Stability of current-carrying states in Josephson-junction arrays

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Analytic and numerical calculations are presented for current-carrying states in frustrated Josephson-junction arrays. In particular, we study a family of states which are a generalization of the "staircase" states proposed by Halsey. For the fully frustrated case, it is possible to obtain exact analytic solutions explicitly in terms of the net supercurrent orientation. We find states that carry currents up to critical values of I_{c1} which are larger than the intrinsic critical currents I_{c0} obtained previously by other authors. However, an analysis of the stability of these new states shows that they are unstable to fluctuations in the phases of the superconducting grains.

I. INTRODUCTION

Two-dimensional Josephson-junction arrays have been a subject of extensive experimental[1–5] and theoretical[6–12] studies. In the simple model generally used to describe these systems, the Hamiltonian is the sum of individual Josephson-junction energies,

$$\mathcal{H}=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j-A_{ij}) , \tag{1}$$

where θ_i is the phase of the complex order parameter of a point superconductor at the site i, $A_{ij}=(2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot d\mathbf{l}$ is the integral of the vector potential $\mathbf{A}$ from site i to j, and Φ_0 is the flux quantum.

The Josephson tunneling current between two sites is given by the relation

$$I_{ij}=(2e/\hbar)J\sin(\theta_i-\theta_j-A_{ij}) .$$

In the limit where the magnetic field induced by supercurrents flowing in the array is negligible compared to the uniform external field $\mathbf{H}$, the sum of the phase factors A_{ij} around a plaquette is equal to

$$2\pi(Ha^2/\Phi_0)=2\pi f ,$$

where f is the number of flux quanta through each plaquette, and a is the lattice constant. The sum of the phase differences around a plaquette n is

$$\begin{aligned}\sum_{ij\in n}\gamma_{ij}&=\sum_{ij\in n}(\theta_i-\theta_j-A_{ij})\\&=-2\pi f(\mathrm{mod}2\pi)\\&=2\pi(\nu_n-f) ,\end{aligned} \tag{2}$$

where ν_n, the vorticity of the plaquette n, is an integer or zero.

The ground state of the system is the configuration of phases $\{\theta_i\}$ which globally minimize the Hamiltonian (1). This is one of the stationary points of $\mathcal{H}$, i.e., configurations $\{\theta_i\}$ from which small deviations leave $\mathcal{H}$ unaltered. At rational fields $f=p/q$, the lowest-energy state is spatially periodic, with a $q\times q$ unit cell, in which unit vorticities are arranged in a regular superlattice.[8,10] In particular, for the "staircase" states proposed by Halsey,[10] vortices lie along diagonal lines (in the $[1\bar{1}]$ direction).

One issue of particular interest is the critical current in these systems. The zero temperature, or "intrinsic" critical current I_{c0} of an array is the largest supercurrent for which a metastable state exists.[5,10] A prominent feature of an array of discrete superconductors is the reduced symmetry compared with an isotropic continuum superconductor. The critical current, for example, is an anisotropic quantity. Apart from presenting numerically determined values of I_{c0} in arbitrary directions, Halsey also gave an analytic argument for its value in the direction along the lines of vortices. This energy difference argument shows that staircase states are unstable to small phase deviations if the phase differences on any staircase are beyond the range $[-\pi/2,\pi/2]$. This sets a limit on the maximum current that a metastable state can carry. More recently, Benz *et al.*[5] presented exact calculations for the critical currents in the [10] direction, for $f=\frac{1}{2}$ and $\frac{1}{3}$. These agree with previous numerical simulations.[8]

In this paper, we present analytic and numerical solutions of current-carrying states, which are a generalization of staircase states. These encompass the results of both Ref. 10 and Ref. 5. In addition to the states obtained in these references, we have also found families of higher-energy states which can carry larger supercurrents, up to "upper" critical values of I_{c1}. Following the procedures of Benedict,[12] we analyzed the stability of these "generalized staircase" states.

Throughout this paper, the effects of thermally activated vortices, domains, and other defects are neglected. This is justified at temperatures far enough below the transition temperature of the array, such as in the experimental condition adopted by Benz *et al.*[5]

II. ANALYTIC RESULTS FOR $f=p/q$

Stationary states of the Hamiltonian (1) satisfy current conservation at all lattice sites.[10] Staircase states are an ansatz for the phase configurations which satisfy this condition. Here, we consider a slight generalization of the staircase states. Suppose that on each staircase running in the $[1\bar{1}]$ direction, the phase differences on all the horizontal links γ_i are equal, and likewise for the vertical phases γ_i', while these two may be different (Fig. 1). Again, we assume a spatial periodicity of $q \times q$ plaquettes. With a change of variables, the phases can be written as

$$\begin{aligned} \gamma_i &= \alpha_i + \delta_i \ , \\ \gamma_i' &= \alpha_i - \delta_i \ , \end{aligned} \tag{3}$$

where $i = 1, 2, \ldots, q$. α_i is the average phase on the ith staircase. The requirement of current conservation readily leads to

$$\cos\alpha_1 \sin\delta_1 = \cos\alpha_2 \sin\delta_2 = \cdots = \cos\alpha_q \sin\delta_q = C \ , \tag{4}$$

where C is a constant.

