Field theory of anyons and the fractional quantum Hall effect

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(Received 26 July 1991; revised manuscript received 13 April 1992)

We present a microscopic theory of the fractional quantum Hall effect and its hierarchy structure on the basis of a Chern-Simons gauge theory. Our Hamiltonian contains a renormalized term $(|\alpha|/m)|\psi|^4$ which plays important roles. It is shown field theoretically that the system of the planar electrons in a strong magnetic field yields a hierarchy of the fractional quantum Hall states, which are a condensed phase of the bosonized electrons and vortices. It is discussed in detail how the Coulomb interaction between electrons plays a crucial role in this phenomenon.

I. INTRODUCTION

The discovery of the fractional quantum Hall effect (FQHE) revealed that the two-dimensional electron system has a rich ground-state structure in a strong magnetic field.[1] Depending on the Landau-level filling factor ν, the system exhibits a beautiful hierarchy of new quantum states. A microscopic theory has been presented to account for the FQHE and its hierarchy by using the Laughlin wave function as a variational function[2] or by exact numerical calculations on a system of a few particles.[3] Although these approaches have produced many useful results, they are still unsatisfactory from a purely theoretical point of view.

Recently, an approach was initiated by Girvin,[4] in which the FQH states are interpreted as a kind of bose condensate of electrons bound to fluxes, and where Laughlin's quasiparticles are identified with vortex solitons arising on such a condensate. This approach is most conveniently realized in Chern-Simons (CS) gauge theories. In fact, based on them Landau-Ginzburg models have been proposed.[5–9] However, these models have some problems. First of all, microscopic derivations of these models are not well established. Next, it is not certain whether the constraint for electrons being in the lowest Landau level is correctly imposed. Furthermore, the role of the Coulomb interaction between electrons is not clear, although the Coulomb interaction must be an essential agent which makes the system incompressible for the FQH states. Finally, there are many undetermined parameters in these models.

Using a CS gauge theory, we have presented a field-theoretical formulation[10] of the FQHE, where the hierarchy structure of the FQH states is derived from a basic Hamiltonian. However, this work also contains some of the problems we have pointed out for the Landau-Ginsburg models. The aim of this paper is to refine our field-theoretical approach by solving these problems to some extent.

Our strategy is to regard planar electrons as anyons with an odd statistics parameter, α/π = odd integer. We are free to represent the anyon by the boson field or the fermion field, and for our purpose it is convenient to use the boson field. Physically, this is to decompose an electron into a boson and a flux quantum. Equivalently, starting with a particle mechanics describing a set of anyons,[11] we naturally arrive at a CS gauge theory which is microscopically equivalent to an ensemble of planar electrons. We analyze the Hamiltonian of this system semiclassically in the thermodynamical limit. In this approach, we first search for the ground state of the mean-field approximation and then take account of small fluctuations around it within the Bogoljubov approximation. The ground state is obtained by minimizing the classical energy of the system: it is a solution of the classical field equations of the Hamiltonian. Thus, we call it the classical ground state.

Some caution is needed in our approach. As is well known, the expansion of the anyon state around the boson states is singular at the short-distance behavior. We can avoid this singularity by modifying the Hamiltonian in such a way that the modified Hamiltonian reproduces the correct short-distance behavior and the exact energy spectrum for known cases perturbatively. We have derived such a Hamiltonian[12] up to the first order of α/π. We also diagonalize the modified Hamiltonian up to the same order of α/π. Thus, strictly speaking, our results are precise in the anyon system up to this order. In the application of our formalism to the real electron system, we assume that the mean-field approximation and the Bogoljubov approximation are valid qualitatively although α/π is not small.

We now summarize the results of our analysis. Our system consists of planar electrons occupying the lowest Landau level in the external magnetic field B. Our field-theoretical formulation realizes various essential features of the system already in the mean-field approximation.

Let us switch off the Coulomb interaction. First of all, there are many nonuniform solutions corresponding to nonuniform distributions of electrons in the lowest Lan-

dau level. These nonuniform solutions represent nontopological and topological vortex solitons, where there exist zero-energy modes associated with the translation of each vortex solitons. (Topological vortices are identified with Laughlin's quasiparticles.) Second, when and only when the filling factor takes a magic number, $\nu = \pi/\alpha$, there is a uniquely constant solution corresponding to the uniform distribution of the electrons. It exists only when the external magnetic field cancels precisely the statistical field of anyons. It is interpreted to represent a condensed state of electrons bound to fluxes. In the absence of the Coulomb interaction, all these solutions have the same energy and the constant solution does not play any particular role. The system is compressible due to the degeneracy of the ground states.

When the Coulomb interaction is switched on, all the nonuniform solutions acquire positive-definite Coulomb energy, while the constant solution does not. Therefore, when $\nu = \pi/\alpha$, the existence of the Coulomb interaction forces us to choose the constant solution as the unique classical ground state with the minimum energy. Because the degeneracy is removed by the Coulomb interaction, the system becomes incompressible. This system exhibits the FQHE. Nonuniform solutions describe excited states with higher Coulomb energy in this system. As we just mentioned, only when the filling factor satisfies the condition $\nu = \pi/\alpha$ does the constant solution exist. On the other hand, for $\nu \neq \pi/\alpha$, the classical ground state is given by a nonuniform solution which minimizes the Coulomb energy. The system is compressible even in the presence of the Coulomb interaction because of the zero-energy modes: obviously, the zero-energy modes are given by the translation of the system itself. As we shall mention soon after, a candidate of the nonuniform ground-state solution at $\nu \neq \pi/\alpha$ is the Wigner crystal made of vortex solitons. In this way, the value of the filling factor ($\nu = \pi/\alpha$) is very special in our formalism.

After choosing the constant solution in the presence of the Coulomb interaction as the classical ground state, we take account of small fluctuations around it. We calculate the corrections due to the small fluctuations to the classical ground-state energy: it reads $-\pi N/(2\sqrt{2})(e^2/\epsilon\ell_B)$, where ϵ is the dielectric constant, $\ell_B = 1/\sqrt{eB}$ is the magnetic length, and N is the number of electrons in the system. (In this paper we use the unit such that $c = 1$ and $\hbar = 1$.) Evaluating the Hall current, we identify this state with the FQH state at the first level of the hierarchy.

The ground state is given by a constant solution at $\nu = \pi/\alpha$ as we have emphasized. This solution is bosonic, and it does not satisfy the Pauli exclusion principle imposed on electrons. One may wonder how this constant solution is related with the Laughlin wave function. We can show[13] that the ground-state wave function is indeed given by the celebrated Laughlin wave function when we take account of the small fluctuations around the constant solution.

Possible excitations in our ground state with uniform density are topological vortex solitons, which may be considered as Laughlin's quasiparticles (vortices) or quasiholes (antivortices). It is curious but a fact that there are only antivortex solitons as classical solutions describing collective modes of electrons in the lowest Landau level. Now, suppose we remove one electron from the FQH state at $\nu = 1/k$ ($k \equiv \alpha/\pi$). Then, k antivortices appear since each vortex carries the electron number $-1/k$, and a state with $N-1$ electrons is formed. Apart from the cyclotron energy of electrons, the energy of the state is higher than the ground-state energy by k times of the Coulomb energy of the antivortex soliton, which is a chemical potential (μ_-) at $\nu = 1/k$ when we approach at $\nu = 1/k$ from $\nu < 1/k$. Numerically, it is given by $\mu_- \simeq 0.13(e^2/\epsilon\ell_B)$ at the filling factor $\nu = \frac{1}{3}$.

We conjecture that vortex solitons are identified with a bound state of $k-1$ antivortices and an electron. Then, when we add one electron to the FQH state, k vortices will effectively appear to make a state with $N+1$ electrons. The energy of the state is higher than the ground-state energy by k times of the Coulomb energy of the vortex soliton, which is a chemical potential (μ_+) at $\nu = 1/k$ when we approach from $\nu > 1/k$. We have a discontinuity in the chemical potential at the magic filling factor, which originates in the difference of the energies of an antivortex and a vortex.

It is very difficult to say something definite for the ground state at $\nu \neq 1/k$. However, we can make the following physical picture. When more electrons are removed from (added to) the FQH state, more antivortices (vortices) must be created. The ground state at the vicinity of the magic filling factor will be described by an ensemble of topological vortices. We would expect that these topological solitons occupy on a triangular lattice to minimize the Coulomb energy, and form a Wigner crystal; see Ref. 14 for more details. Thus, when the filling factor is given by, e.g., $\nu < 1/k$, the addition or removal of an electron implies the removal or addition of k antivortices. Hence, there will be no discontinuity in the chemical potential.

