to the vortex motion. Various models for the relaxation time are considered, allowing for a comparison to some phenomenological models of vortex motion in superconductors, such as the Bardeen-Stephen and Nozières-Vinen models, as well as to models of vortex motion in neutral superfluids. In addition, the transport energy, Nernst effect, and thermopower are calculated for a single vortex. Vortex bending and fluctuations can also be included within this description, resulting in a Langevin-equation description of the vortex motion. The Langevin equation is used to discuss the propagation of helicon waves and the diffusional motion of a vortex line. The results are discussed in light of the rather puzzling sign change of the Hall effect which has been observed in the mixed state of the high-temperature superconductors.

I. INTRODUCTION

The study of vortex motion in type-II superconductors continues to attract the attention of theorists and experimentalists alike, due in part to the rather unusual mixed state transport properties of the high-temperature superconductors. One of the more vexing of these properties is the anomalous behavior of the Hall effect, which is observed to change sign in the superconducting mixed state.[1-6] Such behavior is not expected within the standard models of vortex motion in superconductors, the Bardeen-Stephen model,[7] and the Nozières-Vinen model.[8,9] Indeed, even the low-temperature superconductors exhibited a range of behaviors not in accord with either of the above models, including a sign change[10]; for a review of the early work, we refer the reader to the article by Kim and Stephen.[11] It is the Hall-effect data which constrains models of vortex motion in superconductors, and which represents the greatest challenge to the theorist. This paper is an attempt at understanding vortex motion and the Hall effect in type-II superconductors starting from the time-dependent Ginzburg-Landau equations.

Before discussing the new results contained here, we will first briefly review the phenomenological theories of vortex motion (for a more critical assessment, see Refs. 9 and 11). In the Bardeen-Stephen (BS) model[7] of vortex motion, it is assumed that the vortex may be modeled as a normal core of radius the coherence length. If the applied transport current is $\mathbf{J}_t = e^* n_s \mathbf{v}_{s1}$, with n_s the superfluid density (the density of Cooper pairs), e^* the charge of a pair (which we will take to be positive), and $\mathbf{v}_{s1}$ the uniform superfluid velocity far from the vortex, then the Lorentz force per unit length acting upon an individual vortex is $\mathbf{F} = \phi_0 \mathbf{J}_t \times \mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector which points in the direction of the magnetic field and $\phi_0 = h/e^*$ is the flux quantum (we will take $c=1$ in this paper). This is balanced by a viscous drag force (per unit length) $\mathbf{f} = -\eta \mathbf{v}_L$, where $\mathbf{v}_L$ is the velocity of the vortex line. This drag is due to the dissipation which occurs in the normal core of the vortex, so that $\eta \propto \sigma_{xx}^{(n)}$, with $\sigma_{xx}^{(n)}$ the longitudinal normal-state conductivity. By balancing these two forces, $\mathbf{F} + \mathbf{f} = 0$, we find

$$\phi_0 \mathbf{J}_t \times \mathbf{e}_z = \eta \mathbf{v}_L \ . \tag{1.1}$$

Josephson[12] demonstrated that the motion of the magnetic flux produces an electric field, given by Faraday's law

$$\langle \mathbf{E} \rangle = -\mathbf{v}_L \times \mathbf{B} \ , \tag{1.2}$$

where $\langle \mathbf{E} \rangle$ is the spatially averaged electric field and $\mathbf{B}$ is the magnetic induction field. Assuming that $\mathbf{B} = B\mathbf{e}_z$, and combining Eqs. (1.1) and (1.2), we then have

$$\mathbf{J}_t = \frac{\eta}{\phi_0 B} \langle \mathbf{E} \rangle \ , \tag{1.3}$$

so that the flux flow conductivity is $\sigma_{xx} = \eta/(\phi_0 B) \approx \sigma_{xx}^{(n)}(H_{c2}/B)$. In the BS model the Hall field is entirely due to the Hall field produced in the normal core of the vortex; the corresponding Hall angle is equal to that of a normal metal in a field equal to the field in the core. More importantly, the Hall angle has the same sign as in the normal state.

The Nozières-Vinen (NV) model,[8] on the other hand, incorporates the hydrodynamic Magnus force on the vortex,

$$\mathbf{F}=\phi_0 n_s e^*(\mathbf{v}_{s1}-\mathbf{v}_L)\times\mathbf{e}_z \ . \tag{1.4}$$

The Magnus force must be balanced by viscous drag forces $\mathbf{f}$, so that again $\mathbf{F}+\mathbf{f}=0$. In the absence of any viscous drag ($\mathbf{f}=0$), the vortices would simply drift with the transport current ($\mathbf{v}_L=\mathbf{v}_{s1}$); this would lead to a perfect Hall effect and no longitudinal resistance. The Hall fields generated in this fashion have the same direction as the normal-state Hall fields. By making rather different assumptions regarding the nature of the contact potential at the interface between the superfluid and the normal cores, NV conclude that the viscous force should be of the form $\mathbf{f}=-a\mathbf{v}_{s1}$. The longitudinal resistivity obtained is of the same form as the BS result, whereas NV find that the Hall angle in the mixed state is equal to its value at the upper critical field H_{c2}. However, we still find that the Hall angle has the same sign as in the normal state.

The conclusion is that neither of these phenomenological models is able to explain the sign change in the Hall angle. One difficulty is that both models are strictly speaking only valid at $T=0$; they are also only correct at low magnetic inductions, and should not be applied near H_{c2}. However, it seems unlikely that a finite temperature generalization, or the inclusion of intervortex interactions, would act so as to change the sign of the Hall angle. A more serious difficulty is that both treatments start from a hydrodynamic description, with no reference to the underlying superconducting order parameter. It is unclear whether the inability of the BS and NV models to predict the sign change is due to the hydrodynamic description itself (for instance, the implicit assumption of Galilean invariance), or with the approximations involved in the calculation.

There have also been several attempts at a fully microscopic calculation of the Hall effect for a single vortex, starting from the Bogoliubov–de Gennes equations for a moving vortex.[13,14] The dissipation is provided by quasiparticles which scatter from the time-dependent potential provided by the moving vortex; this is balanced by the Magnus force on the vortex. The Hall effect is entirely due to the Magnus force, so this approach is also unable to explain the sign change of the Hall coefficient. These calculations are limited to very pure superconductors, and do not include band-structure effects which are important in determining the sign of the Hall effect in the normal state.

As an alternative to the hydrodynamic and microscopic approaches, we shall study the Hall effect starting from a time-dependent version of the Ginzburg-Landau equations. This method is intermediate between the hydrodynamic and microscopic approaches, in that the time dependence of the order parameter is explicitly considered, while the effects of the quasiparticles are lumped into an effective conductivity for the "normal fluid." The scheme is to then use this model to systematically study the motion of a single vortex. This program has already been carried out for the longitudinal resistivity by Schmid,[15] Gor'kov and Kopnin,[16,17] and Hu and Thompson.[18] The time-dependent Ginzburg-Landau (TDGL) equations must be generalized somewhat in order to study the Hall effect; with this generalization, we will show that a single vortex has the equation of motion

$$\mathbf{v}_{s1}\times\mathbf{e}_z=\alpha_1\mathbf{v}_L+\alpha_2\mathbf{v}_L\times\mathbf{e}_z \ , \tag{1.5}$$

where α_1 and α_2 are functions of the parameters which appear in the TDGL equations. Such an equation of motion for superconducting vortices was originally proposed by Vinen and Warren;[9] a similar phenomenological model has recently been used by Hagen *et al.*[4] to discuss the sign change of the Hall angle in Y-Ba-Cu-O. We see from Eq. (1.5) that α_1 will determine the longitudinal conductivity, while α_2 determines the Hall conductivity. In particular, if $\alpha_2<0$, then the Hall effect in the vortex state will have a sign which is opposite to the sign of the normal-state Hall effect. Having reduced the problem to this effective equation of motion, one can then pose the question, "What choice of parameters leads to a Hall effect which changes sign?"

The outline of the remainder of the paper is as follows. In Sec. II the time-dependent Ginzburg-Landau equations are presented and discussed. In Sec. III we derive an equation of motion for a single vortex, starting from the time-dependent Ginzburg-Landau equations. From this equation of motion the longitudinal and Hall conductivities are calculated and compared to the predictions of the Bardeen-Stephen and Nozières-Vinen models. We also recover known results for the motion of a rectilinear vortex in a neutral superfluid in the appropriate limit. In Sec. IV we study the thermal transport properties of a single vortex, and calculate the Nernst coefficient and the thermopower. The effects of vortex bending and fluctuations are considered in Sec. V. We will derive a Langevin equation for vortex motion, which we will use to study helicon waves and the diffusive motion of the vortex center of mass. The relevance of these results to the anomalous transport properties of the high-temperature superconductors is discussed in Sec. VI. The London acceleration equation for a charged superfluid is derived from the TDGL equations in Appendix A. Numerical coefficients which enter the vortex equation of motion are calculated using a trial solution for the order parameter in Appendix B. Appendix C is a summary of the definitions of the transport coefficients.

II. TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

A. The model

To begin our discussion of vortex motion in superconductors, we first need an appropriate generalization of the familiar equilibrium Ginzburg-Landau equations to include dynamics. Such generalizations have been the subject of intensive study over the years, starting with Schmid's derivation of TDGL equations.[15,19] Gor'kov and Éliashberg later showed that Schmid's results were only valid in the dirty limit, and derived a modified version of the TDGL equations which are valid when the pair breaking is due to paramagnetic impurities.[20] These equations were further developed to include the effects of electron-phonon scattering on the order-parameter relaxation.[21–23] While there are some important consequences of these generalizations, they result in more complicated and cumbersome dynamic equations; we will therefore adopt the TDGL equations originally proposed by Schmid (with minor modifications) as the prototypical equations of motion. Most of the results in this paper may be generalized in a straightforward, if not tedious, manner to the other more complicated dynamical equations.

Our equation of motion for the superconducting order parameter $\psi(\mathbf{r},t)$ is

$$\hbar\left[\partial_t+i\frac{\tilde{\mu}}{\hbar}\right]\psi=-\Gamma\frac{\delta\mathcal{H}}{\delta\psi^*}\ , \tag{2.1}$$

with the Hamiltonian

$$\mathcal{H}=\int d^3r\left[\frac{\hbar^2}{2m}\left|\left[\nabla-i\frac{e^*}{\hbar}\mathbf{A}\right]\psi\right|^2+a(T)|\psi|^2+\frac{b}{2}|\psi|^4+\frac{1}{8\pi}(\nabla\times\mathbf{A})^2\right]\ . \tag{2.2}$$

In the above equations, $\mathbf{A}$ is the vector potential with $\mathbf{h}=\nabla\times\mathbf{A}$ the microscopic magnetic induction field, $\mathbf{B}=\langle\mathbf{h}\rangle$ is the induction field, m is the effective mass of a Cooper pair, $e^*=2e$ is the charge of a Cooper pair (we take e^* to be positive), $a(T)=a_0(T/T_c-1)$, and $\Gamma=\Gamma_1+i\Gamma_2$ is a complex dimensionless relaxation rate. For anisotropic superconductors (such as the high-temperature superconductors) we would need to allow for an effective-mass tensor m_{ij}; in order to simplify the discussion we shall assume that $m_{ij}=m\delta_{ij}$. This assumption will be relaxed when vortex bending is considered in Sec. V. As long as $\Gamma_1>0$, this equation of motion relaxes to the correct equilibrium Ginzburg-Landau equation. The total chemical potential $\tilde{\mu}$ is given by

$$\tilde{\mu}=\mu+e^*\Phi+\frac{\delta\mathcal{H}}{\delta n_s}\ , \tag{2.3}$$

where μ is the chemical potential, Φ is the electric potential, $n_s=|\psi|^2$ is the superfluid density, and $\delta\mathcal{H}/\delta n_s$ is the kinetic energy of the superfluid. The last contribution is often neglected, although it is essential if one desires a Galilean invariant equation of motion.[24] If we set $(\delta\mathcal{H}/\delta n_s)\psi\approx\delta\mathcal{H}/\delta\psi^*$, then we can rewrite Eq. (2.1) as

$$\hbar\left[\partial_t+i\frac{1}{\hbar}\mu+i\frac{e^*}{\hbar}\Phi\right]\psi=-(\Gamma+i)\frac{\delta\mathcal{H}}{\delta\psi^*}\ . \tag{2.4}$$

Similar equations have been used to study the hydrodynamics of superfluid He^4 near the λ transition.[25–27] By defining a dimensionless order-parameter relaxation time

$$\gamma\equiv\gamma_1+i\gamma_2=\frac{\Gamma_1-i(1+\Gamma_2)}{\Gamma_1^2+(1+\Gamma_2)^2}\ , \tag{2.5}$$

our order-parameter equation of motion becomes[15]

$$\hbar\gamma\left[\partial_t+i\frac{e^*}{\hbar}\tilde{\Phi}\right]\psi=\frac{\hbar^2}{2m}\left[\nabla-i\frac{e^*}{\hbar}\mathbf{A}\right]^2\psi+|a|\psi-b|\psi|^2\psi\ , \tag{2.6}$$

where $\tilde{\Phi}=\Phi+\mu/e^*$. The difference between $\tilde{\Phi}$ and Φ is generally small,[15] and we shall neglect this difference in what follows.