Now, from Eqs. (2) and (3), the phases α_i satisfy

$$\begin{aligned} \alpha_1 &= \alpha_1 \ , \\ \alpha_2 &= \alpha_1 + \pi f - \pi \nu_1 \ , \\ \alpha_3 &= \alpha_2 + \pi f - \pi \nu_2 \\ &= \alpha_1 + 2\pi f - \pi(\nu_1 + \nu_2) \ , \\ &\vdots \\ \alpha_q &= \alpha_1 + (q-1)\pi f - \pi \sum_{i=1}^{q-1} \nu_i \ . \end{aligned} \tag{5}$$

Using the definition of the change of variables (3), the horizontal and vertical components of the net current per junction are, respectively,

$$\begin{aligned} I_h &= \frac{i_{c0}}{q} \sum_{i=1}^{q} \sin\gamma_i \\ &= C i_{c0} + \frac{i_{c0}}{q} \sum_{i=1}^{q} \sin\alpha_i \cos\delta_i \end{aligned} \tag{6}$$

and

$$\begin{aligned} I_v &= \frac{-i_{c0}}{q} \sum_{i=1}^{q} \sin\gamma_i' \\ &= C i_{c0} - \frac{i_{c0}}{q} \sum_{i=1}^{q} \sin\alpha_i \cos\delta_i \ , \end{aligned} \tag{7}$$

where $i_{c0} = 2eJ/\hbar$ is the single junction critical current. And so, $I_h + I_v = 2Ci_{c0}$. Likewise, the energy per lattice site is given by

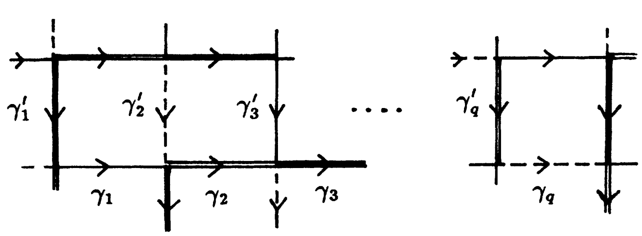

FIG. 1. Phase differences for the generalized staircase state.

$$\begin{aligned} E &= \frac{-J}{q} \sum_{i=1}^{q} (\cos\gamma_i + \cos\gamma_i') \\ &= \frac{-2J}{q} \sum_{i=1}^{q} \cos\alpha_i \cos\delta_i \ . \end{aligned} \tag{8}$$

For a given vorticity configuration $\{\nu_n\}$, where $n = 1, \ldots, q$, all the phases α_i's and δ_i's can be expressed in terms of α_1 and δ_1, according to (4) and (5). α_1 and δ_1 then play the role of two adjustable parameters that vary the magnitude and direction of the net current. Here, we study two special cases.

A. Net current in the $[1\bar{1}]$ direction

In this case, $I_v = -I_h$ and so $C = 0$. Then, Eq. (4) means that either $\cos\alpha_i = 0$ or $\sin\delta_i = 0$ for $i = 1, \ldots, q$.

First, consider $\sin\delta_i = 0$ for all i. The solutions then reduce exactly to staircase states. The zero current ground state is given by[10]

$$\begin{aligned} \alpha_1^0 &= \begin{cases} 0 & \text{for } q \text{ odd} \\ \pi/2q & q \text{ even} \end{cases} \\ \alpha_n^0 &= [\alpha_1^0 + (n-1)\pi f] - \pi \, \mathrm{nint}[\alpha_1^0 + (n-1)\pi f] \ , \end{aligned} \tag{9}$$

where $\mathrm{nint}(x)$ is the nearest integer function. This fixes the vorticities in (5), and ensures that all the phases lie within the range $[-\pi/2, \pi/2]$. Among the q different ground-state phases, for every α_k^0, there exists another phase $\alpha_{k'}^0$ such that $\alpha_k^0 = -\alpha_{k'}^0$. In other words, the phases are "balanced" symmetrically about the $\alpha = 0$ axis.

States carrying a nonzero current in $[1\bar{1}]$ direction are obtained by incrementing all α_i from α_i^0 by the same amount ϵ. If all the phase differences are limited to lie between $-\pi/2$ and $\pi/2$, i.e., within the limits of stability according to Halsey, then the maximum current is reached when $\epsilon = \pi/2q$, for which the intrinsic critical current $I_{c0}([1\bar{1}]) = \sqrt{2} i_{c0}/q$.

Here, we aim at obtaining the full solutions of the problem, leaving the analysis of the stability of each solution till Sec. V. So, if the bound on the phases is disregarded, then the maximum current occurs when $\epsilon = \pi/2$. At this value, the phases are balanced symmetrically about the $\alpha = \pi/2$ axis. Using the results of Ref. 10, we obtain the "upper critical current" as

$$I_{c1}([1\bar{1}]) = \frac{\sqrt{2} i_{c0}}{q} \csc\frac{\pi}{2q} \ . \tag{10}$$

Since the energy contribution of each phase difference, namely, its cosine, is proportional to the projection of each point in the phase diagram onto the $\alpha = 0$ axis, the symmetry of the points means that the total energy sums to zero.