When a sufficient number of topological solitons is created as the filling factor decreases or increases, they would eventually condense. In this way, a new condensed phase of electrons and solitons would appear, which will be the FQH state at the second level of the hierarchy. In order to understand the hierarchy of the FQHE, we make a field theory of the solitons by regarding them as pointlike particles. This can be done with the aid of a singular gauge transformation, which precisely introduces localized vortex solitons (Dirac strings). It is straightforward to construct a second quantized field theory of such localized solitons by introducing a new CS gauge field. The basic physics is that the vortex solitons feel the fluxes of bosonized electrons and of themselves and they behave as anyons. We again search for the ground state of the mean-field approximation by minimizing the energy of the system consisting of electrons and vortices. Only when the total fluxes due to the external magnetic field, electrons, and vortices are canceled does the constant solution exist as a state with the minimum Coulomb energy, thus determining the classical ground state together with the magic filling factor at the second level of the hierarchy. This ground state is incompressible. Examining the Hall current, we identify it with the FQH

state at the second level. We can generalize this scheme to obtain the hierarchy of the FQH states at any levels. Our scheme presents a field-theoretical realization of the hierarchy construction due to Haldane and Halperin.[2] However, we should mention that the derivation of our Hamiltonian of vortices is by no means rigorous but is based on plausible arguments.

This paper is composed as follows. In Sec. II, our basic Hamiltonian is constructed by second quantizing a set of anyons. In Sec. III, the ground states are analyzed in the mean-field approximation when the Coulomb interaction is switched off and on. In Sec. IV, small fluctuations are taken account around the classical ground state in the presence of the Coulomb interaction. In Sec. V, we derive the ground-state wave function in our framework, which is found to coincide with the Laughlin wave function. In Sec. VI, antivortices are obtained as topological solitons in this system. In Sec. VII, we construct a local field theory of the vortex solitons. In Sec. VIII, the hierarchy of the FQH states is derived field theoretically. In Sec. IX, we discuss some remaining problems. We have also prepared two appendixes for some detailed calculations.

II. FIELD THEORY OF ANYONS

In this section we review a nonrelativistic particle mechanics of anyons and its second quantization. In particular, we show how a second-quantized Hamiltonian is modified for a perturbative description of anyons from the boson limit. This is the starting point of our formalism of the FQHE.

We start with a Lagrangian describing a system of N *free* anyons moving nonrelativistically on a plane:

$$L=\sum_{r=1}^{N}\frac{m}{2}\left(\frac{dx_r^j}{dt}\right)^2+\frac{\alpha}{\pi}\sum_{r<s}^{N}\frac{d}{dt}\theta(x_r-x_s). \tag{2.1}$$

Here, m is the anyon mass, (x_r^1,x_r^2) is a vector specifying the location of the rth anyon, and $\theta(x_r-x_s)$ is the azimuthal angle between two anyons, i.e.,

$$\theta(x_r-x_s)=\arctan\left(\frac{x_r^2-x_s^2}{x_r^1-x_s^1}\right). \tag{2.2}$$

It is understood that two anyons never cross each other, i.e., $x_r\neq x_s$, so that the azimuthal angle $\theta(x_r-x_s)$ is well defined. The second term in (2.1) is a total derivative in time, and does not contribute classically. However, quantum mechanically, it produces a phase $e^{i\alpha}$ when two anyons are interchanged. It is natural to consider that the statistics parameter α is defined mod2π because the physical quantities are periodic in α. In Appendix A we argue that it is indeed the case in a finite-temperature theory, and that the term $r=s$ can be included in the sum in (2.1) by making an appropriate regularization for $\theta(x_r-x_r)$.

The canonical momentum is defined as

$$p_r^j=m\dot{x}_r^j+a_j(x_r), \tag{2.3}$$

with

$$\begin{aligned}a_j(x)&=\frac{\alpha}{\pi}\sum_{s}^{N}\partial_j\theta(x-x_s)\\&=-\frac{\alpha}{\pi}\epsilon_{jk}\int d^2y\frac{x^k-y^k}{(x-y)^2}\rho_e(y),\end{aligned} \tag{2.4}$$

where

$$\rho_e(y)=\sum_{s}^{N}\delta^2(y-x_s) \tag{2.5}$$

represents the density of anyons. It is convenient to characterize $a_j(x)$ by the constraint equation

$$\frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j(x)=\rho_e(x), \tag{2.6}$$

together with the gauge condition $\partial_i a_i=0$. Then, the Hamiltonian is given by

$$H=\frac{1}{2m}\sum_{r=1}^{N}[p_r^j-a_j(x_r)]^2, \tag{2.7}$$

together with the constraint equation (2.6).

According to the correspondence principle, second-quantization can easily be done by introducing bose field operators ψ and $\psi^\dagger$, satisfying

$$\begin{aligned}&[\psi(x),\psi^\dagger(y)]=\delta(x-y),\\&[\psi(x),\psi(y)]=[\psi^\dagger(x),\psi^\dagger(y)]=0,\end{aligned} \tag{2.8}$$

and making the replacement such as $p_j\to -i\partial_j$ and $\rho_e\to\psi^\dagger\psi$ in the Hamiltonian (2.7) and the constraint equation (2.6). Then, the second-quantized Hamiltonian reads

$$H=\int d^2x\frac{1}{2m}[(i\partial_j+a_j)\psi]^\dagger(i\partial_j+a_j)\psi, \tag{2.9}$$

together with the constraint equation

$$\frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j=\psi^\dagger\psi, \tag{2.10}$$

and the gauge condition $\partial_i a_i=0$, which determines a_j. Indeed, this may be solved as

$$a_j(x)=-\frac{\alpha}{\pi}\epsilon_{jk}\int d^2y\frac{x^k-y^k}{(x-y)^2}\psi^\dagger\psi(y), \tag{2.11}$$

which corresponds to (2.4) in the particle mechanics.

It is well known[15] that the Hamiltonian (2.9) together with the canonical commutation relations (2.8) leads to the correct quantum-mechanical Schrödinger equation of anyons with this operator ordering. However, in this paper we analyze the anyon system perturbatively in α, in which we expand an anyonic state in terms of bosonic states. In such a case a modification of the Hamiltonian is necessary as we have discussed in detail in Ref. 12. (See also Ref. 16.) Let us briefly review the essence.

We take the Hamiltonian (2.9) in the presence of the external magnetic field, by replacing $(i\partial_j+a_j)$ with $(i\partial_j+a_j-eA_j)$, where A_j is the external electromagnetic potential. We try to solve such a Hamiltonian system perturbatively with respect to the statistics parameter

α/π. We expand the Hamiltonian, the eigenvalue, and the eigenfunction as follows:

$$H(\alpha) = H_0 + \frac{\alpha}{\pi}H_1 + O\Big(\frac{\alpha}{\pi}\Big)^2, \tag{2.12a}$$

$$E(\alpha) = E_0 + \frac{\alpha}{\pi}E_1 + O\Big(\frac{\alpha}{\pi}\Big)^2, \tag{2.12b}$$

$$\Psi(\alpha) = \Psi_0 + \sum_i \left[\frac{\alpha}{\pi}c_i + O\left(\frac{\alpha}{\pi}\right)^2\right]\Psi_i, \tag{2.12c}$$

where the unperturbed quantities H_0, E_0, and Ψ_i are those of the boson limit; $H_0|\Psi_0\rangle = E_0|\Psi_0\rangle$. The first-order correction is given by

$$E_1 = \langle\Psi_0|H_1|\Psi_0\rangle \tag{2.13}$$

in the naive perturbation theory, where

$$H_1 = \frac{1}{m}\int d^2x\,\psi^\dagger(i\partial_k - eA_k)a_k\psi \approx O(\alpha/\pi). \tag{2.14}$$

Now, the two-anyon system can be exactly solved quantum mechanically,[17] and we can check if the first-order perturbation result (2.13) is correct. We can also check it for the many-anyon system[18] where the spectrum has been partially obtained. By an explicit calculation we can prove[12] that instead of the naive expectation (2.13) we have

$$E_1 = \langle\Psi_0|\Big(H_1 + \frac{|\alpha|}{m}\int d^2x : |\psi|^4 :\Big)|\Psi_0\rangle. \tag{2.15}$$

Consequently, in performing a perturbation expansion, we need to adopt the modified Hamiltonian given by

$$H = \int dx\Big[\frac{1}{2m}[(i\partial_j + a_j - eA_j)\psi]^\dagger(i\partial_j + a_j - eA_j)\psi + \frac{|\alpha|}{m} : (\psi^\dagger\psi)^2 :\Big], \tag{2.16}$$

up to the first order of α/π, where the dots $: \mathcal{O} :$ denote the normal ordering of the operator $\mathcal{O}$.

We can understand the origin of the extra term quantum mechanically as follows. Let us consider a two-anyon system in a harmonic potential[16] or in an external magnetic field. In general, anyon wave functions vanish with a fractional power of r as $\psi_{\text{anyon}}(r) \to r^{\alpha/\pi}$ when two anyons come close. However, unperturbed boson states ($\alpha = 0$) do not involve this fractional power of r; their wave functions vanish like r^ℓ with ℓ being angular momentum. Hence, we cannot expand anyon states in terms of the unperturbed boson states which have the standard partial waves. Instead of doing so, we may expand $r^{-\alpha/\pi}\psi_{\text{anyon}}$ by using the standard partial waves. Schematically the expansion looks like $\psi_{\text{anyon}} = r^{\alpha/\pi}(\sum\text{partial waves})$. This expansion leads to a modification of the perturbation series of α/π. Especially, in the first order of α/π, we have to add to the naive Hamiltonian a δ-function-type repulsive force with the strength $2|\alpha|/M$, which leads to the $: (\psi^\dagger\psi)^2 :$ term in (2.16) field theoretically. Namely it is a renormalized interaction term in the perturbation of α/π.