By choosing the complex relaxation time γ appropriately, we can consider a variety of different models. If $\Gamma_2=0$, the order-parameter equation of motion leads to the London acceleration equation for the superfluid velocity, as shown in Appendix A. If $\Gamma_2=-1$ (so that $\gamma_1=\Gamma_1^{-1}$ and $\gamma_2=0$), then we have the TDGL originally derived by Schmid.[15] If $\Gamma_1=\Gamma_2=0$ (so that $\gamma_1=0$ and $\gamma_2=-1$), then we have the Gross-Pitaevskii equation[28] (often called the nonlinear Schrödinger equation) for a charged superfluid at zero temperature. More generally, such an imaginary part of the relaxation time is generated in renormalized theories of the critical dynamics of neutral superfluids.[29] A microscopic derivation of the TDGL by Fukuyama, Ebisawa, and Tsuzuki,[30] leads to a value of γ_2 that depends on details of the band structure of the material, and is generally proportional to the derivative of the density of states at the fermi energy $N'(\epsilon_F)$. In what follows we will consider γ to be arbitrary, and after having derived an equation of motion for a vortex we can then consider specific models for γ.

We also require an equation of motion for the vector potential, which is just Ampère's law

$$\nabla\times\nabla\times\mathbf{A}=4\pi(\mathbf{J}_n+\mathbf{J}_s)\ , \tag{2.7}$$

so that $\nabla\cdot(\mathbf{J}_n+\mathbf{J}_s)=0$. The supercurrent $\mathbf{J}_s$ is given by

$$\mathbf{J}_s=\frac{\hbar e^*}{2mi}(\psi^*\nabla\psi-\psi\nabla\psi^*)-\frac{(e^*)^2}{m}|\psi|^2\mathbf{A}\ , \tag{2.8}$$

while the normal current $\mathbf{J}_n$ is given by

$$\mathbf{J}_n=\sigma^{(n)}\cdot\mathbf{E}=\sigma^{(n)}\cdot(-\nabla\Phi-\partial_t\mathbf{A})\ , \tag{2.9}$$

with $\sigma^{(n)}$ the normal-state conductivity tensor

$$\sigma^{(n)}=\begin{bmatrix}\sigma_{xx}^{(n)} & \sigma_{xy}^{(n)}\\ \sigma_{yx}^{(n)} & \sigma_{xx}^{(n)}\end{bmatrix}\ . \tag{2.10}$$

The Onsager relations and rotational symmetry imply that $\sigma_{yx}^{(n)}(\mathbf{H})=-\sigma_{xy}^{(n)}(\mathbf{H})$, so that the conductivity tensor may be decomposed into a diagonal piece and an antisymmetric piece. The longitudinal normal-state conductivity $\sigma_{xx}^{(n)}$ is generally a weak function of the magnetic field, and we will consider it to be field independent; the Hall conductivity in the low-field limit is of the form $\sigma_{xy}^{(n)}(H)=\omega_c\tau\sigma_{xx}^{(n)}$, with $\omega_c=eH/m$ the cyclotron frequency and τ the scattering time. Since in what follows we will be considering inhomogeneous magnetic fields, the normal-state Hall conductivity is generally a function of position. In equilibrium the electric field is zero, and the TDGL equations reduce to the familiar equilibrium Ginzburg-Landau equations.[31]

The new features in Eqs. (2.6) and (2.7) are the imaginary part in the order-parameter relaxation time, γ_2, and the Hall conductivity for the normal fluid, $\sigma_{xy}^{(n)}$. These terms are crucial for understanding the Hall effect in the mixed state.[32,33] To see why, notice that if γ_2 and $\sigma_{xy}^{(n)}$ are both zero the TDGL equations have an important *particle-hole symmetry:* the equations are invariant under the simultaneous transformations $\psi \rightarrow \psi^*$, $\Phi \rightarrow -\Phi$, and $\mathbf{A} \rightarrow -\mathbf{A}$. Under this transformation the total current $\mathbf{J}=\mathbf{J}_n+\mathbf{J}_s$ changes sign, as does the electric field. If we define the total conductivity tensor $\sigma_{\mu\nu}(\mathbf{H})$ in terms of the spatially averaged current and electric field as $J_\mu(\mathbf{H})=\sigma_{\mu\nu}(\mathbf{H})E_\nu(\mathbf{H})$, then upon reversing the magnetic field $J_\mu(-\mathbf{H})=\sigma_{\mu\nu}(-\mathbf{H})E_\nu(-\mathbf{H})$; but because of the particle-hole symmetry $J_\mu(-\mathbf{H})=-J_\mu(\mathbf{H})$ and $E_\nu(-\mathbf{H})=-E_\nu(\mathbf{H})$, so that $\sigma_{\mu\nu}(-\mathbf{H})=\sigma_{\mu\nu}(\mathbf{H})$; under these conditions the conductivity tensor is even in the magnetic field. However, we know from the Onsager reciprocity relations[34] that in general $\sigma_{\mu\nu}(\mathbf{H})=\sigma_{\nu\mu}(-\mathbf{H})$. Rotational invariance in the plane perpendicular to the applied field requires that the off-diagonal components of the conductivity tensor satisfy $\sigma_{\mu\nu}(\mathbf{H})=-\sigma_{\nu\mu}(\mathbf{H})$; when combined with the Onsager relations, we find that the off-diagonal components of the conductivity tensor must be odd in the magnetic field for a rotationally invariant system. We therefore conclude that if $\gamma_2=\sigma_{xy}^{(n)}=0$, then the Hall conductivity $\sigma_{xy}(\mathbf{H})\equiv 0$. If *either* of these quantities is nonzero, then the particle-hole symmetry is destroyed, and there will be a nonvanishing Hall conductivity. The term γ_2 produces a Hall effect due to the Magnus force on the vortex, while $\sigma_{xy}^{(n)}$ produces a Hall effect due to the transverse response of the normal fluid to the electric fields generated in the vortex core.

B. Dimensionless units

In order to facilitate the calculations it is helpful to recast Eqs. (2.6)–(2.10) into dimensionless units

$$\begin{aligned} &\mathbf{r}=\lambda\mathbf{r}' , \quad t=(\hbar/|a|)t' , \quad \psi=(|a|/b)^{1/2}\psi' , \\ &\mathbf{A}=\sqrt{2}H_c\lambda\,\mathbf{A}' , \quad \Phi=(e^*/|a|)\Phi' , \\ &\sigma=(2m/\hbar)(1/4\pi\kappa^2)\sigma' , \end{aligned} \tag{2.11}$$

where the magnetic penetration depth $\lambda=[mb/4\pi(e^*)^2|a|]^{1/2}$, the coherence length $\xi=\hbar/(2m|a|)^{1/2}$, the Ginzburg-Landau parameter $\kappa=\lambda/\xi$, and the thermodynamic critical field $H_c^2=4\pi|a|^2/b$. In these units the equations become (we will henceforth drop the primes on the dimensionless quantities)

$$(\gamma_1+i\gamma_2)(\partial_t+i\Phi)\psi=\left[\frac{\nabla}{\kappa}-i\,\mathbf{A}\right]^2\psi+\psi-|\psi|^2\psi , \tag{2.12}$$

$$\nabla\times\nabla\times\mathbf{A}=\mathbf{J}_n+\mathbf{J}_s , \tag{2.13}$$

$$\mathbf{J}_n=\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla\Phi-\partial_t\mathbf{A}\right] , \tag{2.14}$$

$$\mathbf{J}_s=\frac{1}{2\kappa i}(\psi^*\nabla\psi-\psi\nabla\psi^*)-|\psi|^2\mathbf{A} . \tag{2.15}$$

The "$\cdot$" in Eq. (2.14) indicates a tensor product. The superfluid velocity, in conventional units, is $\mathbf{v}_s=\mathbf{J}_s/e^*|\psi|^2$. In our dimensionless units this becomes

$$\mathbf{J}_s=\frac{\kappa}{2}f^2\mathbf{v}_s . \tag{2.16}$$

C. Simplification of the TDGL equations

First, we rewrite the complex order parameter in terms of an amplitude and a phase, $\psi(\mathbf{r},t)=f(\mathbf{r},t)\exp[i\chi(\mathbf{r},t)]$. (Note that a moving vortex does not possess cylindrical symmetry, so that the phase variable χ is equal to the angular variable θ only near the center of the vortex.) In terms of the gauge invariant quantities $\mathbf{Q}\equiv\mathbf{A}-\nabla\chi/\kappa$ and $P\equiv\Phi+\partial_t\chi$, the magnetic and electric fields are

$$\mathbf{h}=\nabla\times\mathbf{Q} , \tag{2.17}$$

$$\mathbf{E}=-\frac{1}{\kappa}\nabla P-\partial_t\mathbf{Q} . \tag{2.18}$$

The real part of Eq. (2.12) is

$$\gamma_1\partial_t f-\gamma_2 Pf=\frac{1}{\kappa^2}\nabla^2 f-Q^2 f+f-f^3 , \tag{2.19}$$

while the imaginary part is

$$\gamma_2\partial_t f+\gamma_1 Pf+\frac{1}{\kappa}f\nabla\cdot\mathbf{Q}+\frac{2}{\kappa}\mathbf{Q}\cdot\nabla f=0 , \tag{2.20}$$

and Eqs. (2.13)–(2.15) become

$$\nabla\times\nabla\times\mathbf{Q}=\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla P-\partial_t\mathbf{Q}\right]-f^2\mathbf{Q} . \tag{2.21}$$

To derive an equation for the potential P, first multiply Eq. (2.20) by f:

$$\gamma_1 Pf^2+\frac{1}{\kappa}\nabla\cdot(f^2\mathbf{Q})+\gamma_2 f\partial_t f=0 . \tag{2.22}$$

Next, use the fact that $\nabla\cdot(\mathbf{J}_s+\mathbf{J}_n)=0$ to obtain

$$\nabla\cdot(\sigma^{(n)}\cdot\mathbf{E})-\nabla\cdot(f^2\mathbf{Q})=0 . \tag{2.23}$$

Finally, combining Eqs. (2.22) and (2.23) we obtain

$$\frac{1}{\kappa}\nabla\cdot\left[\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla P-\partial_t\mathbf{Q}\right]\right]+\gamma_1 f^2 P+\gamma_2 f\partial_t f=0 . \tag{2.24}$$

The remainder of the paper is devoted to solving Eqs. (2.19), (2.21), and (2.24) for a moving vortex. If we set γ_2 and $\sigma_{xy}^{(n)}=0$, then our equations are identical to those studied by Schmid,[15] Hu and Thompson,[18] and Gor'kov and Kopnin,[16] in the context of the viscous motion of a single vortex. These equations are therefore a generalization which allows for the possibility of a nonzero Hall conductivity.