Now consider the other possibility, that for some j, $\cos\alpha_j = 0$. In this case, δ_j can take on any value. From (8), the energy is the same for all these values, while the net current varies according to (6) and (7). This corresponds to a horizontal line in the energy-current relation, as observed in the numerical solutions described in Sec.

IV. In general, since there are q different α_i's, there will be q such constant energy lines.

B. Net current in the [11] direction

In this case, $I_v = I_h$. Equations (6) and (7) imply that $I_h = Ci_{c0}$ and

$$\sum_{i=1}^{q} \sin\alpha_i \cos\delta_i = 0 \ . \tag{11}$$

The total net current is then $I = \sqrt{2} I_h = \sqrt{2} C i_{c0}$.

Now, suppose all the phases α_i's take on the ground-state values of α_i^0. As mentioned previously, the phases occur in pairs, with $\alpha_k^0 = -\alpha_{k'}^0$. Equation (4) then requires $\delta_k = \delta_{k'}$. It is clear that such a combination of $\{\alpha_i\}$ and $\{\delta_i\}$ will satisfy the condition (11), in which the terms in the sum are again balanced about zero. Hence, for a state with a net current in the [11] direction, $\alpha_i = \alpha_i^0$ for all i, and the magnitude of current varies according to the δ_i's.

The maximum value of C, and hence $I_{c1}([11])$, can now be determined by considering Eq. (4). Obviously, the largest value of $\cos\alpha_i^0 \sin\delta_i$ for each i occurs when $\sin\delta_i = 1$. And so, the maximum value of C is limited by the smallest of the q values of $\cos\alpha_i^0$. From (9), this occurs at $\alpha_i^0 = \pi/2 - \pi/2q$, giving $C_{\max} = \sin(\pi/2q)$. Hence, the "upper" critical current in the [11] direction is

$$I_{c1}([11]) = \sqrt{2} i_{c0} \sin(\pi/2q) \ . \tag{12}$$

III. ALTERNATIVE ANALYTIC SOLUTIONS FOR $f = \frac{1}{2}$

In the previous section, analytic solutions are given for the phase variables $\{\alpha_i\}$ and $\{\delta_i\}$, as defined in (3), for a general field $f = p/q$. α_i is the average phase on the ith staircase. That makes it easy to utilize results derived for the staircase states by Halsey.

For the fully frustrated case, $f = \frac{1}{2}$, it is also possible to obtain exact analytic solutions given explicitly in terms of the net current orientation θ. For that purpose, it is more convenient to use a new set of variables defined as follows:

$$\begin{aligned} \gamma_1 &= \chi_h + \phi_h, \quad \gamma_2 = -\chi_h + \phi_h \ , \\ \gamma_1' &= \chi_v - \phi_v, \quad \gamma_2' = -\chi_v - \phi_v \ . \end{aligned} \tag{13}$$

ϕ_h and ϕ_v are now the average phase differences on the horizontal and vertical junctions, respectively.

The procedures are similar to those in the last section. With a "staircase-state-like" checkerboard vortex superlattice,[7,8,10]

$$\chi_h + \chi_v = \pi/2 \ . \tag{14}$$

The current conservation requirement leads to

$$\sin\chi_h \cos\phi_h = \sin\chi_v \cos\phi_v \ . \tag{15}$$

The horizontal and vertical components of the net current are, respectively,

$$\begin{aligned} I_h &= i_{c0} \cos\chi_h \sin\phi_h \ , \\ I_v &= i_{c0} \sin\chi_h \sin\phi_v \ , \end{aligned} \tag{16}$$

where (14) has been used to obtain the last expression.

The orientation of the net current θ is thus

$$\tan\theta = I_v / I_h = \tan\chi_h \sin\phi_v / \sin\phi_h \ . \tag{17}$$

Now, with (14), Eq. (15) gives

$$\tan\chi_h = \cos\phi_v / \cos\phi_h \ . \tag{18}$$

Substituting this in (17) then leads to

$$\tan\theta = \sin(2\phi_v) / \sin(2\phi_h) \ . \tag{19}$$

Hence,

$$\phi_v = \tfrac{1}{2} \arcsin[\tan\theta \sin 2\phi_h]$$

or

$$\phi_v = \pi/2 - \tfrac{1}{2} \arcsin[\tan\theta \sin 2\phi_h] \ . \tag{20}$$

The two solutions in (20) correspond, respectively, to the "lower" and "upper" branches described in the next section.

With a fixed current orientation θ, (14), (18), and (20) constitute a full solution to the problem, with ϕ_h acting as the adjustable parameter that determines the magnitude of current.