It should be remarked that the role of the extra term $: (\psi^\dagger\psi)^2 :$ is just to reproduce the correct behavior as $r \to 0$ *in perturbation theory*, and not to introduce the additional repulsive force associated with the anyon statistics. The repulsion due to statistics is already imposed through the CS term.

Strictly speaking, the modified Hamiltonian has been derived only in the first order of the α/π. However, in this paper we assume that this can be used for any values of α/π, especially when $\alpha/\pi =$ an odd integer where the anyon is a fermion. As we shall see in the rest of this paper, this assumption gives us physically quite reasonable results at least qualitatively.

It is straightforward to derive the Lagrangian density from the Hamiltonian. (We do not introduce the external potential A_j in this section for a notational simplicity.) Introducing a field a_0 to implement the constraint equation (2.10), we obtain

$$\mathcal{L} = \psi^\dagger(i\partial_0 + a_0)\psi - \frac{1}{2m}|(i\partial_k + a_k)\psi|^2 - \frac{|\alpha|}{m}(\psi^\dagger\psi)^2 - \frac{1}{4\alpha}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda. \tag{2.17}$$

We call a_μ the statistical field or the CS field.

In this paper it is necessary to consider also a system consisting of two distinguishable anyons. We have in mind electrons and vortex solitons. Let α and β be their own statistics parameters and γ be their *relative* statistics parameter. The Lagrangian governing such a system is

$$L = \sum_r \frac{m}{2}\Big(\frac{dx_r^j}{dt}\Big)^2 + \frac{\alpha}{\pi}\sum_{r<s}\frac{d}{dt}\theta(x_r - x_s) + \sum_r \frac{M}{2}\Big(\frac{dy_r^j}{dt}\Big)^2 + \frac{\beta}{\pi}\sum_{r<s}\frac{d}{dt}\theta(y_r - y_s) + \frac{\gamma}{\pi}\sum_{r,s}\frac{d}{dt}\theta(x_r - y_s). \tag{2.18}$$

Repeating the same procedure as we did for a system of one type of anyon, we can derive the following second-quantized Hamiltonian density:

$$\mathcal{H} = \frac{1}{2m}[(i\partial_j + a_j)\psi]^\dagger(i\partial_j + a_j)\psi + \frac{g_\psi}{2} : (\psi^\dagger\psi)^2 : + \frac{1}{2M}[(i\partial_j + b_j)\phi]^\dagger(i\partial_j + b_j)\phi + \frac{g_\phi}{2} : (\phi^\dagger\phi)^2 : + h : \psi^\dagger\psi\phi^\dagger\phi :, \tag{2.19}$$

where the CS gauge fields a_j and b_j are fixed by the constraint equations

$$\begin{aligned}\epsilon_{ij}\partial_i a_j &= 2\alpha\psi^\dagger\psi + 2\gamma\phi^\dagger\phi,\\ \epsilon_{ij}\partial_i b_j &= 2\beta\phi^\dagger\phi + 2\gamma\psi^\dagger\psi,\end{aligned} \tag{2.20}$$

and the gauge conditions $\partial_i a_i = \partial_i b_i = 0$. The coupling constants g_ψ, g_ϕ, and h are determined as[12]

$$g_\psi = \frac{2|\alpha|}{m}, \quad g_\phi = \frac{2|\beta|}{M}, \quad h = \frac{(m+M)\gamma}{mM}, \tag{2.21}$$

in the first order of perturbation. Just as before, we assume to use the above Hamiltonian for arbitrary values of the statistics parameters α, β, and γ.

The Lagrangian density is easily constructed from the Hamiltonian. Introducing two fields a_0 and b_0 to implement the constraint equations (2.20), we obtain that

$$\begin{aligned}\mathcal{L} &= \psi^\dagger(i\partial_0 + a_0)\psi - \frac{1}{2m}|(i\partial_k + a_k)\psi|^2 \\ &+\phi^\dagger(i\partial_0 + b_0)\phi - \frac{1}{2M}|(i\partial_k + b_k)\phi|^2 \\ &-\frac{g_\psi}{2}(\psi^\dagger\psi)^2 - \frac{g_\phi}{2}(\phi^\dagger\phi)^2 - h\psi^\dagger\psi\phi^\dagger\phi \\ &-\frac{1}{4X}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{4Y}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda \\ &-\frac{1}{4Z}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda. \end{aligned} \tag{2.22}$$

Here, parameters X, Y, and Z are defined by

$$X = \frac{\alpha\beta - \gamma^2}{\beta}, \quad Y = \frac{\alpha\beta - \gamma^2}{\alpha}, \quad Z = \frac{\gamma^2 - \alpha\beta}{2\gamma}. \tag{2.23}$$

In the same way we may include any number of distinguishable anyons into the system, which is actually relevant in discussing the FQHE at the nth level of the hierarchy.

III. PLANAR ELECTRONS AND CLASSICAL GROUND STATES

We consider a system of N electrons in the presence of an external uniform magnetic field B perpendicular to the plane. We assume that the spin degrees of freedom can be ignored since the system is completely polarized by the strong magnetic field. Because such electrons are considered as anyons with α/π being an odd integer, they are described by the Hamiltonian density (2.16) with the inclusion of the Coulomb interaction term $\mathcal{V}$ for electrons. In what follows, for definiteness, we use the notations $\widehat{\psi}$ and $\widehat{a}_k$ for the quantum fields and ψ and a_k for the classical fields.

Thus, our basic Hamiltonian is

$$H = \int d^2x \Big\{ \frac{1}{2m}|D_j\widehat{\psi}|^2 + \frac{|\alpha|}{m} : (\widehat{\psi}^\dagger\widehat{\psi})^2 : + \mathcal{V}[\widehat{\psi}] \Big\}, \tag{3.1}$$

together with the constraint equation (2.10), where $iD_\mu = i\partial_\mu + \widehat{a}_\mu - eA_\mu$. Equivalently, the Lagrangian density reads

$$\begin{aligned}\mathcal{L} &= \psi^\dagger iD_0\psi - \frac{1}{2m}|D_k\psi|^2 - \frac{|\alpha|}{m}(\psi^\dagger\psi)^2 \\ &-\frac{1}{4\alpha}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \mathcal{V}[\psi]. \end{aligned} \tag{3.2}$$

Here and hereafter, we use a notation $|D_k\widehat{\psi}|^2$ for simplicity, but this should always be understood as $(D_k\widehat{\psi})^\dagger(D_k\widehat{\psi})$. Our convention is that an electron carries electric charge $-e$, and the external electromagnetic potential is given by

$$A_j = -\frac{B}{2}\epsilon_{jk}x^k. \tag{3.3}$$

We choose $eB > 0$ and $\alpha > 0$. The Coulomb interaction term is given by

$$\begin{aligned}\mathcal{V}[\widehat{\psi}] = \frac{e^2}{2\epsilon}\int d^2x\, d^2y : \{\widehat{\psi}^\dagger\widehat{\psi}(x) - \rho\} \\ \times\frac{1}{|x-y|}\{\widehat{\psi}^\dagger\widehat{\psi}(y) - \rho\} :, \end{aligned} \tag{3.4}$$

where ϵ is the dielectric constant. Here, we have added a uniform background charge $e\rho$ to the Coulomb term $\mathcal{V}$ for charge neutrality. The boson field $\widehat{\psi}$ is identified with the electron bound to statistical flux, which we call *bosonized* electrons.

It is convenient to rewrite the Hamiltonian in the following way. First, we use the Bogomol'nyi decomposition:

$$|D_k\widehat{\psi}|^2 = |(D_1 - iD_2)\widehat{\psi}|^2 - \epsilon_{jk}\partial_j(\widehat{\psi}^\dagger iD_k\widehat{\psi}) - : \omega_{12}\widehat{\psi}^\dagger\widehat{\psi} :, \tag{3.5}$$

where $\omega_k = \widehat{a}_k - eA_k$ and $\omega_{12} = \partial_1\omega_2 - \partial_2\omega_1$. Substituting this into the Hamiltonian we get that

$$H = \int d^2x \Big[\frac{1}{2m}|(D_1 - iD_2)\widehat{\psi}|^2 + \tfrac{1}{2}\omega_c\widehat{\psi}^\dagger\widehat{\psi} + \mathcal{V}\Big], \tag{3.6}$$

with $\omega_c = eB/m$ being the cyclotron frequency. Note that the term $: (\widehat{\psi}^\dagger\widehat{\psi})^2 :$ has been canceled out in this Hamiltonian.

It should be emphasized that our bosonic Lagrangian describes a microscopic theory of planar electrons interacting each other via the Coulomb interaction $\mathcal{V}$. In this respect it is conceptually different from all the Landau-Ginzburg-type Lagrangians.[5–7] In the latter case, for instance, on the basis of Laughlin's wave functions one "derives" the type of the Lagrangian (3.2) without the Coulomb term $\mathcal{V}$: thus, ψ is not a quantum field operator but is a complicated order parameter containing already the effect of the Coulomb interaction and other interactions. Furthermore, the mass m is not really the mass of the electron but a parameter depending on the Coulomb interaction in their approach.[7]

When the Coulomb interaction between electrons is switched off, we know very well what the ground state of planar electrons in the external magnetic field is. All electrons in the state are in the lowest Landau level, and the ground-state energy is given by

$$E_N = \tfrac{1}{2}\omega_c N, \tag{3.7}$$

where N is the number of electrons. Ground states are highly degenerate. It has been argued physically that this degeneracy must be removed by the Coulomb interaction.