III. VORTEX EQUATION OF MOTION IN THE LIMIT $B \ll H_{c2}$

Since the full nonlinear TDGL equations are complicated, we want to focus attention on the motion of the vortices, as they are the "elementary excitations" of the mixed state. In this paper we will derive an equation of motion for a single vortex; we therefore consider magnetic fields which are slightly above the lower critical field H_{c1}. The vortex motion, and the concomitant motion of magnetic flux, lead to dissipation in the mixed state of type-II superconductors. Several approaches have been adopted in order to study the vortex motion. Schmid[15] constructed a dissipation functional starting from the TDGL, and from energy balance arguments he was able to calculate the flux flow conductivity in terms of the parameters of the TDGL equations. Hu and Thompson[18,35] also used energy balance arguments to calculate the flux-flow conductivity, but they included important backflow contributions which had been neglected by Schmid. This method is intuitive and easily implemented, but is insufficient for our purposes as Hall fields are nondissipative. Instead, we will use the method developed by Gor'kov and Kopnin[16,17] in their study of flux flow; this method has also been used to study the mutual friction of vortices in superfluid HeII near the λ point.[26,27,36,37] There are essentially three steps to the calculation. We first assume that the vortices translate uniformly, so that the order parameter, vector potential, and chemical potential are functions of the quantity $\mathbf{r}-\mathbf{v}_L t$, where $\mathbf{v}_L$ is the vortex line velocity. Next, we assume that these quantities may be expanded in powers of $\mathbf{v}_L$. The terms of $O(1)$ are simply the equilibrium Ginzburg-Landau equations, while the $O(v_L)$ equations are a set of linear, inhomogeneous differential equations. These equations will only have solutions for particular values of $\mathbf{v}_L$. Therefore, the final step is to derive a "solvability condition" for $\mathbf{v}_L$, which is tantamount to deriving an equation of motion for the vortices. This equation of motion, along with Faraday's law for the moving vortices,[12] $\langle \mathbf{E}\rangle = -\mathbf{v}_L\times\mathbf{B}$, lead to the longitudinal and Hall conductivities.

A. Derivation of the solvability condition

First, we assume that f, $\mathbf{Q}$, and P are only functions of $\mathbf{r}-\mathbf{v}_L t$. Therefore we replace all time derivatives in Eqs. (2.19), (2.21), and (2.24) by $-\mathbf{v}_L\cdot\nabla$, and obtain the following set of equations:

$$-\gamma_1\mathbf{v}_L\cdot\nabla f-\gamma_2 Pf=\frac{1}{\kappa^2}\nabla^2 f-Q^2 f+f-f^3\ , \tag{3.1}$$

$$\nabla\times\nabla\times\mathbf{Q}=\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla P+(\mathbf{v}_L\cdot\nabla)\mathbf{Q}\right]-f^2\mathbf{Q}\ , \tag{3.2}$$

$$\frac{1}{\kappa}\nabla\cdot\left\{\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla P+(\mathbf{v}_L\cdot\nabla)\mathbf{Q}\right]\right\} +\gamma_1 f^2 P-\gamma_2 f\mathbf{v}_L\cdot\nabla f=0\ , \tag{3.3}$$

where $P=\Phi-\mathbf{v}_L\cdot\nabla\chi$.

Next, we expand all quantities in powers of the vortex velocity; $f=f_0+f_1$, $\mathbf{Q}=\mathbf{Q}_0+\mathbf{Q}_1$, where f_1 and $\mathbf{Q}_1$ are $O(v_L)$. Note that P is $O(v_L)$, since the electric field vanishes in equilibrium. The $O(1)$ equations are simply the equilibrium Ginzburg-Landau equations,

$$\frac{1}{\kappa^2}\nabla^2 f_0-Q_0^2 f_0+f_0-f_0^3=0\ , \tag{3.4}$$

$$\nabla\times\nabla\times\mathbf{Q}_0+f_0^2\mathbf{Q}_0=0\ , \tag{3.5}$$

with the equilibrium supercurrent given by

$$\mathbf{J}_0=-f_0^2\mathbf{Q}_0\ . \tag{3.6}$$

Next, we need the $O(v_L)$ equations. In terms of the quantities $f_v\equiv\mathbf{v}_L\cdot\nabla f_0$, $\mathbf{Q}_v\equiv(\mathbf{v}_L\cdot\nabla)\mathbf{Q}_0$, these are

$$\frac{1}{\kappa^2}\nabla^2 f_1-Q_0^2 f_1-2f_0\mathbf{Q}_0\cdot\mathbf{Q}_1 +f_1-3f_0^2 f_1+\gamma_2 P f_0=-\gamma_1 f_v\ , \tag{3.7}$$

$$\nabla\times\nabla\times\mathbf{Q}_1+f_0^2\mathbf{Q}_1+2f_0 f_1\mathbf{Q}_0+\frac{1}{\kappa}\sigma^{(n)}\cdot\nabla P=\sigma^{(n)}\cdot\mathbf{Q}_v\ , \tag{3.8}$$

$$-\frac{1}{\kappa^2}\nabla\cdot(\sigma^{(n)}\cdot\nabla P)+\gamma_1 f_0^2 P=\gamma_2 f_0 f_v-\frac{1}{\kappa}\nabla\cdot(\sigma^{(n)}\cdot\mathbf{Q}_v)\ , \tag{3.9}$$

with the current $\mathbf{J}_1=\mathbf{J}_{1s}+\mathbf{J}_{1n}$, where

$$\mathbf{J}_{1s}=-f_0^2\mathbf{Q}_1-2f_0 f_1\mathbf{Q}_0\ , \tag{3.10}$$

$$\mathbf{J}_{1n}=\sigma^{(n)}\cdot\left[-\frac{1}{\kappa}\nabla P+\sigma^{(n)}\cdot\mathbf{Q}_v\right]\ . \tag{3.11}$$

Far from the center of the vortex $\mathbf{J}_{1n}\to 0$, and $\mathbf{J}_{1s}$ is equal to the applied transport current $\mathbf{J}_t$. Equations (3.7)–(3.9) are a set of inhomogeneous linear equations which must be solved in order to determine the vortex velocity.

We next derive a solvability condition for the linear inhomogeneous equations which will determine the vortex velocity. Following Gor'kov and Kopnin[17] we note that the *time-independent* Ginzburg-Landau equations possess a translational invariance, so that if $f_0(\mathbf{r})$ and $\mathbf{Q}_0(\mathbf{r})$ are solutions, so are $f_0(\mathbf{r}+\mathbf{d})$ and $\mathbf{Q}_0(\mathbf{r}+\mathbf{d})$, with $\mathbf{d}$ an arbitrary translation vector.[38] If $\mathbf{d}$ is an infinitesimal translation, then we have $f_0(\mathbf{r}+\mathbf{d})=f_0(\mathbf{r})+\mathbf{d}\cdot\nabla f_0(\mathbf{r})+\cdots$, so that the quantities $f_d\equiv\mathbf{d}\cdot\nabla f_0$ and $\mathbf{Q}_d\equiv(\mathbf{d}\cdot\nabla)\mathbf{Q}_0$ will solve the linear equations (3.7) and (3.8) without the inhomogeneous terms on the right-hand side and with $P=0$:

$$\frac{1}{\kappa^2}\nabla^2 f_d-Q_0^2 f_d-2f_0\mathbf{Q}_0\cdot\mathbf{Q}_d+f_d-3f_0^2 f_d=0\ , \tag{3.12}$$

$$\nabla\times\nabla\times\mathbf{Q}_d+f_0^2\mathbf{Q}_d+2f_0 f_d\mathbf{Q}_0=0\ , \tag{3.13}$$

with the current

$$\mathbf{J}_d\equiv(\mathbf{d}\cdot\nabla)\mathbf{J}_0=-2f_0 f_d\mathbf{Q}_0-f_0^2\mathbf{Q}_d\ . \tag{3.14}$$

To derive the solvability condition, we multiply Eq. (3.7) by f_d and integrate over a cylindrical volume. We then integrate the term $f_d\nabla^2 f_1$ by parts twice, discarding the surface terms, and we use Eq. (3.12) for f_d to eliminate the nonlinear terms in the integral. We finally obtain

$$\int d^2r(2f_0f_1\mathbf{Q}_0\cdot\mathbf{Q}_d-2f_0f_d\mathbf{Q}_0\cdot\mathbf{Q}_1+\gamma_1f_vf_d+\gamma_2Pf_0f_d)=0\ . \quad (3.15)$$

Using Eq. (3.10) for $\mathbf{J}_{1s}$ and Eq. (3.14) for $\mathbf{J}_d$, we may express Eq. (3.15) in terms of the currents as

$$\int d^2r(\mathbf{J}_d\cdot\mathbf{Q}_1-\mathbf{J}_{1s}\cdot\mathbf{Q}_d)=-\int d^2r(\gamma_1f_vf_d+\gamma_2Pf_0f_d)\ . \quad (3.16)$$

Further simplification occurs in the large-κ limit,[17] where $\mathbf{Q}_d\approx-\nabla\chi_d/\kappa$, $\mathbf{Q}_1\approx-\nabla\chi_1/\kappa$. Inserting these expressions into the left-hand side of Eq. (3.16), we have

$$\int d^2r(\mathbf{J}_d\cdot\mathbf{Q}_1-\mathbf{J}_{1s}\cdot\mathbf{Q}_d)=\frac{1}{\kappa}\int d^2r(\mathbf{J}_{1s}\cdot\nabla\chi_d-\mathbf{J}_d\cdot\nabla\chi_1)$$
$$=\frac{1}{\kappa}\int d^2r[\nabla\cdot(\mathbf{J}_{1s}\chi_d)-\nabla\cdot(\mathbf{J}_d\chi_1)]-\frac{1}{\kappa}\int d^2r\chi_d\nabla\cdot\mathbf{J}_{1s}\ , \quad (3.17)$$

where we have used the fact that $\nabla\cdot\mathbf{J}_d=0$. The first integral on the right hand side of Eq. (3.17) may be converted into a surface integral. The second term may be rewritten using $\nabla\cdot\mathbf{J}_{1s}=-\nabla\cdot\mathbf{J}_{1n}$, and then by using Eq. (3.9) we find

$$\frac{1}{\kappa}\int d^2r\chi_d\nabla\cdot\mathbf{J}_{1s}=-\frac{1}{\kappa}\int d^2r\chi_d\nabla\cdot\mathbf{J}_{1n}=\int d^2r\chi_d(\gamma_1f_0^2P-\gamma_2f_0f_v)\ . \quad (3.18)$$

Combining Eqs. (3.16), (3.17), and (3.18), we have

$$\frac{1}{\kappa}\int d\mathbf{S}\cdot(\mathbf{J}_{1s}\chi_d-\mathbf{J}_d\chi_1)=-\int d^2r(\gamma_1f_vf_d-\gamma_1\chi_df_0^2P+\gamma_2Pf_0f_d+\gamma_2\chi_df_0f_v)\ . \quad (3.19)$$

This is our solvability condition for steady vortex motion, which is exact to linear order in the vortex velocity. If this solvability condition does not hold, then the linear inhomogeneous equations have no solutions, and steady vortex motion is impossible. The remainder of this section is devoted to evaluating Eq. (3.19).

B. Coordinate system and core fields

The coordinate system which will be used for evaluating Eq. (3.19) is defined in Fig. 1. The applied transport current $\mathbf{J}_t$ is assumed to be in the x direction; the magnetic field is in the z direction (out of the page); the vortex moves at an angle θ_H with respect to the $-y$ direction, so that the averaged electric field $\langle\mathbf{E}\rangle$ makes an angle θ_H with the x axis. We will use (r,θ,z) as our cylindrical coordinates, with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$, respectively. The displacement vector $\mathbf{d}$ makes an angle ϕ with respect to the x axis. The explicit forms of $\mathbf{J}_t$, $\mathbf{v}_L$, and $\mathbf{d}$ in this cylindrical coordinate system are

$$\mathbf{J}_t=\mathbf{J}_t[\cos\theta\mathbf{e}_r-\sin\theta\mathbf{e}_\theta]\ , \quad (3.20)$$

$$\mathbf{v}_L=-v_L[\sin(\theta-\theta_H)\mathbf{e}_r+\cos(\theta-\theta_H)\mathbf{e}_\theta]\ , \quad (3.21)$$

$$\mathbf{d}=d[\cos(\theta-\phi)\mathbf{e}_r-\sin(\theta-\phi)\mathbf{e}_\theta]\ . \quad (3.22)$$

In this coordinate system, we have $\mathbf{v}_L\cdot\mathbf{d}=-v_Ld\sin(\phi-\theta_H)$ and $(\mathbf{v}_L\times\mathbf{e}_z)\cdot\mathbf{d}=-v_Ld\cos(\phi-\theta_H)$.