IV. NUMERICAL RESULTS FOR $f = \frac{1}{2}$ AND $\frac{1}{3}$

Since stationary states of the Hamiltonian satisfy current conservation, they can be represented by systems of "loop currents" flowing around in every plaquette. The equations which determine these loop currents may be derived from Eq. (2), by noting that

$$\sin\gamma_{ij} = I_{(ij,L)} - I_{(ij,R)} \ ,$$

where the two terms on the right-hand side denote the loop currents (in units of i_{c0}) to the left and right of the link ij. This leads to

$$\sum_m \arcsin(I_n - I_m) = 2\pi(\nu_n - f) \ , \tag{21}$$

where the sum is over plaquettes neighboring to n. In this equation, arcsin is a multivalued function, and its range extends from $-\infty$ to $+\infty$.

A net current can be imposed to flow across a periodic system of $L_x \times L_y$ plaquettes, with the following twisted periodic boundary conditions:

$$\begin{aligned} I(x + I_x, y) &= I(x,y) - L_x I_v \ , \\ I(x, y + L_y) &= I(x,y) + L_y I_h \ , \end{aligned} \tag{22}$$

where I_v and I_h are the average vertical and horizontal components of the net current per junction. More details of the numerical method and the loop current formalism will be given in another publication.

Figures 2(a) and 2(b) show the energy-current relation for numerical solutions of (21), for $f = \frac{1}{2}$ and $\frac{1}{3}$ and the same vortex pattern $\{\nu_n\}$ as the corresponding staircase states, namely with unit vortices along diagonal lines ($[1\bar{1}]$). The graphs are symmetric about the $E=0$ axis, and only the lower halves are shown. At every orientation of the net current, there exist two branches. The "lower" branch [for example, curve ACD on Fig. 2(a)] corresponds to solutions obtained in Ref. 5, and is responsible for the intrinsic critical current I_{c0}. The "upper" branch (curve DEG) consists of higher-energy states with currents extending up to a maximum of I_{c1}. The high-energy end of the lower branch is degenerate with the low-energy end of the upper branch. For clarity, some of the zero current states are shown in Figs. 3 and 4. Another general feature is that for a current in the direction of the lines of vortices, i.e., $\theta=-0.25\pi$, the low-energy branch joins on smoothly to the high-energy branch as current is increased from zero.

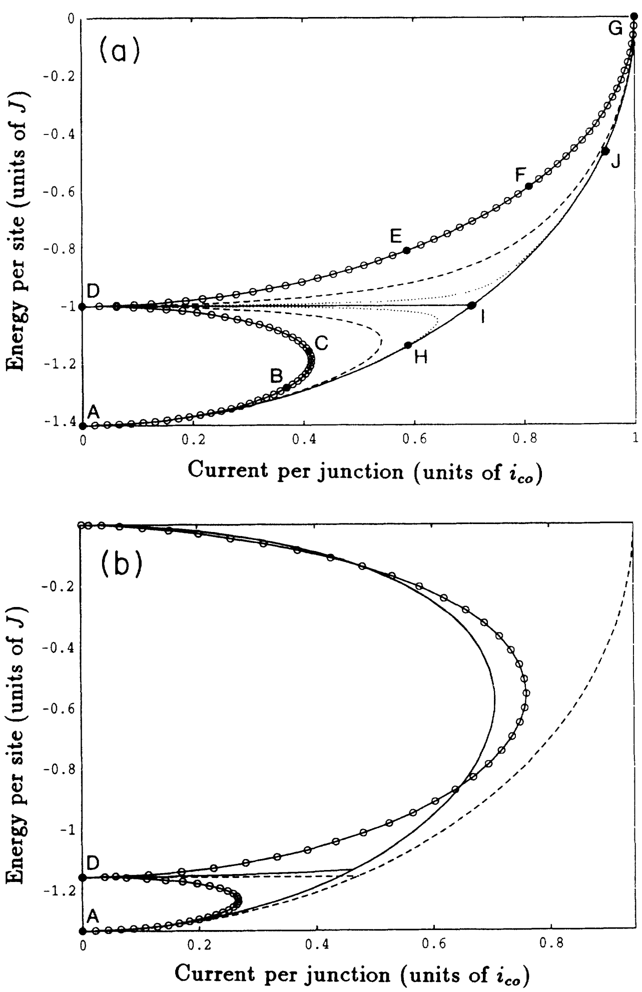

FIG. 2. Numerical results of the energy-current relation in the frustrated Josephson-junction array, calculated for various current orientations. Lines of vortices are parallel to the $[1\bar{1}]$ diagonal ($\theta=-0.25\pi$), and the current orientations are denoted by (a) $f=\frac{1}{2}$, $\theta=0$ (○), 0.20π (dashed); 0.24π (dotted), 0.25π (plain curve); (b) $f=\frac{1}{3}$, $\theta=0$ (○), 0.25π (plain curve), -0.25π (dashed curve).

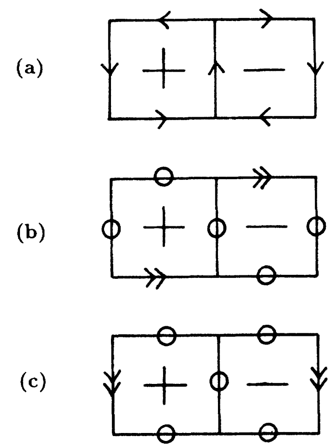

FIG. 3. Some zero current metastable states for $f=\frac{1}{2}$. + denotes unit vorticity (or a "charge" of $1-f$), and − denotes zero vorticity (or a charge of $-f$). The arrows indicate the phase differences on the junctions: ⟶: $\pi/4$, ⟶⟶: π, ○: 0. (a) is the ground state [point A in Fig. 2(a)], while (b) and (c) are two degenerate excited states (point D).