In our field-theoretical formalism let us see how we can understand the ground-state degeneracy and its removal depending on the absence and the presence of the Coulomb interaction. In the semiclassical approach we take account of small fluctuations around the ground state ψ of the mean-field approximation by setting

$$\widehat{\psi} = \psi + \eta. \tag{3.8}$$

In this section we discuss the ground state ψ of the mean-field approximation.

The ground state is described by the c-number function ψ, satisfying $\int d^2x|\psi(x)|^2 = N$, which minimizes the classical energy of the system. The classical energy is given by

$$E(\psi) = \int d^2x\Big[\frac{1}{2m}|(D_1 - iD_2)\psi|^2 + \tfrac{1}{2}\omega_c N + \mathcal{V}(\psi)\Big], \tag{3.9}$$

where a_k in D_k is a c-number function given by (2.10).

Let us neglect the Coulomb term $\mathcal{V}$ temporarily. Then, the classical ground state is given by solving the self-dual equation

$$(D_1 - iD_2)\psi = 0, \tag{3.10}$$

since it minimizes the classical energy (3.9). All the solutions have the same energy; thus, the corresponding states are degenerate. It is interesting that the "classical" energy gives precisely the exact result (3.7). This suggests that there should be no corrections due to the fluctuations η, which we shall prove in the next section.

When the external magnetic field does not exist, Jackiw and Pi[19,15] have analyzed this self-dual equation in detail. In this case, this equation is reduced to the Liouville equation, and all the solutions can be obtained explicitly. Physically, they represent an ensemble of nontopological vortices.

In the presence of the external magnetic field, it is impossible to solve all the solutions explicitly. Nevertheless, we can easily derive some essential features of the solutions. First of all, the self-dual equation is translational invariant: The translation of the system is compensated by the gauge transformation in A_k and ψ. Second, there are topological vortex solutions in addition to nontopological vortex solutions.[20] Nontopological solitons exist at any filling factor ν, while topological solitons exist only when the constant solution exists. This is the case if and only if the filling factor ν takes a particular number given by

$$\nu \equiv \frac{2\pi\rho}{eB} = \frac{\pi}{\alpha}, \tag{3.11}$$

as is easily derived from (2.10) and (3.10), and the constant solution is given by

$$\psi = \sqrt{\rho}, \qquad a_k = eA_k. \tag{3.12}$$

The equation $a_k = eA_k$ means that the statistical field and the external magnetic field cancel each others precisely. Vortex solitons are characterized by the asymptotic behaviors as $|x| \to \infty$:

$$\psi \to 0 \tag{3.13a}$$

for nontopological vortices, and

$$\psi \to \sqrt{\rho}e^{in\theta} \tag{3.13b}$$

for topological vortices with the flux $2\pi n$, where n is an integer and θ is the azimuthal angle. See also Sec. VI. The positions of vortices can be taken arbitrarily and represent zero-energy modes of the classical solutions. Furthermore, other nonuniform solutions besides vortices might exist. All these nonuniform solutions have at least two zero-energy modes associated with the translation of the system itself. In any case, we have nonuniform solutions as well as the uniform (constant) solution. They are all degenerate in the absence of the Coulomb interaction. Therefore, the system is compressible.

Now, let us switch on the Coulomb interaction between electrons. It is a crucial fact that all the nonuniform solutions acquire the positive-definite Coulomb energy through the Coulomb term (3.4), while the constant solution does not. As we have explained, when $\nu \neq \pi/\alpha$, the constant solution does not exist. Thus, a nonuniform solution with the minimum Coulomb energy is a ground state. However, as we have pointed out, the zero-energy modes necessarily exist associated with the translational invariance of the system itself. Hence, the system is still compressible.

The situation is entirely different for the case $\nu = \pi/\alpha$. In this case the constant solution exists and minimizes the Coulomb energy, and gives the unique ground state. Indeed, there are no zero-energy modes associated with translational symmetry. The translation of the system simply produces a phase factor $e^{i\Lambda}$ in $\psi = \sqrt{\rho}$. The phase factor, however, is removed by the gauge transformation in the CS gauge field. Indeed, as we show in Sec. IV, no zero-energy modes exist associated with small fluctuations η around the classical ground state. Although there are other excitations given by topological vortex solitons, their excitations cost a Coulomb energy. Therefore, when $\nu = \pi/\alpha$, there exists the unique ground state of the constant solution. This is the explanation of the incompressibility of the ground state at $\nu = \pi/\alpha$ in the mean-field approximation.

We have shown that, in the presence of the Coulomb interaction $\mathcal{V}$, the ground state of the Hamiltonian at $\nu = \nu^{(0)} \equiv \pi/\alpha$ is given by the classical solution (3.12) which describes a condensed phase of bosonized electrons. In Appendix B we show that, when the external electric field E_k is applied, the Hall current is induced such that

$$e\langle J_j\rangle = -\frac{ie}{m}\langle\widehat{\psi}^\dagger \overleftrightarrow{D}_j \widehat{\psi}\rangle = \frac{e^2}{2\alpha}\epsilon_{jk}E_k. \tag{3.14}$$

Hence, the Hall conductance is given by $\sigma_{xy} = e^2/2\alpha = \nu^{(0)}(e^2/2\pi)$, as is expected. We may identify our ground state with the FQH state at filling factor $\nu = \nu^{(0)}$.

IV. BOGOLJUBOV APPROXIMATION

In this section we only consider the case when the filling factor takes a special value given by (3.11). As we have argued in the preceding section, there exists the unique nondegenerate ground state ($\psi = \sqrt{\rho}$) with the minimum Coulomb energy in the mean-field approximation. We now consider small fluctuations around it.

We substitute $\widehat{\psi} = \sqrt{\rho} + \eta$ into the Hamiltonian density (3.5), where

$$\eta = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}\neq\mathbf{0}} a_{\mathbf{p}}e^{i\mathbf{p}\mathbf{x}}, \tag{4.1}$$

with V being the volume of the system, and

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \delta_{\mathbf{pq}}, \qquad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \tag{4.2}$$

We eliminate the gauge potentials by using the constraint condition (2.10) or

$$a_i - eA_i = -\frac{\alpha}{\pi}\epsilon_{ij}\int d^2y \frac{x^j - y^j}{(x-y)^2}(\psi^\dagger\psi - \rho). \tag{4.3}$$

Keeping only the terms up to quadratic in η and $\eta^\dagger$, we obtain

$$H = \frac{\omega_c}{2}N + \int d^2x\Big(\frac{1}{2m}\partial_k\eta^\dagger\partial_k\eta + \frac{\alpha\rho}{m} : (\eta+\eta^\dagger)^2 :\Big) + \int\int d^2x\, d^2y \frac{2\alpha^2\rho^2}{m}[\eta(x)+\eta(x)^\dagger]\frac{\ln|x-y|}{-2\pi}[\eta(y)+\eta(y)^\dagger]$$
$$+ \int\int d^2x\, d^2y \frac{e^2\rho}{2\epsilon} : [\eta(x)+\eta(x)^\dagger]\frac{1}{|x-y|}[\eta(y)+\eta(y)^\dagger] : . \tag{4.4}$$

Using (4.1), the Hamiltonian reads

$$H = \frac{\omega_c}{2}N + \sum_{\mathbf{p}\neq 0}\Big(\epsilon_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + U_{\mathbf{p}}(2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}}a_{-\mathbf{p}}) + \frac{2\alpha^2\rho^2}{m|\mathbf{p}|^2}\Big), \tag{4.5}$$

where

$$\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}, \qquad U_{\mathbf{p}} = \frac{2\alpha^2\rho^2}{m\mathbf{p}^2} + \frac{\pi e^2\rho}{\epsilon|\mathbf{p}|} + \frac{\alpha\rho}{m}. \tag{4.6}$$

By way of the Bogoljubov transformation

$$a_{\mathbf{p}} = g_{\mathbf{p}}b_{\mathbf{p}} - h_{\mathbf{p}}b_{-\mathbf{p}}^\dagger, \tag{4.7}$$

with $h_{\mathbf{p}}^2 = g_{\mathbf{p}}^2 - 1$ and

$$g_{\mathbf{p}}^2 = \frac{1}{2}\Big(\frac{\epsilon_{\mathbf{p}} + 2U_{\mathbf{p}}}{\sqrt{\epsilon_{\mathbf{p}}^2 + 4\epsilon_{\mathbf{p}}U_{\mathbf{p}}}} + 1\Big), \tag{4.8}$$

it is diagonalized as

$$H = \frac{\omega_c}{2}N + E_\eta + \sum_{\mathbf{p}\neq 0}E_{\mathbf{p}}b_{\mathbf{p}}^\dagger b_{\mathbf{p}}, \tag{4.9}$$

with

$$[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = \delta_{\mathbf{pq}}, \qquad [b_{\mathbf{p}}, b_{\mathbf{q}}] = [b_{\mathbf{p}}^\dagger, b_{\mathbf{q}}^\dagger] = 0. \tag{4.10}$$