Before evaluating the solvability condition, we first want to simplify the equations for the order parameter, vector potential, and scalar potential, in order to bring them into a more manageable form. The equilibrium order parameter $f_0(r)$ and vector potential $\mathbf{Q}_0(r)=Q_0(r)\mathbf{e}_\theta$ satisfy Eqs. (3.4) and (3.5), which in the cylindrical coordinates defined above become

$$\frac{1}{\kappa^2}\frac{1}{r}\frac{d}{dr}\left[r\frac{df_0}{dr}\right]-Q_0^2f_0+f_0-f_0^3=0\ , \quad (3.23)$$

$$\frac{d}{dr}\frac{1}{r}\frac{d(rQ_0)}{dr}-f_0^2Q_0=0\ , \quad (3.24)$$

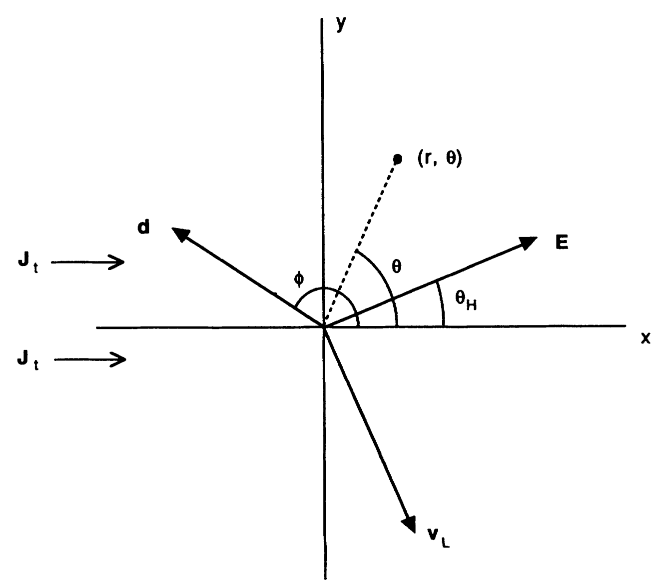

FIG. 1. Definition of the coordinate system (r,θ) and the relationship between the uniform transport current $\mathbf{J}_t$, the vortex velocity $\mathbf{v}_L$, the average electric field $\mathbf{E}$, the Hall angle θ_H, and the arbitrary translation vector $\mathbf{d}$. The magnetic field is out of the page.

with $Q_0(r) \sim -1/(\kappa r)$ as $r \to 0$.

The equation for the gauge-invariant scalar potential is rather more complicated. It can be simplified somewhat if we ignore spatial derivatives of the normal-state Hall conductivity, which should be small (especially near the center of the vortex). Then $\nabla\cdot(\sigma^{(n)}\cdot\nabla P)=\sigma_{xx}^{(n)}\nabla^2 P$. Using the coordinate system defined above and $\mathbf{h}_0=\nabla\times\mathbf{Q}_0$, Eq. (3.9) then becomes

$$\frac{\sigma_{xx}^{(n)}}{\kappa^2}\nabla^2 P-\gamma_1 f_0^2 P = \left[\gamma_2 f_0\frac{\partial f_0}{\partial r}-\frac{\sigma_{xy}^{(n)}}{\kappa}\frac{\partial h_0}{\partial r}\right]v_L\sin(\theta-\theta_H) . \quad (3.25)$$

Now we notice that as $r\to 0$, $P\approx -\mathbf{v}_L\cdot\nabla\theta = v_L\cos(\theta-\theta_H)/r$. We therefore decompose $P(\mathbf{r})$ as

$$P(\mathbf{r})=v_L[p_1(r)\cos(\theta-\theta_H)+p_2(r)\sin(\theta-\theta_H)] . \quad (3.26)$$

The contribution $p_1(r)$ satisfies a homogeneous equation

$$\frac{\sigma_{xx}^{(n)}}{\kappa^2}\frac{d}{dr}\frac{1}{r}\frac{d(rp_1)}{dr}-\gamma_1 f_0^2 p_1=0 , \quad (3.27)$$

with the boundary condition $p_1(r)\sim 1/r$ as $r\to 0$. The contribution p_2 is due to the particle-hole asymmetry, and satisfies an inhomogeneous equation

$$\frac{\sigma_{xx}^{(n)}}{\kappa^2}\frac{d}{dr}\frac{1}{r}\frac{d(rp_2)}{dr}-\gamma_1 f_0^2 p_2=\gamma_2 f_0\frac{df_0}{dr}-\frac{\sigma_{xy}^{(n)}}{\kappa}\frac{dh_0}{dr} , \quad (3.28)$$

with the boundary condition $p_2(r=0)=0$. We also have $p_1(r)$, $p_2(r)\to 0$ as $r\to\infty$. This means that the homogeneous contribution to $p_2(r)$ is identically zero, so that $p_2(r)$ is $O(\gamma_2,\sigma_{xy}^{(n)})$. Note that the equations for p_1 and p_2 are decoupled; if we had included the terms involving spatial gradients of the Hall conductivity then these two equations would be coupled together, an inessential complication for our purposes.

It is now possible to determine the electric field at the core of the vortex, $\mathbf{E}(0)$. First, notice that as $r\to 0$, the scalar and vector potentials have the following behaviors:

$$Q_0(r)\approx -\frac{1}{\kappa r}+\tfrac{1}{2}h_0(0)r , \quad (3.29)$$

$$p_1(r)\approx\frac{1}{r}-p_1^{(1)}r , \quad p_2(r)\approx -p_2^{(1)}r , \quad (3.30)$$

where $h_0(0)$ is the magnetic field at the center of the vortex, and where $p_1^{(1)}$ and $p_2^{(1)}$ are constants which are determined from the solution of Eqs. (3.27) and (3.28). In terms of these constants, the electric field at the center of the vortex is

$$\mathbf{E}(0)=\left[-\frac{1}{\kappa}\nabla P+(\mathbf{v}_L\cdot\nabla)\mathbf{Q}_0\right]_{r=0} = -\frac{p_2^{(1)}}{\kappa}\mathbf{v}_L-\left[\frac{p_1^{(1)}}{\kappa}+\tfrac{1}{2}h_0(0)\right]\mathbf{v}_L\times\mathbf{e}_z , \quad (3.31)$$

where we have used Eqs. (3.26), (3.29), and (3.30). Explicit expressions for $p_1^{(1)}$ and $p_2^{(1)}$ may be found in Appendix B. The electric field at the core of the vortex is not parallel to the averaged electric field, $\langle\mathbf{E}\rangle$, except for the particle-hole symmetric case ($p_2^{(1)}=0$).

C. Evaluation of the solvability condition

We start by evaluating the left-hand side of Eq. (3.19). The surface integral can be expressed in terms of the applied transport current at the boundaries, since at the boundaries $\mathbf{J}_{1s}(r=\infty,\theta)=\mathbf{J}_t$, $\mathbf{J}_d\cdot\mathbf{e}_r=d\sin(\theta-\phi)/(\kappa r^2)$, $\chi_d\equiv\mathbf{d}\cdot\nabla\theta=-d\sin(\theta-\phi)/r$ and $\chi_1=\kappa J_t r\cos\theta$. Substituting these expressions into the left-hand side of Eq. (3.19), and performing the remaining angular integral, we find

$$\frac{1}{\kappa}\int d\mathbf{S}\cdot[\mathbf{J}_{1s}\chi_d-\mathbf{J}_d\chi_1]=-\frac{2\pi}{\kappa}(\mathbf{J}_t\times\mathbf{e}_z)\cdot\mathbf{d} . \quad (3.32)$$

This term represents the driving force on the vortex, due to the applied transport current. It is balanced by the viscous forces on the right-hand side of the solvability condition, Eq. (3.19).

The next step is to evaluate the right-hand side of Eq. (3.19). The first term is

$$\gamma_1\int d^2r\, f_v f_d=\pi\mathbf{v}_L\cdot\mathbf{d}\gamma_1\int_0^\infty (f_0')^2 r\,dr , \quad (3.33)$$

where the prime denotes a derivative with respect to r. The second term is

$$\gamma_1\int d^2r\,\chi_d f_0^2 P=-\pi\mathbf{v}_L\cdot\mathbf{d}\gamma_1\int_0^\infty f_0^2 p_1 dr + \pi(\mathbf{v}_L\times\mathbf{e}_z)\cdot\mathbf{d}\gamma_1\int_0^\infty f_0^2 p_2 dr . \quad (3.34)$$

The third term is

$$\gamma_2\int d^2r\,P f_0 f_d=-\pi(\mathbf{v}_L\times\mathbf{e}_z)\cdot\mathbf{d}\frac{\gamma_2}{2}\int_0^\infty (f_0^2)' p_1 r\,dr - \pi\mathbf{v}_L\cdot\mathbf{d}\frac{\gamma_2}{2}\int_0^\infty (f_0^2)' p_2 r\,dr . \quad (3.35)$$

For the fourth term we have

$$\gamma_2\int d^2r\,\chi_d f_0 f_v=-\frac{\pi}{2}\gamma_2(\mathbf{v}_L\times\mathbf{e}_z)\cdot\mathbf{d} . \quad (3.36)$$

Collecting together the various terms on the right-hand sides of Eqs. (3.33)–(3.36), equating them with the driving force on the left-hand side of Eq. (3.32), and recalling that the displacement vector $\mathbf{d}$ is arbitrary, we obtain the following equation of motion for the vortex:

$$\mathbf{J}_t\times\mathbf{e}_z=\frac{\alpha_1\kappa}{2}\mathbf{v}_L+\frac{\alpha_2\kappa}{2}\mathbf{v}_L\times\mathbf{e}_z , \quad (3.37)$$

where the constants α_1 and α_2 are given by

$$\alpha_1=\gamma_1\int_0^\infty (f_0')^2 r\,dr+\gamma_1\int_0^\infty f_0^2 p_1 dr - \frac{\gamma_2}{2}\int_0^\infty (f_0^2)' p_2 r\,dr , \quad (3.38)$$

$$\alpha_2=-\frac{\gamma_2}{2}\int_0^\infty (f_0^2)'p_1 r\,dr-\frac{\gamma_2}{2}-\gamma_1\int_0^\infty f_0^2 p_2 dr\ . \tag{3.39}$$

Alternative expressions for α_1 and α_2 may be obtained by using Eqs. (3.27) and (3.28) for $p_1(r)$ and $p_2(r)$, as follows:

$$\gamma_1\int_0^\infty f_0^2 p_1 dr=\frac{\sigma_{xx}^{(n)}}{\kappa^2}\int_0^\infty \frac{d}{dr}\frac{1}{r}\frac{d(rp_1)}{dr}dr=\frac{2\sigma_{xx}^{(n)}}{\kappa^2}p_1^{(1)}\ , \tag{3.40}$$

$$\begin{aligned}\gamma_1\int_0^\infty f_0^2 p_2 dr&=\frac{\sigma_{xx}^{(n)}}{\kappa^2}\int_0^\infty \frac{d}{dr}\frac{1}{r}\frac{d(rp_2)}{dr}dr\\&\quad-\frac{\gamma_2}{2}-\frac{1}{\kappa}\sigma_{xy}^{(n)}h_0(0)\\&=\frac{2\sigma_{xx}^{(n)}}{\kappa^2}p_2^{(1)}-\frac{\gamma_2}{2}-\frac{1}{\kappa}\sigma_{xy}^{(n)}(0)h_0(0)\ ,\end{aligned} \tag{3.41}$$

where $\sigma_{xy}^{(n)}(0)=\sigma_{xy}^{(n)}[h_0(0)]$ is the normal-state Hall conductivity in a magnetic field equal to the field in the vortex core. Then we have

$$\alpha_1=\gamma_1\int_0^\infty (f_0')^2 r\,dr+\frac{2\sigma_{xx}^{(n)}}{\kappa^2}p_1^{(1)}-\frac{\gamma_2}{2}\int_0^\infty (f_0^2)'p_2 r\,dr\ , \tag{3.42}$$

$$\alpha_2=-\frac{2\sigma_{xx}^{(n)}}{\kappa^2}p_2^{(1)}+\frac{1}{\kappa}\sigma_{xy}^{(n)}h_0(0)-\frac{\gamma_2}{2}\int_0^\infty (f_0^2)'p_1 r\,dr\ . \tag{3.43}$$

The last integral in Eq. (3.42) is generally quite small, being $\mathcal{O}(\gamma_2^2,\gamma_2\sigma_{xy}^{(n)})$, and will be dropped from now on. Finally, by using Eq. (2.16) for the superfluid velocity, we can also write Eq. (3.37) in terms of the superfluid velocity at the boundaries (where $f_0=1$), $\mathbf{v}_{s1}=2\mathbf{J}_t/\kappa$, as

$$\mathbf{v}_{s1}\times\mathbf{e}_z=\alpha_1\mathbf{v}_L+\alpha_2\mathbf{v}_L\times\mathbf{e}_z\ . \tag{3.44}$$

Equations (3.37)–(3.44) are the primary results of this paper.