$f=\frac{1}{2}$ is special in that at this value of the magnetic field, $I_{c1}(\theta)$ is equal to $1.0 i_{c0}$ for all θ. Furthermore, all the states are degenerate at I_{c1}.

Polar plots of $I_{c1}(\theta)$ are shown in Fig. 5 for $f=\frac{1}{2}$ and $\frac{1}{3}$. The plots of I_{c0} are also included for comparison. Both plots are symmetrical about the [11] and $[1\bar{1}]$ axes. The $f=\frac{1}{2}$ curves, in addition, are symmetrical about the [10] axis. In the case of $f=\frac{1}{3}$, the asymmetry about the [10] axis is very slight for I_{c0}, but is quite apparent for I_{c1}. The plot of I_{c0} for $f=\frac{1}{2}$ is identical to that obtained by Halsey.[10] However, there is a discrepancy in the $f=\frac{1}{3}$ case, in that our result indicates that I_{c0} in the [11] direction, i.e., perpendicular to the vortices ($0.462 i_{c0}$), is slight-

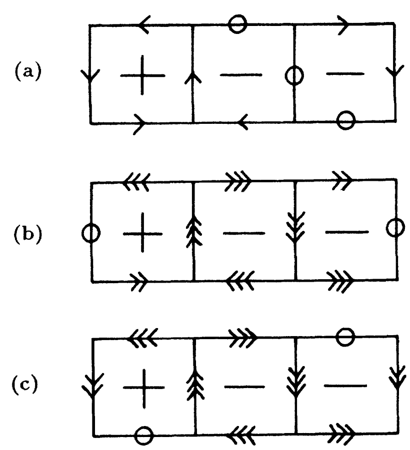

FIG. 4. Some zero current metastable states for $f=\frac{1}{3}$. The phase differences are indicated by ⟶: $\pi/3$, ⟶⟶: π, ⟶⟶⟶: $\pi/6$, ○: 0. (a) is the ground state [point A in Fig. 2(b)], while (b) and (c) are two degenerate excited states (point D).

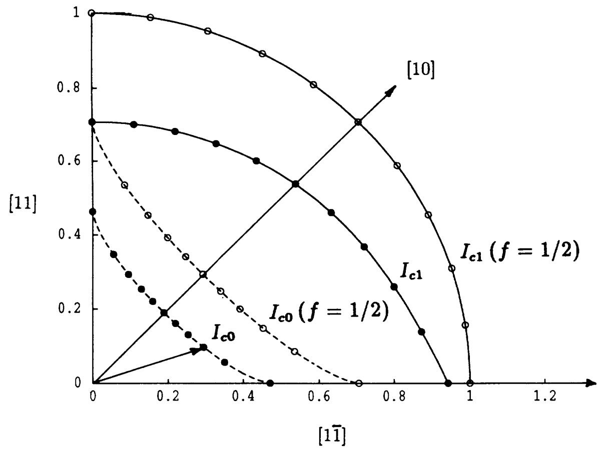

FIG. 5. Polar plots of the critical currents I_{c0} (plane curve) and I_{c1} (dashed curve) for $f=\frac{1}{2}$ (open circle) and $f=\frac{1}{3}$ (closed circle). The units are $2eJ/\hbar$ or i_{c0}.

ly smaller than in the $[1\bar{1}]$ direction ($I_{c0}(f=\frac{1}{3}, [1\bar{1}])=0.471 i_{c0}$), while the opposite is obtained by Halsey. We believe that in this matter, our numerical method is more accurate than the instability method that Halsey used.

V. STABILITY OF THE CURRENT-CARRYING STATES

Benedict[12] carried out a stability analysis of the zero current staircase states.

Here, we apply a straightforward extension of that analysis to include the current-carrying states described in the previous sections. This involves finding the eigenvalues of the stability, or Hessian, matrix which has elements

$$M_{x',y';x,y}=\frac{\partial^2 \mathcal{H}}{\partial\theta_{x',y'}\partial\theta_{x,y}} . \tag{23}$$

We denote the phase differences on horizontal and vertical links as follows:

$$\begin{aligned}\gamma_{x,y,x',y'}&=\theta_{x,y}-\theta_{x',y'}-A_{x,y;x',y'} ,\\ \gamma^x_{x,y}&=\gamma_{x,y;x+1,y} ,\\ \gamma^y_{x,y}&=\gamma_{x,y;x,y+1} .\end{aligned} \tag{24}$$

For the generalized staircase states, these satisfy the relations

$$\begin{aligned}\gamma^x_{x,y}&=\gamma^x_{x+y,0}\equiv\gamma^x_{x+y} ,\\ \gamma^y_{x,y}&=\gamma^y_{x+y,0}\equiv\gamma^y_{x+y} .\end{aligned} \tag{25}$$