The ground state $|g\rangle$ is such that

$$b_{\mathbf{p}}|g\rangle = 0. \tag{4.11}$$

Here,

$$E_{\mathbf{p}}^2 = \epsilon_{\mathbf{p}}^2 + 4\epsilon_{\mathbf{p}}U_{\mathbf{p}}$$
$$= \Big(\frac{\mathbf{p}^2}{2m} + \omega_c\Big)^2 + \frac{2\pi e^2\rho}{\epsilon m}|\mathbf{p}| \tag{4.12}$$

and

$$E_\eta = \sum_{\mathbf{p}\neq 0}\Big[\frac{1}{2}\Big(E_{\mathbf{p}} - \frac{\mathbf{p}^2}{2m} - \omega_c\Big) - \frac{\pi e^2\rho}{\epsilon|\mathbf{p}|}\Big]. \tag{4.13}$$

The correction due to the fluctuation η to the ground-state energy is given by E_η. This is proportional to the total number N of the electrons in the system, and it is explicitly calculated as

$$E_\eta = N\Delta E, \tag{4.14}$$

where

$$\Delta E = -\frac{\pi}{2\sqrt{2}}\frac{e^2}{\epsilon\ell_B} - \frac{\pi}{16\omega_c}\frac{e^4}{\epsilon^2\ell_B^2} + O(e^6). \tag{4.15}$$

It is the Coulomb energy per a particle of the ground state $|g\rangle$.

The excitation spectrum is given by (4.12). There is a gap energy ω_c. The mode represents the excitation of the center-of-mass motion across the Landau-level gap. This shows the absence of the zero-energy modes associated with the small fluctuations around $\psi = \sqrt{\rho}$ whether $e^2 = 0$ or not. In the preceding section we have argued that when $e^2 = 0$ the uniform state is degenerate with many nonuniform states. A transition from the uniform state to a nonuniform state is impossible by small fluctuations. It occurs, for instance, by exciting topological vortex solitons. Thus, the actual gap energy of the FQH state is not given by the dispersion relation (4.12), but the creation energy of topological solitons. This is discussed in Sec. VI.

In the presence of the Coulomb interaction ($e^2 \neq 0$), there is a finite correction ($\Delta E \neq 0$) to the ground-state energy $\frac{1}{2}\omega_c N$ in the mean-field approximation. On the other hand, there is no correction ($\Delta E = 0$) in the absence of the Coulomb interaction ($e^2 = 0$). These properties are what we expect physically since the system is reduced to an ensemble of noninteracting electrons when $e^2 = 0$. It should be mentioned that we have obtained these results due to the extra $: (\widehat{\psi}^\dagger\widehat{\psi})^2 :$ term we added in the modified Hamiltonian (3.1). Indeed, without the extra term we would obtain a diverging result for ΔE even in the case $e^2 = 0$. The extra term is necessary for expanding anyon state ($\alpha \neq 0$) in terms of bosonic states, as we have remarked in Sec. II.

In these calculations the unperturbed bosonic states are given by the states implied by the Hamiltonian (4.5) or (4.9) when we set $\alpha = 0$ ($\omega_c = 0$) therein. From (4.12) it is obvious that the state represents a superfluid with

the phonon dispersion $E_{\mathbf{p}} \propto e\sqrt{\rho|\boldsymbol{p}|}$ as $|\boldsymbol{p}| \to 0$. We should mention that what we have done in this section is the calculation of perturbative corrections in the first order of α/π to the unperturbed state of the superfluid under the condition $eB/2\pi\rho = \alpha/\pi$. We have found that the condensed state continues to exist as a stable ground state with the gap energy $\omega_c = (2\rho/m)\alpha$ up to the order of α/π. Similarly, the Coulomb energy ΔE we have found is correct to the same order of α/π. In the physical application to the FQHE in the electron system the parameter α/π is not small, and in general ΔE will receive higher-order corrections in α/π. Furthermore, there are corrections due to virtual excitations of topological solitons. Therefore, it is rather natural that our numerical result (4.15) does not coincide with the previous results obtained in the Laughlin theory.[21] However, it is interesting to note that their results on ΔE at $\alpha/\pi = 3$, 5, and 7 have a tendency to approach our result as α/π decreases.

V. GROUND-STATE WAVE FUNCTION

As we have emphasized, the ground state is given by the constant solution at $\nu = \pi/\alpha$ in the mean-field approximation. It is important to see how this constant solution is related with the Laughlin wave function. To make this paper self-contained, we now give a brief discussion on this point based on a work given in Ref. 13.

The ground-state wave function is defined by

$$\Psi_g(x_1, \ldots, x_N) \equiv \langle 0|\psi(x_1)\ldots\psi(x_N)|g\rangle, \tag{5.1}$$

where $\psi(x)|0\rangle = 0$, and $|g\rangle$ is the ground state given by (4.11).

It should be commented that the classical field ψ represents the expectation value of the operator $\widehat{\psi}$ with the ground state $|g\rangle$, $\psi = \langle g|\widehat{\psi}|g\rangle$. In the mean-field approximation the wave function is reduced to the product of the c-number functions:

$$\Psi_g(x_1, \ldots, x_N) \Longrightarrow \psi(x_1)\ldots\psi(x_N), \tag{5.2}$$

and it is bosonic. It does not satisfy the Pauli exclusion principle of electrons. To restore it we need to go beyond the classical approximation.

Before discussing small fluctuations around ψ, we recall that the ground-state wave functions are exactly obtained in the general N anyon system in the absence of the Coulomb interaction.[18] We can reproduce their results in the following way. When all anyons are in the lowest Landau level, the state $|g\rangle$ satisfies the self-dual equation

$$(D_1 - iD_2)\widehat{\psi}|g\rangle = 0, \tag{5.3}$$

which results from the Hamiltonian (3.6) with $\mathcal{V} = 0$. Note that this equation is precisely the one that determines the ground state in the model of Girvin *et al.*[22] Their Hamiltonian contains a contact interaction term $:(\psi^\dagger\psi)^2:$ by definition, while ours contains it to perform a consistent perturbation. Thus, their origins are quite different. Nevertheless, their mathematical role is the same in yielding the self-dual equation (5.3). It should be noted that the contact interaction term does not affect the anyon wave functions as far as they vanish when two anyons come to the same point.

Introducing the complex coordinate $z = x + iy$ for each electron, we rewrite the self-dual equation as

$$\Big(\frac{\partial}{\partial z_r} + \frac{eB}{4}\bar{z}_r - \frac{\alpha}{2\pi}\sum_{s\neq r}\frac{1}{z_r - z_s}\Big)\Psi_g = 0. \tag{5.4}$$

This equation is easily solved as[22]

$$\Psi_g = f(\bar{z})\prod_{r>s}|z_r - z_s|^{\alpha/\pi}\exp\Big[-\frac{eB}{4}\sum|z_r|^2\Big]. \tag{5.5}$$

Here, f is an *arbitrary* entire function of $\bar{z}_r$. This arbitrariness corresponds to the degeneracy of the ground states in the absence of the interaction. Formula (5.5) agrees with the well-known quantum-mechanical result[18] for the ground states of the bosonized anyons. [To see the wave function of anyons it is necessary to perform a singular gauge transformation which results in replacing $|z_r - z_s|^{\alpha/\pi}$ with $(\bar{z}_r - \bar{z}_s)^{\alpha/\pi}$. In our convention the conjugate coordinate $\bar{z}$ appears instead of z.]

In the presence of the Coulomb interaction, it is impossible to determine the ground-state wave function precisely. However, we can do it perturbatively. Because the Coulomb interaction removes the degeneracy, the ground-state wave function of the interacting case must approach one of the entire function $f(\bar{z})$ in the limit of the vanishing coupling constant ($e^2 \to 0$). Our idea is to determine this function from our unique mean-field solution.

Within the present approximation the ground-state wave function is obtained as follows. Substituting $\widehat{\psi} = \sqrt{\rho} + \eta$ into its definition (5.1), we find that

$$\Psi_g = \rho^{N/2}\Big(\langle 0|g\rangle + \rho^{-1}\sum_{r<s}\langle 0|\eta(x_r)\eta(x_s)|g\rangle + \cdots\Big), \tag{5.6}$$

where the terms not displayed explicitly are represented by the products of $\rho^{-1}\langle 0|\eta(x)\eta(y)|g\rangle$. The ground state $|g\rangle$ is defined by (4.11), which implies

$$a_{\mathbf{p}}|g\rangle = -\frac{h_{\mathbf{p}}}{g_{\mathbf{p}}}a^\dagger_{-\mathbf{p}}|g\rangle, \tag{5.7}$$

because of (4.7), or

$$\langle 0|\eta(x)\eta(y)|g\rangle = g(x-y)\langle 0|g\rangle, \tag{5.8}$$

with

$$g(x-y) \equiv \frac{1}{V}\sum_{\mathbf{p}}\Big(-\frac{h_{\mathbf{p}}}{g_{\mathbf{p}}}\Big)e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}. \tag{5.9}$$

Here, $h_{\mathbf{p}}$ and $g_{\mathbf{p}}$ are defined by (4.8) and (4.6).