To calculate the conductivities, we use Faraday's law, $\langle\mathbf{E}\rangle=-\mathbf{v}_L\times\mathbf{B}$, to obtain

$$\mathbf{J}_t=\frac{\alpha_1\kappa}{2B}\langle\mathbf{E}\rangle+\frac{\alpha_2\kappa}{2B}\langle\mathbf{E}\rangle\times\mathbf{e}_z\ . \tag{3.45}$$

We therefore obtain the longitudinal conductivity

$$\sigma_{xx}=\frac{\alpha_1\kappa}{2B} \tag{3.46}$$

and the transverse or Hall conductivity

$$\sigma_{xy}=\frac{\alpha_2\kappa}{2B}\ . \tag{3.47}$$

Returning to conventional units, we have

$$\sigma_{xx}=\frac{2m}{\hbar}\frac{\alpha_1}{8\pi\kappa^2}\frac{H_{c2}}{B} \tag{3.48}$$

and

$$\sigma_{xy}=\frac{2m}{\hbar}\frac{\alpha_2}{8\pi\kappa^2}\frac{H_{c2}}{B}\ , \tag{3.49}$$

with the corresponding Hall angle

$$\tan\theta_H=\frac{\alpha_2}{\alpha_1}\ . \tag{3.50}$$

Therefore, we find that the Hall angle is *independent* of magnetic field near H_{c1}.

D. Comparison to previous work

Neutral superfluids. The dynamics of a vortex in a neutral superfluid described by "model-A" dynamics[39] (a nonconserved order parameter without coupling to a conserved density) has been considered by Onuki.[40] This is a limiting case of the above results, obtained by taking $\sigma_{xx}^{(n)}\to\infty$, $\kappa\to\infty$, and $\sigma_{xy}^{(n)}=0$; then $p_1(r)=1/r$ and $p_2(r)=0$. Using the expressions for α_1 and α_2 in Eqs. (3.38) and (3.39), we find

$$\alpha_1=\gamma_1\int_0^\infty\left[(f_0')^2+\frac{f_0^2}{r^2}\right]dr\ , \tag{3.51}$$

$$\alpha_2=-\gamma_2\ . \tag{3.52}$$

For the Galilean invariant case $\Gamma_2=0$; if in addition we assume that the dissipation is small so that $\Gamma_1\ll 1$, then $\gamma_1\approx\Gamma_1$ and $\gamma_2\approx-1$ (see Sec. II A above). Combining these results, we arrive at the equation of motion derived by Onuki,[40]

$$(\mathbf{v}_{s1}-\mathbf{v}_L)\times\mathbf{e}_z=\alpha_1\mathbf{v}_L\ . \tag{3.53}$$

The left-hand side is the Magnus force acting on the vortex, and the right-hand side is the viscous drag on the vortex. If, in addition, $\Gamma_1=0$ (so that our order-parameter equation of motion corresponds to the Gross-Pitaevskii equation), then $\alpha_1=0$, and the vortex will drift with the local superfluid velocity.

An interesting feature of Eq. (3.51) for α_1 is that the integral is logarithmically divergent [since $f_0(r)\approx1$ for $r\gg1$]. According to Onuki, the divergence should be cut off at a length L, which is either the intervortex spacing or the wavelength of second sound.[40] However, Neu[41] has recently shown that the damping coefficient α_1 is actually velocity dependent, indicating a general breakdown of linear response in two dimensions for the superfluid described by model-A dynamics (similar results for the drag on a disclination line in a nematic liquid crystal have been obtained by Ryskin and Kremenetsky[42]). The point is that for a *moving* vortex, there is a length scale set by the velocity of the vortex, which in conventional units is $L=\hbar/(2m\gamma_1 v_L)$, and it is this length which cuts off the divergence of the friction coefficient. Therefore $\alpha_1\approx\gamma_1\ln(L/\xi)$ is perfectly well defined, in contrast to the equilibrium energy of the vortex, which is logarithmically divergent and will depend on the system size (or the intervortex separation). A more realistic model of a superfluid includes a coupling of the order parameter to the entropy density ("model-F" dynamics[39]); as shown by Onuki,[26,27] this coupling removes the divergence in α_1 by causing the dissipation to occur in the core of the vortex. Something similar occurs for the charged superfluid, where there is an additional length scale set by the conductivity of the normal fluid, as discussed by Hu and Thompson;[18] in conventional units, this length is $\zeta_{\rm HT}=(4\pi\hbar\sigma_{xx}^{(n)}/2m\gamma_1)^{1/2}\lambda$. As shown in Appendix B, if the normal fluid has a very high conductivity so that $\zeta_{\rm HT}\gg\xi$, then $\alpha_1\approx\gamma_1\ln(\zeta_{\rm HT}/\xi)$; the length $\zeta_{\rm HT}$ cuts off the logarithmic divergence in α_1. We therefore expect linear response to hold for more realistic models of neutral superfluids and for charged superfluids (superconductors).

Paramagnetic impurities. For a superconductor containing a high concentration of paramagnetic impurities, Gor'kov and Éliashberg[20] have shown that the parameters in the TDGL are related such that $\zeta_{\rm HT}=\xi/\sqrt{12}$, and $\kappa^{-2}=48\pi\hbar\sigma_{xx}^{(n)}/(2m\gamma_1)$. Kupriyanov and Likharev[43] numerically integrated Eqs. (3.23), (3.24), and (3.27), and found $\alpha_1=0.438\gamma_1$. Hu[44] used a hybrid method which combined a trial order-parameter solution with numerical integration and also found $\alpha_1=0.438\gamma_1$, corrected the earlier work of Hu and Thompson.[18] The trial order-parameter solution discussed in Appendix B gives a value of $\alpha_1=0.436\gamma_1$. Therefore, it appears that at least in the large-κ limit the trial order parameter allows us to calculate the transport coefficients to within 1%. The results may be extended to lower values of κ by finding the optimal value of $\xi_v(\kappa)$ using the variational principle discussed in the Appendix. Substituting our value of α_1 into our expression for the longitudinal conductivity, Eq. (3.48), we find

$$\sigma_{xx}=2.62\sigma_{xx}^{(n)}\frac{H_{c2}}{B}\ , \tag{3.54}$$

in agreement with the results of Kupriyanov and Likharev[43] and Hu.[44] For the Hall conductivity we obtain (in conventional units)

$$\sigma_{xy}=6\sigma_{xx}^{(n)}\frac{\alpha_2}{\gamma_1}\frac{H_{c2}}{B}\ , \tag{3.55}$$

where

$$\alpha_2=-0.140\gamma_2-0.186\left[\gamma_2+\frac{2\pi\hbar}{m}\sigma_{xy}^{(n)}(0)\right]+\frac{\pi\hbar}{m}\ln\kappa\ \sigma_{xy}^{(n)}(0)\frac{h_0(0)}{H_{c1}}\ . \tag{3.56}$$

There are two features of this result which are worth emphasizing. First, the Hall conductivity contains two contributions, one from the imaginary part of the order-parameter relaxation time γ_2, and one from the normal-state Hall conductivity. If $\Gamma_2=0$ [which would produce the London acceleration equation (see Appendix A)], then $\gamma_2<0$, and these two contributions have the same sign, leading to a Hall effect in the mixed state of the same sign as in the normal state. To get a sign change we at least require that $\gamma_2>0$. If, in addition, $\Gamma_1\ll1$, then $\gamma_2\approx-1$; if we ignore the contribution from the normal-state Hall conductivity, then $\alpha_2=0.326$, quite different from the value of $\alpha_2=1$ which we obtain for the neutral superfluid discussed above. Due to the screening effect of the normal fluid, the vortex does not experience the full Magnus force as it would in a neutral superfluid.

Dirty limit. As shown by Schmid,[15] in the dirty limit the dimensionless normal-state conductivity is proportional to the real part of the order-parameter relaxation time: $\sigma_{xx}^{(n)}=0.173\gamma_1$ ($\sigma_{xx}^{(n)}/\gamma_1=\sum$ in Schmid's notation). Using the results of Appendix B, we find that $\alpha_1=0.508\gamma_1$. Therefore, in conventional units we find for the longitudinal conductivity

$$\sigma_{xx}=1.47\sigma_{xx}^{(n)}\frac{H_{c2}}{B}\ . \tag{3.57}$$

Schmid obtained a similar result but with a prefactor of 1.56; we have not been able to track down the source of this small discrepancy. For the Hall conductivity, we have

$$\sigma_{xy}=2.89\sigma_{xx}^{(n)}\frac{\alpha_2}{\gamma_1}\frac{H_{c2}}{B}\ , \tag{3.58}$$

where we again use the results of Appendix B to find

$$\alpha_2=-0.178\gamma_2-0.258\left[\gamma_2+\frac{2\pi\hbar}{m}\sigma_{xy}^{(n)}(0)\right]+\frac{\pi\hbar}{m}\ln\kappa\ \sigma_{xy}^{(n)}(0)\frac{h_0(0)}{H_{c1}}\ . \tag{3.59}$$

Fukuyama, Ebisawa, and Tsuzuki[30] have derived TDGL equations from the microscopic BCS theory, including terms which break particle-hole symmetry, and find

$$\gamma_1=\frac{\pi}{8k_BT_c}\frac{\hbar^2}{2m\xi^2(0)}\ , \tag{3.60}$$

$$\gamma_2=\alpha\left[\frac{k_BT_c}{\epsilon_F}\right]\gamma_1\ , \tag{3.61}$$

where m is the effective mass of a Cooper pair in the x-y plane, $\xi(0)$ is the zero temperature correlation length in the x-y plane, ϵ_F if the Fermi energy, and α is a dimensionless parameter introduced by Fukuyama, Ebisawa, and Tsuzuki which characterizes the electronic structure of the material. The sign of the Hall conductivity therefore depends on the sign of α; in this picture the sign change would be a consequence of the detailed electronic structure of the material.