Defining the variables A_i and B_i as the cosines of the phase differences, $A_i=\cos\gamma^x_i$, and $B_i=\cos\gamma^y_i$, the Hessian matrix (23), when evaluated for a generalized staircase state, gives

$$\begin{aligned}M_{x',y';x,y}=J\{&A_{x+y-1}(\delta_{x',x}\delta_{y',y}-\delta_{x',x-1}\delta_{y',y})\\ &+A_{x+y}(\delta_{x',x}\delta_{y',y}-\delta_{x',x+1}\delta_{y',y})\\ &+B_{x+y-1}(\delta_{x',x}\delta_{y',y}-\delta_{x',x}\delta_{y',y-1})\\ &+B_{x+y}(\delta_{x',x}\delta_{y',y}-\delta_{x',x}\delta_{y',y+1})\} .\end{aligned} \tag{26}$$

For a magnetic field $f=p/q$, the phase differences are periodic with a $q\times q$ unit cell, and so $A_{i+q}=A_i$, and $B_{i+q}=B_i$.

The eigenvalue equation for the stability matrix is

$$\sum_{x',y'} M_{x,y;x',y'}v_{x',y'}=\omega v_{x,y}$$

or

$$\begin{aligned}&A_{x+y-1}(v_{x,y}-v_{x-1,y})+A_{x+y}(v_{x,y}-v_{x+1,y})\\ &\quad+B_{x+y-1}(v_{x,y}-v_{x,y-1})\\ &\qquad+B_{x+y}(v_{x,y}-v_{x,y+1})=(\omega/J)v_{x,y} .\end{aligned} \tag{27}$$

Because of the particular structure of the Hessian, it is convenient to employ a change in coordinate variables;

$$s=x-y, \quad t=x+y . \tag{28}$$

In the new coordinates, the eigenvalue equation (27) is

$$\begin{aligned}&A_{t-1}(v_{s,t}-v_{s-1,t-1})+A_t(v_{s,t}-v_{s+1,t+1})\\ &\quad+B_{t-1}(v_{s,t}-v_{s+1,t-1})\\ &\qquad+B_t(v_{s,t}-v_{s-1,t+1})=(\omega/J)v_{s,t} .\end{aligned} \tag{29}$$

Furthermore, the periodicity of the Hessian means that the eigenfunctions have the form

$$v_{s,t}=e^{ik_\parallel s}e^{ik_\perp t}C_t(\mathbf{k}) . \tag{30}$$

The Brillouin zone is defined by

$$\begin{aligned}&k_\parallel\in[-\pi/2,\pi/2] ,\\ &k_\perp\in\begin{cases}[-\pi/q,\pi/q] & \text{for } q \text{ even},\\ [-\pi/2q,\pi/2q] & q \text{ odd} .\end{cases}\end{aligned} \tag{31}$$

The Bloch function C_t has period q, and satisfies the reduced equation

$$\begin{aligned}&(A_{t-1}+A_t+B_{t-1}+B_t-\omega^{(\nu)}/J)C_t^{(\nu)}\\ &\quad-e^{-ik_\perp}(A_{t-1}e^{-ik_\parallel}+B_{t-1}e^{ik_\parallel})C_{t-1}^{(\nu)}\\ &\qquad-e^{ik_\perp}(A_te^{ik_\parallel}+B_te^{-ik_\parallel})C_{t+1}^{(\nu)}=0 .\end{aligned} \tag{32}$$

Since A_i and B_i have a period of q, it follows then that both $\omega^{(\nu)}$ and the $C_t^{(\nu)}$'s are q valued, as indicated by the q-valued "band index" ν.

In general, the reduced equation (32) can be solved numerically to obtain the eigenvalues. For $f=\frac{1}{2}$, in addition, exact diagonalization can be done analytically. In that case, there are two distinct values for A_i and B_i. Some manipulation gives the eigenvalues as

$$\omega^{(\pm)}/J=A_1+A_2+B_1+B_2\pm|D| ,$$

where

$$|D|^2 = A_1^2 + A_2^2 + B_1^2 + B_2^2 + 2(A_1B_1 + A_2B_2)\cos(2k_\parallel) + 2(A_1B_2 + A_2B_1)\cos(2k_\perp) + 2A_1A_2\cos(2k_\parallel + 2k_\perp) + 2B_1B_2\cos(2k_\parallel - 2k_\perp)\ . \tag{33}$$

The $\pm$ sign corresponds to the two "bands" of eigenvalues.

This is shown for a few cases in Fig. 6. There, the eigenvalues are evaluated at various orientations and magnitudes of the net current, using the analytic expressions of Sec. III. One common feature shared by all the graphs is that there always exists a zero eigenvalue at the zone center $\Gamma(0,0)$. This corresponds to a Goldstone mode in which all the phases rotate by the same amount. Figures 6(a) and 6(b) show the results for a few points on the lower and upper branches, respectively, at $\theta=0$. On the lower branch, the eigenvalues are non-negative, while negative eigenvalues exist for the whole of the upper branch. The negative eigenvalues label those modes of fluctuations which cause instability. Hence, all of the solutions belonging to the upper branch are in fact unstable, while the opposite is true for the lower branch. This, together with the results of the previous section, confirm the values of the intrinsic critical current I_{c0} obtained by other authors.