Let us calculate the ground-state wave function explicitly up to the order of α/π. It is easy to see that $g(x)/\rho = O(\alpha/\pi)$, and hence

$$\Psi_g = \rho^{N/2}\langle 0|g\rangle\Big(1+\rho^{-1}\sum_{r<s} g(x_r-x_s)+o(\rho^{-2}g^2)\Big)$$
$$= \rho^{N/2}\langle 0|g\rangle\Big(\prod_{r<s} e^{g(x_r-x_s)/\rho}+o(\alpha^2/\pi^2)\Big). \quad (5.10)$$

In the limit $e^2 \to 0$ we can make an integration in (5.9):

$$\frac{g(x)}{\rho} = -\frac{1}{2\pi^2\rho}\int d^2p \frac{eB}{\boldsymbol{p}^2+eB}e^{i\boldsymbol{p}\boldsymbol{x}}$$
$$= -\frac{\alpha}{\pi}K_0(|x|\sqrt{eB})$$
$$= \frac{\alpha}{\pi}[\gamma_E+\ln(|x|\sqrt{eB}/2)]+o(\alpha^2/\pi^2), \quad (5.11)$$

where we used the fact that $eB = 2\alpha\rho$. Here, $K_0(y)$ is the modified Bessel function and γ_E is the Euler constant. Consequently, we get

$$\Psi_g = C\prod_{r>s}|z_r-z_s|^{\alpha/\pi}, \quad (5.12)$$

with C being a constant. This is to be compared with the wave function (5.5).

We comment on the exponential factor in (5.5), which may be rewritten as

$$\prod_r \exp\Big[-\frac{eB}{4}|x_r|^2\Big]$$
$$= \exp\Big[-\frac{eB}{4}\Big(\sum_r \frac{x_r}{\sqrt{N}}\Big)^2 - \frac{eB}{4N}\sum_{r>s}|x_r-x_s|^2\Big], \quad (5.13)$$

where we have separated the coordinate of the center of mass from those of the relative motions. Because of the translational invariance in our mean-field approximation, our wave function does not involve the term depending on the center-of-mass coordinate. Furthermore, the remaining terms $\frac{eB}{4N}|x_r-x_s|$ do not appear in the large limit of N. Consequently, we have found that the ground-state wave function agrees precisely with the Laughlin wave function in the limit of the vanishing coupling constant. The correction term in the first order of e^2 is found in Ref. 13.

VI. TOPOLOGICAL VORTEX SOLITONS

We have shown that at the magic filling factor the classical ground state is given uniquely by the constant solution (3.12) in the presence of the Coulomb interaction. It represents a condensed phase of electrons bound to fluxes. In the condensed phase of the bosonized electrons, it is expected that there are vortex solitons carrying the statistical flux. A vortex sitting at the origin of the coordinate is characterized by the phase of the bosonized electron field asymptotically as

$$\psi \simeq \sqrt{\rho}e^{in\theta} \qquad (r\to\infty), \quad (6.1)$$

where n is an integer and (r,θ) is the polar coordinate.

Because of the constraint equation (2.10) or

$$\int d^2x\,\psi^\dagger\psi = \frac{1}{2\alpha}\oint dx^k a_k, \quad (6.2)$$

there is a relation between the statistical charge Q and the statistical flux Φ of the vortex soliton. In our normalization the charge density and the number density are the same. These quantities are trivially calculated as

$$Q = \frac{1}{2\alpha}\Phi = \frac{\pi}{\alpha}n, \quad (6.3)$$

where

$$Q = \int d^2x(\psi^\dagger\psi-\rho) \quad (6.4)$$

and

$$\Phi = \oint dx^k(a_k - eA_k). \quad (6.5)$$

Here, we have defined Q and Φ by subtracting the ground-state contributions, $\rho = (e/2\alpha)\epsilon_{ij}\partial_i A_j$.

The electric charges of these vortices are given by

$$-eQ = -\frac{\pi}{\alpha}en, \quad (6.6)$$

since a vortex is made of Q of bosonized electrons; recall that electrons carry the charge $-e$. The mass of the vortex can be determined[20,23] by calculating the kinetic energy of the vortex soliton. We may also determine it by examining a time-dependent solution describing a vortex soliton making cyclotron motion.[24] In any of these analyses we get

$$M = m|Q|. \quad (6.7)$$

The interpretation of this formula is simple: since a vortex is made up of Q of the bosonized electron, its mass is the sum of those of each constituent. It is notable that the charge-mass ratio of the vortex soliton is the same as that of the bosonized electron: $|eQ/M| = e/m$.

Vortex solitons may be identified with Laughlin's quasiparticles ($n=1$) or quasiholes ($n=-1$). They are collective excitations of electrons in the lowest Landau level. To see this, it is enough to show that there are vortex solutions to the self-dual equation (3.10). It is curious but a fact that there are only antivortices ($n<0$) as classical soliton solutions to the self-dual equation, as we shall soon see. Although there are vortex solutions ($n>0$) to the Euler-Lagrangian equations of the system (3.6), they are not self-dual;[20] in this case the creation energy of the vortex is of order of the cyclotron energy ω_c, and hence it cannot be a collective mode of the electrons in the lowest Landau level.

The existence of self-dual vortex solutions can be analyzed as follows (see details in Ref. 20). We parametrize the bosonized electron field as

$$\psi = \sqrt{\rho}e^{u(r)+in\theta}. \quad (6.8)$$

Substituting this ansatz into the self-duality equation (3.10), we find that

$$a_k - eA_k = n\partial_k\theta - \epsilon_{kl}\partial_l u(r). \quad (6.9)$$

We then substitute (6.8) and (6.9) into the constraint equation (2.10):

$$\frac{d^2u}{dz^2} = -\frac{1}{z}\frac{du}{dz} - \frac{\partial U}{\partial u}, \tag{6.10}$$

where

$$U = 2u - e^{2u}, \tag{6.11}$$

and z is the rescaled radius defined by $r = z/\sqrt{\alpha\rho} = \sqrt{2}z\ell_B$.

The boundary condition at the vortex center reads

$$\lim_{z\to 0} u(z) = -\infty, \tag{6.12a}$$

$$\lim_{z\to 0} u'(z) = -\frac{n}{z}, \tag{6.12b}$$

for $n \neq 0$, while at infinity it is

$$\lim_{z\to\infty} u(z) = 0. \tag{6.13}$$

The conditions (6.12a) and (6.12b) follow from the requirements $\psi \to 0$ and $a_k \to eA_k$, respectively, at the vortex center ($r = 0$) in order to avoid the multivaluedness and the singularity at that point. The condition (6.13) is necessary since the vortex configuration should approach the classical ground state asymptotically.

Near the vortex center $z = 0$, approximate solutions are easily found:

$$u(z) \sim -\frac{z^2}{2} - n\ln\frac{z}{z_0}. \tag{6.14}$$

Because of the boundary conditions (6.12a) it is necessary that $n < 0$. Hence, only antivortices (quasiholes) are possible. It can be proved[20] that there are indeed antivortex solutions. We have performed a numerical computation of antivortices with $n = -1$ and -2. The method reads as follows. We integrate Eq. (6.10) by assuming an appropriate initial data z_0 in the asymptotic solution (6.14) at the vortex core, and examine whether or not an obtained solution satisfies the boundary condition (6.13) at infinity. We repeat this process by changing the initial data. We find that $z_0 = 0.828\,791$ for $n = -1$ and $z_0 = 1.036\,66$ for $n = -2$. See Fig. 1.

A vortex soliton is an extended object whose size is $z \approx 1$ or $r \approx \ell_B$, the magnetic length. Because the vortex solution approaches very rapidly its background constant outside of its size, it is expected that there is a solution containing several of these antivortices which is approximately given by their superpositions. This can be proved mathematically.[25] The positions of the antivortices are free parameters and they become zero-energy modes of the classical solution.