IV. THERMOMAGNETIC EFFECTS IN THE LIMIT $B \ll H_{c2}$

A. The transport energy

In addition to producing dissipation, moving vortices also transport energy in a direction which is parallel to their velocity. This leads to thermomagnetic effects in the mixed state which are significantly enhanced over their normal-state counterparts. In order to calculate these effects, we first note that the energy current, which can be derived from energy conservation,[15] is given by (in dimensionless units)

$$\mathbf{J}^h=\mathbf{E}\times\mathbf{h}-\mathbf{E}\times\mathbf{B}-\frac{1}{2\kappa}\left[\left[\frac{\nabla}{\kappa}-i\,\mathbf{A}\right]\psi(\partial_t-i\Phi)\psi^* + \left[\frac{\nabla}{\kappa}+i\,\mathbf{A}\right]\psi^*(\partial_t+i\Phi)\psi\right], \quad (4.1)$$

where the unit of heat current is $(H_c^2/4\pi)(\hbar/m)(\kappa^2/\lambda)$. In Eq. (4.1) we have subtracted the energy current of the uniform background induction field, which was not considered by Schmid. Expressing this in terms of the gauge-invariant quantities $\mathbf{Q}$ and P, we have

$$\mathbf{J}^h=\left[-\frac{1}{\kappa}\nabla P-\partial_t\mathbf{Q}\right]\times(\nabla\times\mathbf{Q}) +\frac{1}{\kappa}\left[-\frac{1}{\kappa}(\nabla f)\partial_t f+P\mathbf{Q}f^2\right]-\mathbf{v}_L B^2 . \quad (4.2)$$

We again assume that the vortex translates uniformly, and expand the order parameter and the potentials in powers of the velocity, to obtain the local heat current

$$\mathbf{J}^h=\left[-\frac{1}{\kappa}\nabla P+\mathbf{v}_L\cdot\nabla\mathbf{Q}_0\right]\times(\nabla\times\mathbf{Q}_0) +\frac{1}{\kappa}\left[-\frac{1}{\kappa}(\mathbf{v}_L\cdot\nabla f_0)(\nabla f_0)+P\mathbf{Q}_0 f_0^2\right]-\mathbf{v}_L B^2 . \quad (4.3)$$

Using $\nabla\times\nabla\times\mathbf{Q}_0+f_0^2\mathbf{Q}_0=0$, the first and last terms in Eq. (4.3) may be combined:

$$(\nabla P)\times(\nabla\times\mathbf{Q}_0)+P\nabla\times\nabla\times\mathbf{Q}_0=\nabla\times(P\nabla\times\mathbf{Q}_0) , \quad (4.4)$$

where a vector identity has been used. The second term on the left-hand side of Eq. (4.3) may be written as

$$(\mathbf{v}_L\cdot\nabla\mathbf{Q}_0)\times(\nabla\times\mathbf{Q}_0) =\nabla\times[(\mathbf{v}_L\cdot\mathbf{Q}_0)\nabla\times\mathbf{Q}_0]+\mathbf{v}_L(\nabla\times\mathbf{Q}_0)^2 -(\mathbf{v}_L\cdot\mathbf{Q}_0)\nabla\times\nabla\times\mathbf{Q}_0 , \quad (4.5)$$

where we have again used several vector identities. Combining Eqs. (4.3)–(4.5), we have

$$\mathbf{J}^h=\nabla\times\left[\left[-\frac{1}{\kappa}P+\mathbf{v}_L\cdot\mathbf{Q}_0\right]\nabla\times\mathbf{Q}_0\right] +\frac{1}{\kappa^2}(\mathbf{v}_L\cdot\nabla f_0)(\nabla f_0)+f_0^2(\mathbf{v}_L\cdot\mathbf{Q}_0)\mathbf{Q}_0 +\mathbf{v}_L h_0^2-\mathbf{v}_L B^2 . \quad (4.6)$$

Finally, we average overall space, and note that the first term in Eq. (4.6) is a surface term, which vanishes. Then we have

$$\langle\mathbf{J}^h\rangle=\int d^2r\,\mathbf{J}^h(\mathbf{r})=\tfrac{1}{2}nU_\phi\mathbf{v}_L , \quad (4.7)$$

where n is the vortex density (equal to B/ϕ_0 in conventional units and $(\kappa/2\pi)B$ in dimensionless units) and U_ϕ is the transport energy per vortex (the $\frac{1}{2}$ is due to the choice of dimensions); this combination is equal to

$$nU_\phi=2\pi\int_0^\infty\left[\frac{1}{\kappa^2}(f_0')^2+f_0^2Q_0^2+2h_0^2\right]r\,dr-2B^2 . \quad (4.8)$$

The first two terms in the integral in Eq. (4.8) are the kinetic energy of the superfluid (the factor of $\frac{1}{2}$ coming from an angular average), while the third term is twice the magnetic-field energy. Recently, Doria, Gubernatis, and Rainer[45] have derived a "virial theorem" which shows that this combination is precisely equal to $2\mathbf{H}\cdot\mathbf{B}$. Therefore, we find quite generally that $U_\phi=-(2\pi/\kappa)8\pi M$, where $\mathbf{M}=(\mathbf{B}-\mathbf{H})/4\pi$ is the spatially averaged magnetization of the sample. This result is true throughout the mixed state. Near H_{c1}, $B\approx 0$, so that $M\approx -H_{c1}/4\pi$. Then $U_\phi\approx(4\pi/\kappa)H_{c1}$, which is the line energy ϵ_1 of the vortex.[31] The line energy is calculated using a trial order parameter in Appendix B; for large κ, we find that the transport energy per vortex near H_{c1} is

$$U_\phi\approx\frac{2\pi}{\kappa^2}(\ln\kappa+0.519)=\left[\frac{\phi_0}{4\pi\lambda}\right]^2(\ln\kappa+0.519) , \quad (4.9)$$

where the units have been reinstated in the last line of Eq. (4.9). Therefore, the moving vortex transports an amount of energy equal to the vortex line energy.[46]

Before discussing the various thermomagnetic effects, we should mention that there have been several previous attempts to calculate the transport energy in the low induction limit. De Lange[47] and Kopnin[48] both calculated the transport energy but neglected to account for the contribution coming from the electromagnetic field. Hu[49] included the contribution from the electromagnetic field, but unfortunately never provided an explicit derivation of his result.

B. Thermomagnetic effects

We are now in a position to calculate the thermomagnetic effects; the definitions are summarized in Appendix C. First, we combine Eq. (4.7) with Faraday's Law for the moving vortices, $\langle\mathbf{E}\rangle=-\mathbf{v}_L\times\mathbf{B}$, to obtain

$$\langle\mathbf{J}^h\rangle=(U_\phi/B)\langle\mathbf{E}\rangle\times\mathbf{e}_z , \quad (4.10)$$

so that the vortex contribution to the transport coefficient $\alpha_{xy}=U_\phi/B$. The normal fluid contribution to α_{xy} is generally several orders of magnitude smaller than the vortex contribution, and it will therefore be omitted. Since the energy current for a vortex is always perpendicular to the electric field, the moving vortices do not contribute to the transport coefficient α_{xx}; any contribution to α_{xx} arises solely from the normal fluid flow. We may therefore write $\alpha_{xx}=T\epsilon^{(n)}/\rho_{xx}^{(n)}$, with $\epsilon^{(n)}$ the normal-state thermopower and $\rho_{xx}^{(n)}$ the normal-state resistivity. The longitudinal thermomagnetic effects arise primarily from the motion of the normal fluid, while the transverse thermomagnetic effects are predominantly due to vortex motion.

We have, for the Nernst coefficient ν,

$$\nu=\frac{1}{TH}\rho_{xx}\alpha_{xy}=\frac{\phi_0}{TH}\frac{\hbar}{2m}\frac{\ln\kappa}{\alpha_1} , \tag{4.11}$$

where we have used Eq. (3.48) for the resistivity and Eq. (4.9) for the transport energy. This is the Nernst coefficient for a *single* vortex; for a collection of $n=B/\phi_0$ noninteracting vortices, this result should be multiplied by n. The thermopower is given by

$$\epsilon=\frac{1}{T}\rho_{xx}\alpha_{xx}+H\nu\tan\theta_H=(\rho_{xx}/\rho_{xx}^{(n)})\epsilon^{(n)}+H\nu\tan\theta_H , \tag{4.12}$$

with ρ_{xx} the flux-flow conductivity calculated above. The vortex motion does contribute to the thermopower if there is a nonzero Hall angle, but under most circumstances the first term in Eq. (4.12) is much larger than the second term. Therefore the thermopower in the mixed state will generally track the behavior of the flux-flow resistivity, and not the behavior of the Hall angle. This is in agreement with recent measurements of the thermopower in the high-temperature superconductors;[50,51] similar conclusions have been reached starting from a phenomenological model.[52]

V. VORTEX BENDING AND FLUCTUATIONS

So far we have limited our discussion to the motion of rectilinear vortices, without thermal fluctuations. However, it is straightforward to generalize the technique discussed in Sec. III above to situations in which the vortices are bent along the z direction. This has been carried out by Gor'kov and Kopnin;[16,17] the addition of a complex relaxation time does not change their derivation, so we will only outline the derivation here and refer the reader to the original literature for the details. First, since we have in mind the problem of vortex motion in high-temperature superconductors, we want to allow for an anisotropic effective mass in the Ginzburg-Landau Hamiltonian; the effective mass is m in the x-y plane, and m_z along the z direction. In this notation, ξ, λ, and $\kappa=\lambda/\xi$ will denote the correlation length, penetration depth, and Ginzburg-Landau parameter in the x-y plane. Next, we label the vortex position by $\mathbf{r}_L(z,t)$; $\mathbf{r}$ will be a position vector in the x-y plane. We assume that both the order parameter and the vector potential, which are functions of the position $(\mathbf{r},z)$ and time t, are functions only of the distance away from the vortex at time t; e.g., $f(\mathbf{r},z,t)=f(\mathbf{r}-\mathbf{r}_L)$. We again expand the order parameter, vector potential, and scalar potential in powers of the velocity $\partial_t\mathbf{r}_L$ and the curvature $\partial_z\mathbf{r}_L$ of the vortex; substituting these expansions into the TDGL equations, we find that the zeroth-order terms produce the equilibrium Ginzburg-Landau equations, while the first-order terms produce a set of linear inhomogeneous differential equations. Utilizing the translational invariance of the equilibrium equations, we derive a solvability condition. After evaluating this solvability condition, we arrive at the following equation of motion for the vortex (in conventional units):

$$\eta_1\partial_t\mathbf{r}_L+\eta_2(\partial_t\mathbf{r}_L)\times\mathbf{e}_z=\phi_0\mathbf{J}_t\times\mathbf{e}_z+\tilde{\epsilon}_1\frac{\partial^2\mathbf{r}_L}{\partial z^2} , \tag{5.1}$$

with

$$\eta_1=\frac{2m}{\hbar}\left[\frac{\phi_0}{4\pi\lambda}\right]^2\alpha_1 , \quad \eta_2=\frac{2m}{\hbar}\left[\frac{\phi_0}{4\pi\lambda}\right]^2\alpha_2 , \tag{5.2}$$

and where $\tilde{\epsilon}_1=m\epsilon_1/m_z$, with ϵ_1 the line energy of the vortex (see Appendix B).[53]

This equation of motion can be used to study the propagation of helicon waves, which are elliptically polarized waves which propagate along the z direction. Setting $\mathbf{J}_t=0$, and searching for solutions of the form

$$\mathbf{r}_L(z,t)=\mathbf{u}_0e^{i(kz-\omega t)} , \tag{5.3}$$

we find (in conventional units)

$$\omega_\pm=\frac{\pm\alpha_2-i\alpha_1}{\alpha_1^2+\alpha_2^2}\left[\frac{\hbar}{2m_z}\ln\kappa\right]k^2 . \tag{5.4}$$

If we set $\alpha_1=0$ and $\alpha_2=1$, then we obtain the well-known dispersion relation for helicon waves in an ideal incompressible fluid.[31] However, under most circumstances $\alpha_2\ll\alpha_1$, so these waves are overdamped, and therefore rather difficult to observe.

It is possible to include the effects of thermal fluctuations by appealing to the fluctuation-dissipation theorem. To the right-hand side of Eq. (5.1) we add a fluctuating force $\boldsymbol{\zeta}(z,t)$, which is chosen in such a manner so as to guarantee that the correct equilibrium correlations are obtained. The resulting equation of motion (in conventional units) is

$$\eta_1\partial_t\mathbf{r}_L+\eta_2(\partial_t\mathbf{r}_L)\times\mathbf{e}_z=-\frac{\delta H_L}{\delta\mathbf{r}_L}+\boldsymbol{\zeta} , \tag{5.5}$$

where the "Hamiltonian" for the vortex line is

$$H_L=\int dz\left[\frac{\tilde{\epsilon}_1}{2}\left[\frac{\partial\mathbf{r}_L}{\partial z}\right]^2-\phi_0(\mathbf{J}_t\times\mathbf{e}_z)\cdot\mathbf{r}_L\right] , \tag{5.6}$$

and where the noise term has the correlations

$$\begin{aligned}&\langle\zeta_i(z,t)\rangle=0 ,\\&\langle\zeta_i(z,t)\zeta_j(z',t')\rangle=2\eta_1k_BT\delta_{ij}\delta(z-z')\delta(t-t') ,\end{aligned} \tag{5.7}$$

with the brackets denoting an average with respect to the noise distribution. We see that the first term in the Hamiltonian is the bending energy of the vortex while the second term is essentially "$\mathbf{J}_t\cdot\mathbf{A}$"; i.e., the interaction energy between the magnetic field and the transport current. Similar Langevin equations for vortices in superfluid HeII have been derived starting from model-F dynamics by Onuki,[26,27] Kawasaki,[54] and Ohta, Ohta, and Kawasaki.[55] Also, Ambegaokar *et al.*[56] have used Langevin equations for point vortices in two dimensions to study vortex dynamics near the Kosterlitz-Thouless vortex unbinding transition.