However, it may be argued that in a current-biased set-up, only those fluctuations that do not change the net current should be allowed. The type of fluctuation considered by Halsey in the argument leading to the bounds of the phase differences, for example, does not conserve the net current. Therefore, we proceed to investigate the effect on the net current of each of the fluctuation modes.

For $f=\frac{1}{2}$ there are two distinct Bloch functions C_1 and C_2. Corresponding to the eigenvalues given in (33), they are

$$\begin{bmatrix} C_1^{(\pm)} \\ C_2^{(\pm)} \end{bmatrix} = K \begin{bmatrix} 1 \\ \mp Z^{-1/2}(Z^*)^{1/2} \end{bmatrix}\ , \tag{34}$$

where $(+)$ and $(-)$ are the band indices, K is a normalization constant, and

$$Z = e^{ik_\perp}(A_1 e^{ik_\parallel} + B_1 e^{-ik_\parallel}) + e^{-ik_\perp}(A_2 e^{-ik_\parallel} + B_2 e^{ik_\parallel})\ . \tag{35}$$

Writing $Z = |Z|e^{i\eta}$, then the eigenfunctions (30) are

$$v_{s,t}^{(\pm)} = \begin{cases} Ke^{i(k_\parallel s + k_\perp t)} & \text{odd } t \\ \mp Ke^{i(k_\parallel s + k_\perp t - \eta)} & \text{even } t\ . \end{cases} \tag{36}$$

This means that different values of A_i and B_i, and hence different magnitudes and orientations of current, only change the eigenvectors by a constant phase factor.

Suppose the phases for a generalized staircase state are $\{\theta_{s,t}^0\}$. The currents on each horizontal and vertical link are then