Until now we have not included the Coulomb interaction. It is possible but very difficult to solve vortex solutions in the presence of the Coulomb interaction. However, since the Coulomb interaction is very small with respect to the kinetic energy (i.e., $\frac{1}{2}\omega_c$ for each constituent electrons), we may treat it as a perturbation. Then, we may estimate the Coulomb energy of vortices simply by using the above self-dual vortex solutions:

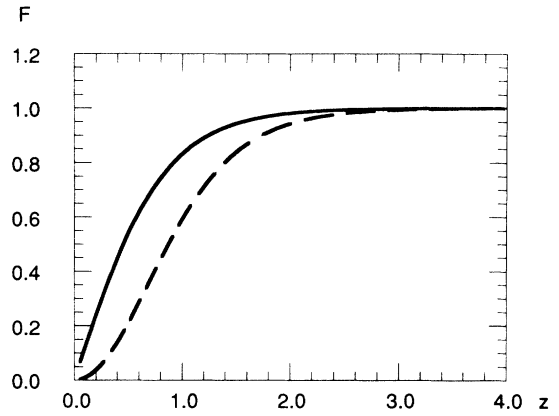

FIG. 1. Numerical solutions for topological vortices with $n = -1$ (solid line) and $n = -2$ (dashed line).

$$\langle\mathcal{V}\rangle = C\frac{e^2}{\epsilon\ell_B}, \tag{6.15}$$

where

$$C = \frac{\sqrt{2}}{4\alpha^2}\int\int d^2z_1 d^2z_2\ \{F^2(z_1) - 1\} \times\frac{1}{|z_1 - z_2|}\ \{F^2(z_2) - 1\}, \tag{6.16}$$

with $F(z) = e^{u(z)}$. By a numerical calculation we find that $C_{-1} \simeq 0.044$ for the antivortex with $n = -1$ and $C_{-2} \simeq 0.14$ for the antivortex with $n = -2$, where we have chosen $\alpha/\pi = 3$ for definiteness (i.e., $\nu = \frac{1}{3}$). These Coulomb energies are the creation energies of the antivortices. Since $C_{-2} > 2C_{-1}$, it is concluded that two antivortices with $n = -1$ are much more easily excited than a single antivortex with $n = -2$. Hence, in what follows, we only consider an antivortex with $n = -1$.

We have shown the existence of antivortices (quasiholes) as solutions to the self-dual equation. Now, we consider the excitation energy (gap energy) due to these antivortices. Suppose that the ground state $|g\rangle$ contains N electrons. We remove one electron from this Hall fluid. Then there appear k antivortices since each vortex carries the electric charge $-e/k$, where $k = \alpha/\pi$. The energy of the state is given by

$$(\tfrac{1}{2}\omega_c + \Delta E)(N-1) + k\langle\mathcal{V}\rangle,$$

where the energy due to the Coulomb interaction between different vortices is neglected. Let us then dope one electron back to the system. The doped electron will bound to the antivortices by the Coulomb interaction and make an excited state, or it will annihilate the antivortices and recover the ground state $|g\rangle$. Therefore, the excited state consists of one electron and k antivortices, with the energy being

$$E' \equiv (\tfrac{1}{2}\omega_c + \Delta E)N + k\langle\mathcal{V}\rangle + E_B, \tag{6.17}$$

where E_B is the bound energy of one electron and k antivortices. We estimate the gap energy as

$$E_{\rm gap} \equiv E' - (\tfrac{1}{2}\omega_c + \Delta E)N$$
$$= k\langle\mathcal{V}\rangle + E_B \tag{6.18a}$$
$$\simeq 3C_{-1}\frac{e^2}{\epsilon\ell_B} \simeq 0.13\frac{e^2}{\epsilon\ell_B} \tag{6.18b}$$

at $k = 3$ ($\nu = \frac{1}{3}$) by neglecting this binding energy. It should be recalled that the gap energy has been calculated as $E_{\rm gap} \simeq 0.097(e^2/\epsilon\ell_B)$, which is obtained by Monte Carlo methods based on Lauglin's wave function.[26,27]

As we have stated, there are no vortex soliton solutions in the self-dual equation. However, we conjecture that a vortexlike object is formed as a bound state of one electron and $k-1$ antivortices at the filling factor $\nu = 1/k$. The object has the electric charge $-e/k$ and one unit of the CS magnetic flux. Thus, we may identify a vortex excitation. Then its energy $\langle\mathcal{V}\rangle_v$ is given

$$\langle\mathcal{V}\rangle_v = (k-1)\langle\mathcal{V}\rangle + E'_B, \tag{6.19}$$

where E'_B is the binding energy between one electron and $k-1$ antivortices.

When we add one electron to the Hall state at $\nu = 1/k$, k such vortices appear. On the other hand, when we remove one electron, k antivortices appear. The chemical potentials are given by

$$\mu_+ = \tfrac{1}{2}\omega_c + \Delta E + k\langle\mathcal{V}\rangle_v, \tag{6.20a}$$
$$\mu_- = -\tfrac{1}{2}\omega_c - \Delta E + k\langle\mathcal{V}\rangle. \tag{6.20b}$$

Hence, the discontinuity of the chemical potential is given by

$$k(\langle\mathcal{V}\rangle_v - \langle\mathcal{V}\rangle) = (k-2)\langle\mathcal{V}\rangle + E'_B, \tag{6.21}$$

which is $(k-2)/k$ times of the gap energy $E_{\rm gap}$ when the binding energy is neglected.

We have shown that the gap energy at $\nu = 1/k$ is given by (6.18a), i.e., $E_{\rm gap} = k\langle\mathcal{V}\rangle + E_B$. This is the Coulomb energy of k antivortices and the binding energy between one electron and k antivortices. Equivalently, this can be regarded as a binding energy of a vortex and an antivortex, where a vortex is composed of one electron and $k-1$ antivortices. Therefore, the gap energy is physically the same as the so-called magnetoroton energy.[28]

VII. FIELD THEORY OF VORTEX SOLITONS

We have shown that there are topological excitations of antivortices on the FQH state at $\nu = \pi/\alpha$. When one electron is removed from the ground state, α/π antivortices must be created to form a state with $N-1$ electrons. As more electrons are removed, more antivortices will be created. The ground state at the vicinity of the filling factor $\nu = \pi/\alpha$ must be described by such an ensemble of the vortices. In order to minimize the Coulomb energy they would form a Wigner crystal, as we have discussed in a separate paper.[14]

When a sufficient number of antivortices are created as the filling factor decreases, the antivortices condense. We then obtain a new condensed phase of electrons and antivortices. In order to describe such a phase, it is convenient to construct an effective field theory of antivortices. This is the main topic of this section. The present theory may be considered as a nonrelativistic version of the formalism given in Ref. 10. In the next section, using the effective field theory of vortices, we discuss the hierarchy structure of the FQH states.

We would like to construct a local field theory of vortex solitons with vorticity $n = -1$. Although vortex solitons are extended objects, we take the pointlike limit. Then their local field theory describes correctly the effects whose scale is larger than the scale of the vortex soliton. The scale is given by the magnetic length and is independent of α.

A localized vortex is a flux concentrated in a small domain. Such a flux is easily introduced by considering a singular gauge transformation such that

$$\psi \to e^{if}\psi, \qquad a_\mu \to a_\mu + \partial_\mu f, \tag{7.1}$$

where $f(x) = -\sum_{r=1}^{N_v}\theta(x - z_r)$, and $\theta(x - z_r)$ is the azimuthal angle defined by (2.2). We get

$$\epsilon_{ij}\partial_i a_j \to \epsilon_{ij}\partial_i a_j - 2\pi\rho_v(x), \tag{7.2}$$

where

$$\rho_v(x) = \sum_{r=1}^{N_v}\delta^2(x - z_r) \tag{7.3}$$

describes a set of local vortices sitting at $x^k = z_r^k(t)$ in the two-dimensional space at time t, where $r = 1, 2, \ldots, N_v$, N_v = total number of vortices.

Performing this singular gauge transformation to the Lagrangian density (3.2), we obtain

$$\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}^{\rm vortex}, \tag{7.4}$$

with

$$\Delta\mathcal{L}^{\rm vortex} = \frac{\pi}{\alpha}a_\mu K^\mu + \widehat{\alpha}_\phi G, \tag{7.5}$$

where $\widehat{\alpha}_\phi$ is given by the reciprocal relation in terms of α,

$$\widehat{\alpha}_\phi = -\frac{\pi^2}{\alpha}. \tag{7.6}$$

Here,

$$K^\mu = (1/2\pi)\sum_{r=1}^{N_v}\epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta(x - z_r)$$
$$= \sum_{r=1}^{N_v}\dot{z}_r^\mu\delta^2(x - z_r), \tag{7.7}$$

which represents world lines of the local vortices; $z^\mu = (t, z^k)$. On the other hand,

$$G = (1/4\pi^2)\sum_{r,s}\epsilon^{\mu\nu\lambda}\partial_\mu\theta(x - z_r)\partial_\nu\partial_\lambda\theta(x - z_s)$$
$$= (1/2\pi)\sum_s K^\mu\partial_\mu\theta(x - z_s). \tag{7.8}$$

This quantity is not well defined when two vortices coin-

cide, i.e., for $r = s$. In Appendix A it is argued that this coincidence can be neglected, and that the quantity $\widehat{\alpha}_\phi$ is only defined mod2π. This $\widehat{\alpha}_\phi$ represents the statistics of the Laughlin quasiparticles. Therefore, we can replace $\widehat{\alpha}_\phi$ in formula (7.6) with α_ϕ which obeys the generalized reciplocal relation,[10]

$$\alpha_\phi = -\frac{\pi^2}{\alpha} - 2\pi p, \tag{7.9}$$

with p being an integer.

Integrating over the two-dimensional space we get

$$\int d^2x\, \Delta\mathcal{L}^{\text{vortex}} = \frac{\pi}{\alpha}\sum_{r=1}^{N_v} a_\mu \frac{dz_r^\mu}{dt} + \frac{\alpha_\phi}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r - z_s), \tag{7.10}$$

which describes how local vortices interact with the background CS field a_k and among themselves.