We can use the Langevin equation to determine the distance that a single vortex line $\mathbf{r}_L(z,t)$ wanders perpendicular to the z axis in a time t, in the absence of a transport current. For simplicity we will assume that $\eta_2=0$, so that the motion is purely diffusive. Clearly the center of mass does not move, i.e., $\langle \mathbf{r}_L(z,t)\rangle=0$. For the mean-square displacement we have

$$\langle |\mathbf{r}_L(z,t)-\mathbf{r}_L(0,0)|^2\rangle = 8\eta_1 k_B T\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1-e^{i(kz-\omega t)}}{(\tilde{\epsilon}_1 k^2)^2+(\eta_1\omega)^2} = \frac{2k_BT|z|}{\tilde{\epsilon}_1}f(z^2/4D|t|)\ , \quad (5.8)$$

where $D=\tilde{\epsilon}_1/\eta_1=(\hbar/m_z)(\ln\kappa/2\alpha_1)$ is the diffusion constant for the vortex motion, and where the scaling function $f(x)$ is given by

$$f(x)=\frac{1}{\sqrt{\pi x}}e^{-x}+\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-y^2/4x}\sin y\frac{dy}{y}\ . \quad (5.9)$$

The last integral in Eq. (5.9) can be expressed in terms of generalized hypergeometric functions, but this is not particularly useful for our purposes. We are primarily concerned with the limiting behavior of the scaling function. For $x\to 0$, the $\sin y$ may be expanded in a power series and we find

$$f(x)\sim\frac{1}{\sqrt{\pi x}}[1+x+O(x^2)]\ , \quad (5.10)$$

while for $x\to\infty$ the integral may be calculated by steepest descents with the result

$$f(x)\sim 1+\frac{2}{\sqrt{\pi x}}e^{-x}[1+O(x^{-1})]\ . \quad (5.11)$$

We see that at equal times $(x\to\infty)$ the mean-square displacement as a function of z scales as $|z|^{1/2}$, so that the vortex "diffuses" along the z direction, as was first discussed by Nelson.[57,58] On the other hand, if we focus on the fixed value of $z=0$, then we see that the mean-square displacement scales with time as

$$\langle |\mathbf{r}_L(0,t)-\mathbf{r}_L(0,0)|^2\rangle=\frac{2k_BT}{\tilde{\epsilon}_1}\left[\frac{4D|t|}{\pi}\right]^{1/2}\ . \quad (5.12)$$

This is quite different from the result we would obtain for point vortices diffusing in two dimensions, where the mean-square displacement would scale as $|t|$. The difference is due to the restraining effect of the line tension of the vortex.

We should stress that our vortex equation of motion was derived in the absence of vortex pinning. Pinning may be included in a phenomenological fashion by including a pinning potential $V_p(\mathbf{r}_L,z)$ in the Hamiltonian for the vortex line:

$$H_L=\int dz\left[\frac{\tilde{\epsilon}_1}{2}\left[\frac{\partial\mathbf{r}_L}{\partial z}\right]^2+V_p(\mathbf{r}_L,z)-\phi_0(\mathbf{J}_t\times\mathbf{e}_z)\cdot\mathbf{r}_L\right]\ . \quad (5.13)$$

The equation of motion would still be given by Eq. (5.5). Numerical studies of this equation have recently been carried out by Enomoto and collaborators.[59]

VI. DISCUSSION AND SUMMARY

In this paper we have derived an equation of motion for a single vortex in a type-II superconductor in the large-κ limit, starting from a set of generalized TDGL equations. This in turn allowed us to calculate the Hall conductivity for a single vortex moving in response to an applied transport current. There are two important features of the results which are worth emphasizing. First, there are two contributions to the Hall conductivity σ_{xy} (and therefore to the Hall angle θ_H), one from the imaginary part of the order-parameter relaxation time γ_2, and the other from the Hall conductivity of the normal fluid in the core of the vortex, $\sigma_{xy}^{(n)}(0)$. This is different from both the Bardeen-Stephen (BS) and Nozières-Vinen (NV) models, in which θ_H is determined entirely by the normal-state Hall conductivity. Second, θ_H is independent of the magnetic field. In this regard our result resembles the behavior of θ_H obtained in the NV model, but with a magnitude which depends on details such as the normal-state conductivity, the order-parameter relaxation time, and so on. This is quite different from the predictions of the BS model, in which the Hall angle is linear in magnetic field. Jing and Ong[60] have recently measured the flux-flow Hall conductivity in $NbSe_2$, a material which can be prepared with comparatively few macroscopic inhomogeneities which would serve to pin vortices. They find that θ_H in the vortex state is field independent, in agreement with the NV model; however, their results would also be consistent with the conclusions of this paper. On the other hand, Hagen *et al.*[4] have measured the flux-flow Hall effect in thin films of the high T_c superconductor $Tl_2Ba_2CaCu_2O_8$ (which should also have relatively weak pinning due to enhanced thermal fluctuations) and find that θ_H has a field-independent component with a complicated temperature dependence and a component which is linear in the magnetic field and which resembles the normal-state Hall angle, in apparent contradiction with our results. However, it is important to bear in mind that our derivation was for a single vortex; i.e., for H close to H_{c1}. It is possible that at higher magnetic inductions the contribution to σ_{xy} from the Hall conductivity of the normal fluid in the vortex core will become magnetic-field dependent (as the magnetic field in the core would then be a function of H). When this is added to the field-independent contribution from γ_2, θ_H would have exactly the form suggested by the experiments of Hagen *et al.* The complicated temperature dependence of the field-independent term would be encapsulated in γ_2.

As discussed in the introduction, this work was motivated by a number of experimental observations of an anomalous sign change in the Hall conductivity in several of the high-temperature superconductors. With results in hand, it is now time to see whether our calculations shed any light on these puzzling observations. First, as previously noted, the Hall conductivity of the vortex state will have a sign opposite to that in the normal state if $\alpha_2<0$. From the explicit calculations of α_2 in Appendix B, we see from Eq. (B15) that if $\gamma_2>0$, there is at least the possibility that $\alpha_2<0$, whereas if $\gamma_2<0$, it appears that α_2 is always positive. Therefore the issue of the sign change of the Hall conductivity hinges on whether γ_2 is positive or negative. If we choose γ_2 so that we generate the London acceleration equation in the hydrodynamic limit (ensuring Galilean invariance), then we must take $\gamma_2=-1$, and the Hall effect does not change sign. This is in accord with the simple picture that the vortex motion which results from the Magnus force produces a Hall field which is in the same direction as the normal-state Hall field. If, however, we imagine deriving the TDGL equations from the microscopic theory, along the lines of the work by Fukuyama, Ebisawa, and Tsuzuki[30] then the order parameter relaxation time depends upon the detailed electronic structure of the material, and it is quite possible that $\gamma_2>0$, producing a sign change in the Hall conductivity.

There has been a recent suggestion by Wang and Ting[61] that pinning forces acting on a vortex may produce backflow currents which act in such a way so as to change the sign of the Hall angle. It appears difficult to incorporate this effect into the present calculation. However, there are several *a priori* objections to this mechanism which are worth mentioning. First, the observed sign change occurs at relatively high temperatures (close to T_{c2}); one might expect that at these temperatures thermal fluctuations would tend to overwhelm the pinning forces, rendering them ineffective. In fact, experimental studies of the Ettingshausen effect[62] and the Nernst effect[63] in Y-Ba-Cu-O near T_{c2} appear to indicate extensive flux flow, consistent with the idea that pinning is insignificant in this regime. Second, from their model Wang and Ting predict that as the temperature is lowered the pinning should cause the longitudinal and Hall resistivities to vanish at different temperatures, for a fixed value of the magnetic field. However, the recent measurements of Luo *et al.*[6] indicate that these resistivities vanish at the same temperature. Third, Wang and Ting begin with a hydrodynamic model for the superfluid velocity (with the attendant shortcomings), and incorporate the effect of pinning by including a pinning force which depends on the fluid velocity in the normal core. This is rather peculiar, in that one expects the pinning force to be position dependent, but not velocity dependent. A crucial test of the Wang and Ting theory would be to see if a Hall resistivity which is initially positive in the mixed state of some material can be induced to change sign as pinning sites are artificially introduced (by ion bombardment, for example).

Are there other measurements which might be useful in sorting out the sign change problem? One might hope that measurements of the thermopower would be useful, as the thermopower owes its existence to particle-hole asymmetry. Unfortunately, the thermopower in the mixed state is dominated by the normal-state contribution, and is therefore proportional to the flux-flow resistivity [see the discussion in Sec. IV B and Eq. (4.12)]. Therefore the thermopower provides very little additional information which would be useful in piecing together the puzzle. The observation of helicon waves in the vortex state would also be interesting; a change in the sign of the Hall effect would cause the polarization of the helicon waves to change direction. However, given that these waves are heavily overdamped, the prospects for observing this effect appear dim. If the sign change is indeed a consequence of the electronic structure of the material, then a more sophisticated theory should be able to predict the existence of the sign change based on, say, band-structure calculations. There is clearly a need for greater theoretical understanding of the interplay between materials properties and vortex motion in superconductors.

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APPENDIX A: LONDON ACCELERATION EQUATION

In this appendix we will derive the London acceleration equation for a charged superfluid starting from the order-parameter equation of motion, Eq. (2.4). In the London approximation, we assume that the superfluid density is constant throughout the fluid, and write the order parameter as $\psi(\mathbf{r},t)=n_s^{1/2}\exp[i\chi(\mathbf{r},t)]$, with $n_s=|a(T)|/b$. Substituting this into Eq. (2.4), and writing for the relaxation rate $\Gamma=\Gamma_1+i\Gamma_2$, with Γ_1 and Γ_2 both real, we have for the imaginary part

$$\hbar\partial_t\chi+\mu+e^*\Phi+(1+\Gamma_2)\frac{m}{2}v_s^2=\frac{\hbar\Gamma_1}{2}\nabla\cdot\mathbf{v}_s \ , \tag{A1}$$

where the superfluid velocity is

$$\mathbf{v}_s=\frac{\hbar}{m}\left[\nabla\chi-\frac{e^*}{\hbar}\mathbf{A}\right] . \tag{A2}$$

Taking the gradient of Eq. (A1), recalling that $\mathbf{E}=-\nabla\Phi-\partial_t\mathbf{A}$, and defining a viscosity coefficient $\zeta_3=\hbar\Gamma_1/2m$, we have

$$\partial_t\mathbf{v}_s+(1+\Gamma_2)\nabla(\tfrac{1}{2}v_s^2)=-\frac{1}{m}\nabla\mu+\frac{e^*}{m}\mathbf{E}+\zeta_3\nabla(\nabla\cdot\mathbf{v}_s) . \tag{A3}$$

If $\Gamma_2=0$, then we can put this into a more familiar form by noting that $\nabla\times\mathbf{v}_s=-(e^*/m)\mathbf{B}$, so that

$$\begin{aligned}\nabla(\tfrac{1}{2}v_s^2)&=(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s+\mathbf{v}_s\times(\nabla\times\mathbf{v}_s)\\&=(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s-\frac{e^*}{m}\mathbf{v}_s\times\mathbf{B} .\end{aligned} \tag{A4}$$

Substituting this into Eq. (A3), we obtain the final form of the London acceleration equation, with a dissipative term[64]

$$\begin{aligned}\partial_t\mathbf{v}_s+(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s=&-\frac{1}{m}\nabla\mu+\frac{e^*}{m}(\mathbf{E}+\mathbf{v}_s\times\mathbf{B})\\&+\zeta_3\nabla(\nabla\cdot\mathbf{v}_s) ,\end{aligned} \tag{A5}$$

with the supercurrent being given by $\mathbf{J}_s=e^*n_s\mathbf{v}_s$.