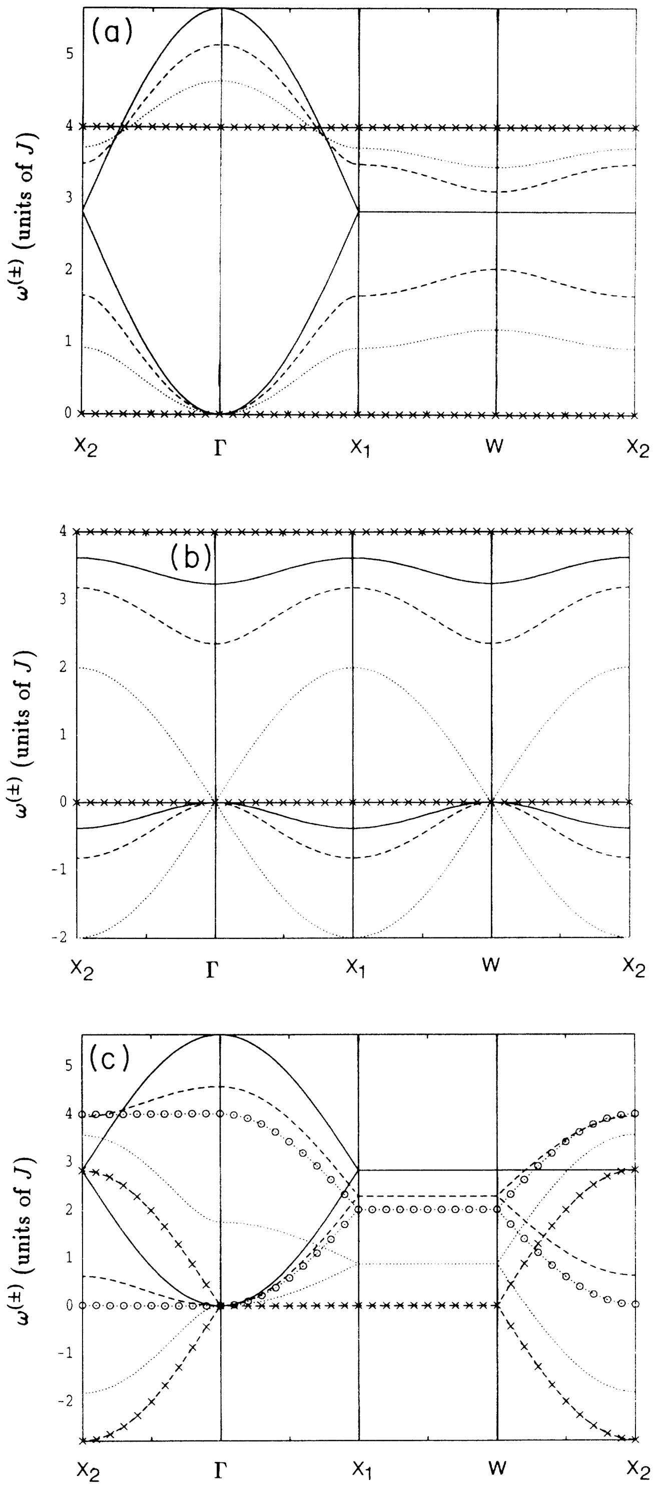

FIG. 6. Eigenvalues of the stability matrix evaluated along four line segments of the Brillouin zone, for $f=\frac{1}{2}$. The points $(k_\parallel, k_\perp)$ are labeled as $X_2(0,\pi/2)$, $\Gamma(0,0)$, $X_1(\pi/2,0)$, and $W(\pi/2,\pi/2)$. The curves correspond to the following points labeled in Fig. 2(a): (a) $\theta=0$, "lower" branch, A, $\phi_h=0$ (plain curve), B, $\phi_h=0.2\pi$ (dashed), C, $\phi_h=0.3\pi$ (dotted), D, $\phi_h=0.5\pi$ (crossed); (b) $\theta=0$, "upper" branch, D (crossed), E (plain), F (dashed), G (dotted); (c) $\theta=0.25\pi$, A (plain), H (dashed), I (○), J (dotted), G (crossed).

$$I_h^0(s,t)=i_{c0}\sin(\theta^0_{s,t}-\theta^0_{s+1,t+1}-A_{s,t;s+1,t+1}) ,$$
$$I_v^0(s,t)=i_{c0}\sin(\theta^0_{s,t}-\theta^0_{s-1;t+1}-A_{s,t;s-1,t+1}) . \quad (37)$$

Upon small changes $\delta\theta_{s,t}$ in the phases, the currents will change by

$$\delta I_h(s,t)\sim i_{c0}\cos(\theta^0_{s,t}-\theta^0_{s+1;t+1}-A_{s,t;s+1,t+1}$$
$$\times(\delta\theta_{s,t}-\delta\theta_{s+1,t+1})$$
$$=i_{c0}A_t(\delta\theta_{s,t}-\delta\theta_{s+1,t+1}) \quad (38)$$

and

$$\delta I_v(s,t)\sim i_{c0}\cos(\theta^0_{s,t}-\theta^0_{s-1,t+1}-A_{s,t;s-1,t+1})$$
$$\times(\delta\theta_{s,t}-\delta\theta_{s-1,t+1})$$
$$=i_{c0}B_t(\delta\theta_{s,t}-\delta\theta_{s-1,t+1}) \quad (39)$$

to first order. If the fluctuations are given by the real part of (36), then (38) is of the form

$$\delta I_h^{(+)}(s,t)=\begin{cases} i_{c0}A_t2K\cos(k_\parallel s+k_\perp t+\xi_1)\cos(\xi_1) & \text{odd } t \\ -i_{c0}A_t2K\cos(k_\parallel s+k_\perp t+\xi_1)\cos(\xi_2) & \text{even } t\ , \end{cases}$$
$$\delta I_h^{(-)}(s,t)=\begin{cases} i_{c0}A_t2K\sin(k_\parallel s+k_\perp t+\xi_1)\sin(\xi_1) & \text{odd } t \\ i_{c0}A_22K\sin(k_\parallel s+k_\perp t+\xi_1)\sin(\xi_2) & \text{even } t\ , \end{cases} \quad (40)$$

where ξ_1 and ξ_2 are constant phase terms. Expressions for $\delta I_v^{(+)}$ and $\delta I_v^{(-)}$ are similar.

Thus, deviations in the current components are periodic in s and t. It can easily be shown that on summing these deviations over a period, the result is

$$\delta I_h^{(+)}=\begin{cases} Ki_{c0}(A_1-A_2) & \text{for } \mathbf{k}=(0,0) \\ 0 & \text{otherwise}\ , \end{cases}$$
$$\delta I_h^{(-)}=0 \quad \text{all } (k_\parallel,k_\perp)\ , \quad (41)$$

and

$$\delta I_v^{(+)}=\begin{cases} Ki_{c0}(B_1-B_2) & \text{for } \mathbf{k}=(0,0) \\ 0 & \text{otherwise}\ , \end{cases}$$
$$\delta I_v^{(-)}=0 \quad \text{all}(k_\parallel,k_\perp)\ . \quad (42)$$

Hence, to first order, apart from the $(+)$ eigenvector at the Brillouin zone center $\Gamma(0,0)$, the current is unchanged in all the fluctuation modes. It is thus conclusive that even under the condition of a fixed current bias, all the states in the "upper" branches in $f=\frac{1}{2}$ are unstable to phase fluctuations. We expect that this is representative of the behavior of any other value of the field f.

VI. CONCLUSION

We have studied current-carrying states, of the frustrated Josephson-junction array, that have a "staircase-state-like" vortex superlattice. Comprehensive analytic and numerical solutions are presented. These extend the work of other authors to cover the whole range of orientation of the net current. The current orientation is particularly simple to impose in the numerical calculations using the "loop current" formalism. We found states which carry currents up to an "upper" critical value I_{c1} which is higher than the intrinsic critical value I_{c0}. A stability analysis is carried out in detail for $f=\frac{1}{2}$. It is shown that these states are in fact unstable to fluctuation in the phases, even under the condition of a current bias. The stable solutions are those studied by Halsey and Benz *et al.* The values of I_{c0} obtained previously are thus confirmed, and in some cases more accurately determined.

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