If $\Gamma_2\neq0$, then Eq. (A3) has no simple hydrodynamic interpretation. In fact, Eq. (A3) is similar to the equation used by Vinen and Warren in their study of vortex motion in dirty materials.[65]

APPENDIX B: VARIATIONAL CALCULATION OF THE CONSTANTS

In this appendix we will calculate the various constants which appear in the transport coefficients, which involve the solution of Eqs. (3.23), (3.24), (3.27), and (3.28). There are no known exact solutions to this set of equations, so generally they must be solved numerically. However, approximate closed-form solutions may be obtained by using a trial form for the amplitude of the order parameter $f_0(r)$; since the equations for the vector potential and the chemical potential are linear, a sufficiently clever choice for $f_0(r)$ will allow the remaining two equations to be solved exactly. This is the method originally due to Schmid,[15] who assumed an approximate order-parameter profile (in dimensionless units) of the form

$$f_0(r)=\frac{\kappa r}{[(\kappa r)^2+\xi_v^2]^{1/2}} , \tag{B1}$$

where ξ_v is a parameter which measures the healing length of the order parameter and is numerically close to one. An optimal choice for ξ_v is that which minimizes the free energy; the dependence of ξ_v on κ is then[66]

$$1=\frac{\sqrt{2}}{\xi_v}[1-K_0^2(\xi_v/\kappa)/K_1^2(\xi_v/\kappa)]^{1/2} , \tag{B2}$$

with $K_0(z)$ and $K_1(z)$ the standard modified Bessel functions. In the limit of large κ, this reduces to $\xi_v=\sqrt{2}$, which is the value used by Schmid,[15] whereas $\xi_v=0.935$ when $\kappa=1/\sqrt{2}$.[66] Upon substituting Eq. (B1) into Eqs. (3.24) for the vector potential and Eq. (3.27) for the chemical potential, we obtain the following analytic solutions:[15,18,66]

$$p_1(r)=\frac{R}{\xi_v r}\frac{K_1(R/\zeta)}{K_1(\xi_v/\zeta)} , \tag{B3}$$

$$Q_0(r)=-\frac{R}{\xi_v \kappa r}\frac{K_1(R/\kappa)}{K_1(\xi_v/\kappa)} , \tag{B4}$$

with the local magnetic field

$$h_0(r)=\frac{1}{r}\frac{\partial[rQ_0(r)]}{\partial r} =\frac{1}{\xi_v}\frac{K_0(R/\kappa)}{K_1(\xi_v/\kappa)} , \tag{B5}$$

where we have defined $R\equiv[(\kappa r)^2+\xi_v^2]^{1/2}$ and $\zeta=(\sigma_{xx}^{(n)}/\gamma_1)^{1/2}$.[67] These quantities have the limiting behaviors

$$p_1(r)=\begin{cases}\dfrac{1}{r}-\dfrac{\kappa^2}{2\xi_v\zeta}\dfrac{K_0(\xi_v/\zeta)}{K_1(\xi_v/\zeta)}r+O(r^3) , & \text{for } r\ll\xi_v/\kappa ,\\[2ex] \dfrac{\kappa}{\xi_v}\dfrac{K_1(\kappa r/\zeta)}{K_1(\xi_v/\zeta)} , & \text{for } r\gg\xi_v/\kappa ,\end{cases} \tag{B6}$$

$$h_0(r)=\begin{cases}h_0(0)-\dfrac{\kappa}{2\xi_v^2}r^2+\mathcal{O}(r^4) , & \text{for } r\ll\xi_v/\kappa ,\\[2ex] \dfrac{1}{\xi_v}\dfrac{K_0(r)}{K_1(\xi_v/\kappa)} , & \text{for } r\gg\xi_v/\kappa ,\end{cases} \tag{B7}$$

where $h_0(0)$ is the field at the center of the vortex,

$$h_0(0)=\frac{1}{\xi_v}\frac{K_0(\xi_v/\kappa)}{K_1(\xi_v/\kappa)}=\frac{1}{\kappa}(\ln\kappa-0.231) , \tag{B8}$$

and where the last line is correct in the large-κ limit.[66] The core field $h_0(0)\approx 2H_{c1}$ in the large-κ limit. It is also easily verified that

$$B=2\pi\int_0^\infty h_0(r)rdr=\frac{2\pi}{\kappa} , \tag{B9}$$

as required by flux quantization.

The equation for $p_2(r)$, Eq. (3.28), does not appear to have an analytic solution. However, as we are primarily interested in the $r\to 0$ behavior, we seek an approximate solution as follows.[68] First, we define a new function $h_2(r)=p_2(r)+A/r$, where A is a constant to be determined. Substituting into Eq. (3.28), we have

$$\frac{\sigma_{xx}^{(n)}}{\kappa^2}\frac{d}{dr}\frac{1}{r}\frac{d(rh_2)}{dr}-\gamma_1 f_0^2 h_2 =\gamma_2 f_0\frac{df_0}{dr}-\frac{\sigma_{xy}^{(n)}}{\kappa}\frac{dh_0}{dr}-A\gamma_1\frac{f_0^2}{r} . \tag{B10}$$

Now all of the terms on the right-hand side of this equation are proportional to r as $r\to 0$. Therefore, we choose A to eliminate these linear terms, so that the remaining terms are $O(r^3)$ as $r\to 0$. Using the small-r behavior of f_0 and h_0, we find that we must choose

$$A=\frac{1}{\gamma_1}\left[\gamma_2+\frac{1}{\kappa^2}\sigma_{xy}^{(n)}(0)\right] . \tag{B11}$$

With this choice of A, Eq. (B10) is approximately homogeneous for small r. The solution is therefore

$$h_2(r)=C\frac{R}{\xi_v r}\frac{K_1(R/\zeta)}{K_1(\xi_v/\zeta)} , \tag{B12}$$

where C is a constant which must be determined from the boundary conditions. In order that $p_2(0)=0$, we must have $C=A$; our approximate solution for small r is therefore

$$p_2(r)=A\left[\frac{R}{\xi_v r}\frac{K_1(R/\zeta)}{K_1(\xi_v/\zeta)}-\frac{1}{r}\right] \approx-\frac{\kappa^2}{2\gamma_1\xi_v\zeta}\frac{K_0(\xi_v/\zeta)}{K_1(\xi_v/\zeta)}\left[\gamma_2+\frac{1}{\kappa^2}\sigma_{xy}^{(n)}(0)\right]r . \tag{B13}$$

We expect that this expansion will capture the small-r behavior in the limit that $\zeta\to 0$.

We are now in a position to calculate the coefficients α_1 and α_2, which are defined in Eqs. (3.38) and (3.39). Performing the integrals, for α_1 we obtain

$$\alpha_1=\frac{\gamma_1}{4}+\frac{\gamma_1\zeta}{\xi_v}\frac{K_0(\xi_v/\zeta)}{K_1(\xi_v/\zeta)} , \quad \text{(B14)}$$

which has the limiting behavior

$$\alpha_1=\begin{cases}\gamma_1\left[\frac{1}{4}+\frac{\zeta}{\xi_v}-\frac{\zeta^2}{2\xi_v^2}+O(\zeta^3)\right] , & \text{for } \zeta\ll\xi_v , \\ \gamma_1[\ln(\zeta/\xi_v)+0.365+O(\zeta^{-2})] , & \text{for } \zeta\gg\xi_v .\end{cases} \quad \text{(B15)}$$

For α_2 we have

$$\alpha_2=-\frac{\gamma_2}{2}I(\xi_v/\zeta)-\frac{\zeta}{\xi_v}\frac{K_0(\xi_v/\zeta)}{K_1(\xi_v/\zeta)}\left[\gamma_2+\frac{1}{\kappa^2}\sigma_{xy}^{(n)}(0)\right] +\frac{1}{\kappa}\sigma_{xy}^{(n)}(0)h_0(0) , \quad \text{(B16)}$$

where the integral $I(z)$ is given by

$$I(z)=\frac{2z}{K_1(z)}\int_z^\infty\frac{K_1(x)}{x^2}dx . \quad \text{(B17)}$$

This integral has the limiting behavior

$$I(z)=\begin{cases}1-\{\frac{1}{2}(\ln z)^2-[\ln(2)-\gamma]\ln z\}z^2+O[z^4(\ln z)^2] , & \text{for } z\ll 1 , \\ \frac{2K_0(z)}{zK_1(z)}-\frac{4}{z^2}+\frac{16K_2(z)}{z^3K_1(z)} , & \text{for } z\gg 1 .\end{cases} \quad \text{(B18)}$$

Using the trial order-parameter solution it is also possible to calculate the line energy ϵ_1 in the large-κ limit[15,66]

$$\epsilon_1\approx\frac{2\pi}{\kappa^2}(\ln\kappa+0.519) . \quad \text{(B19)}$$

APPENDIX C: DEFINITION OF THE TRANSPORT COEFFICIENTS

In this appendix we summarize the definitions of the transport coefficients, for completeness. A full discussion may be found in Ref. (34). In the presence of an electric field $\mathbf{E}$ and a temperature gradient ∇T, the electrical current $\mathbf{J}$ and the heat current $\mathbf{J}^h$ are written as

$$\begin{aligned}
J_x&=\sigma_{xx}E_x+\sigma_{xy}E_y+\frac{\alpha_{xx}}{T}\frac{\partial T}{\partial x}+\frac{\alpha_{xy}}{T}\frac{\partial T}{\partial y} , \\
J_y&=-\sigma_{xy}E_x+\sigma_{xx}E_y-\frac{\alpha_{xy}}{T}\frac{\partial T}{\partial x}+\frac{\alpha_{xx}}{T}\frac{\partial T}{\partial y} , \\
J_x^h&=-\alpha_{xx}E_x-\alpha_{xy}E_y-\kappa_{xx}\frac{\partial T}{\partial x}-\kappa_{xy}\frac{\partial T}{\partial y} , \\
J_y^h&=\alpha_{xy}E_x-\alpha_{xx}E_y+\kappa_{xy}\frac{\partial T}{\partial x}-\kappa_{xx}\frac{\partial T}{\partial y} ,
\end{aligned} \quad \text{(C1)}$$

where the Onsager relations and rotational symmetry have been used to simplify the equations.[34]

The isothermal Nernst coefficient is

$$\nu=E_y/H(\partial T/\partial x) , \quad \text{(C2)}$$

under the conditions $J_x=J_y=\partial T/\partial y=0$. Then by solving Eqs.(C1), we find that the Nernst coefficient can be expressed as

$$\nu=\frac{1}{TH}[\alpha_{xy}\rho_{xx}-\alpha_{xx}\rho_{xy}] , \quad \text{(C3)}$$

where the resistivities are expressed in terms of the conductivities as

$$\begin{aligned}\rho_{xx}&=\sigma_{xx}/(\sigma_{xx}^2+\sigma_{xy}^2) , \\ \rho_{xy}&=\sigma_{xy}/(\sigma_{xx}^2+\sigma_{xy}^2) .\end{aligned} \quad \text{(C4)}$$

Under most experimental conditions, the second term in Eq. (C2) is much smaller than the first, so for most purposes we have

$$\nu\approx\frac{1}{TH}\alpha_{xy}\rho_{xx} . \quad \text{(C5)}$$

The Ettingshausen coefficient $\mathscr{E}$ as defined as

$$\mathscr{E}=(\partial T/\partial y)/HJ_x , \quad \text{(C6)}$$

under the conditions $J_y^h=J_y=\partial T/\partial x=0$. We then find that

$$\kappa_{xx}\mathscr{E}=T\nu , \quad \text{(C7)}$$

which is a consequence of the Onsager relations.

The absolute thermopower ϵ is defined as

$$\epsilon=-E_x/(\partial T/\partial x) , \quad \text{(C8)}$$

under the conditions $J_x=J_y=\partial T/\partial y=0$. We then find

$$\epsilon=\frac{1}{T}[\rho_{xx}\alpha_{xx}+\rho_{xy}\alpha_{xy}] . \quad \text{(C9)}$$

Using Eq. (C5), the thermopower can be rewritten as

$$\epsilon=\frac{1}{T}\rho_{xx}\alpha_{xx}+H\nu\tan\theta_H , \quad \text{(C10)}$$

where $\tan\theta_H=\rho_{xy}/\rho_{xx}$ is the Hall angle.

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