So far we have treated the vortex solitons as point particles, and have analyzed their interaction terms, i.e., potential terms. In order to find their kinetic terms we have to carefully analyze so-called collective coordinates of the solitons.[29] But it is natural to guess that the terms are given by $\frac{1}{2}M\dot{z}^2$ in our nonrelativistic formulation, where M is the mass of the soliton, as given by (6.7). Hereafter, we assume this kinetic term.

Therefore, the particle mechanics of local vortices is described by

$$\begin{aligned}\mathcal{L}^{\text{vortex}} &= \sum_{r=1}^{N_v}\frac{M}{2}\Big(\frac{dz_r^k}{dt}\Big)^2 + \int d^2x\,\Delta\mathcal{L} \\ &= \sum_{r=1}^{N_v}\Big[\frac{M}{2}\Big(\frac{dz_r^k}{dt}\Big)^2 + \frac{\pi}{\alpha}a_k\frac{dz_r^k}{dt} + \frac{\pi}{\alpha}a_0(z_r)\Big] \\ &\quad + \frac{\alpha_\phi}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r - z_s),\end{aligned} \tag{7.11}$$

where $\theta(z_r - z_s)$ is the azimuthal angle (2.2) between the rth and sth vortices. The total Lagrangian is given by $\mathcal{L} + \mathcal{L}^{\text{vortex}}$ with (3.2) and (7.11). Taking the variation δa_0 we obtain a constraint equation:

$$\psi^\dagger\psi + \frac{\pi}{\alpha}\rho_v = \frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j, \tag{7.12}$$

which determines a_j. Here, we assume that ψ is simply a classical field of the bosonized electron, not a quantum operator. After second quantization of the vortices we regard ψ as the quantum operator.

When the particle-mechanical Lagrangian of vortices is known, its second quantization proceeds precisely as we described in Sec. II. First, the particle-mechanical Hamiltonian reads

$$H^{\text{vortex}} = \frac{1}{2M}\sum_{r=1}^{N_v}\Big(p_r^k - \frac{\pi}{\alpha}a_k - c_k(z_r)\Big)^2, \tag{7.13}$$

together with the constraint condition

$$\frac{1}{2\alpha_\phi}\epsilon_{jk}\partial_j c_k = \rho_v, \tag{7.14}$$

which follows from

$$c_j(x) = \frac{\alpha_\phi}{\pi}\sum_s^{N_v}\partial_j\theta(x - z_s). \tag{7.15}$$

In second quantizing the system of vortices, it is necessary to introduce a field operator ϕ which annihilates vortices with $n = -1$. Then, the second-quantized Hamiltonian density is

$$\mathcal{H}^{\text{vortex}} = \frac{1}{2M}|D_k^{(1)}\phi|^2 + \frac{g_\phi}{2} : (\phi^\dagger\phi)^2 : + h : \psi^\dagger\psi\phi^\dagger\phi : \tag{7.16}$$

with (2.21) and

$$iD_k^{(1)} = i\partial_k + c_k + \frac{\pi}{\alpha}a_k. \tag{7.17}$$

Here, a_k and c_k are determined by the constraint equations

$$\psi^\dagger\psi + \frac{\pi}{\alpha}\phi^\dagger\phi = \frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j, \tag{7.18a}$$

$$\phi^\dagger\phi = \frac{1}{2\alpha_\phi}\epsilon_{ij}\partial_i c_j, \tag{7.18b}$$

which follow from (7.12) and (7.14) by the correspondence principle. When we set

$$b^k = c^k + \frac{\pi}{\alpha}a^k, \tag{7.19}$$

the total Hamiltonian $H + H^{\text{vortex}}$ has the standard form given in (2.19), and the constraint equations (7.18a) and (7.18b) read

$$\begin{aligned}\epsilon_{ij}\partial_i a_j &= 2\alpha\psi^\dagger\psi + 2\pi\phi^\dagger\phi, \\ \epsilon_{ij}\partial_i b_j &= 2\pi\psi^\dagger\psi - 4p\pi\phi^\dagger\phi.\end{aligned} \tag{7.20}$$

Comparing these expressions with (2.20), we find that the bosonized electrons have their own statistics parameter α, the vortices their own statistics parameter β, and they have the relative statistics parameter γ, where

$$\beta = -2p\pi, \qquad \gamma = \pi. \tag{7.21}$$

Hence, vortices are actually bosons. Nevertheless, the relative statistics between an electron and a vortex is fermionic.

The Hamiltonian density

$$\mathcal{H}_{\text{FQHE}}^{(1)} = \mathcal{H} + \mathcal{H}^{\text{vortex}} \tag{7.22}$$

with (3.1) and (7.16) describes the total system consisting of vortex solitons as well as bosonized electrons. Using the Bogomol'nyi decomposition (3.5) we rewrite it as

$$\begin{aligned}H_{\text{FQHE}}^{(1)} &= \int d^2x\,\mathcal{H}_{\text{FQHE}}^{(1)} \\ &= \int d^2x\Big[\frac{1}{2m}|(D_1 - iD_2)\psi|^2 + \tfrac{1}{2}\omega_c N + \mathcal{V}\Big] \\ &\quad + \int d^2x\Big[\frac{1}{2M}|(D_1^{(1)} - iD_2^{(1)})\phi|^2\Big].\end{aligned} \tag{7.23}$$

This Hamiltonian is very interesting. First, let us switch off the Coulomb interaction. Although our system contains both electrons and vortices apparently as independent degrees of freedom, this Hamiltonian implies that the ground-state energy of the system receives the contribution only from the kinetic energy of the electrons, i.e., $\frac{1}{2}\omega_c N$. Second, the Coulomb interaction acts directly only on electrons. Nevertheless, we can see that vortices also interact through the Coulomb interaction. These features are physically correct because vortices are really collective modes of electrons in the lowest Landau level.

The derivation of the Hamiltonian (7.23) is by no means rigorous, and it is a plausible one. We have included the kinetic term of the vortices by hand. We have included the interaction terms of ψ and ϕ as required to perform a consistent perturbation; see (7.16). It may well be that there are other interaction terms which may result, e.g., from the finite size of actual vortices. However, the form of the Hamiltonian is severely restricted by the condition that vortices are really collective modes of electrons in the lowest Landau level. Our effective Hamiltonian (7.23) satisfies this criterion as we have emphasized. In this sense it is plausible and it seems to be practically correct.

VIII. HIERARCHY OF THE FQH STATES

In the preceding section, we derived the Hamiltonian density (7.23) describing the system of electrons and vortices. We can repeat the same procedure for this Hamiltonian $\mathcal{H}^{(1)}_{\rm FQHE}$ as we did for $\mathcal{H}$ in order to obtain the ground state. Thus, we search for the classical ground state which minimizes the energy. When the Coulomb interaction is neglected, it is a solution of the self-dual equations

$$(D_1 - iD_2)\psi = 0, \qquad (D_1^{(1)} - iD_2^{(1)})\phi = 0. \tag{8.1}$$

In general, there are nonuniform solutions containing zero-energy modes associated with the translational invariance. In addition to these nonuniform solutions, there exist two constant solutions, when and only when the filling factor takes two special values such that $\nu = \nu^{(0)} \equiv (\pi/\alpha)$ or $\nu = \nu^{(1)}$ with

$$\nu^{(1)} = \frac{\pi}{\alpha}\Big(1 + \frac{\pi^2}{\alpha\alpha_\phi}\Big) = \frac{1}{k + \dfrac{1}{2p}}. \tag{8.2}$$

Here, $k \equiv \alpha/\pi$, and p is the parameter which appears in the generalized reciprocal relation (6.5). Both energies of these two states are given by $E = \frac{1}{2}\omega_c N$, as should be the case.

In the presence of the Coulomb interaction, all the nonuniform solutions acquire positive-definite Coulomb energy. The constant solutions become the true ground states with the minimum Coulomb energy when they exist. The solution at $\nu = \nu^{(0)}$ corresponds to the condensed phase of only bosonized electrons, while the solution at $\nu = \nu^{(1)}$ corresponds to the condensed phase of both bosonized electrons and vortices, and is given by

$$\psi = \sqrt{\rho}, \qquad \phi = \sqrt{\frac{\rho}{2p}}, \tag{8.3}$$

$$a_k = -\frac{\alpha}{\pi}c_k = eA_k.$$

The fact that this classical ground state is rigid against small fluctuations can be argued as before.

We comment that the state at $\nu = \nu^{(1)}$ contains the vortices with the density $|\phi|^2 = \rho/2p$; namely, the vortex density is less than the electron density by the factor $1/2p$. As we have mentioned before, the vortex size is of the order of the magnetic length ℓ_B. Therefore, the vortices never overlap with each other. For instance, the filling of the vortices is $\frac{1}{7}$ when the filling of the electrons is $\frac{2}{7}$. Hence, our effective Hamiltonian of the pointlike vortices is consistent in this phase given by (8.3).

In the condensed phase of the ψ and ϕ quanta, there are again vortex solitons (daughter vortices) which are characterized by the phase of the field ϕ;

$$\psi \simeq \sqrt{\rho}, \qquad \phi \simeq e^{in\theta}\sqrt{\frac{\rho}{2p}}. \tag{8.4}$$

We can introduce local daughter vortices by considering a singular gauge transformation such that

$$\phi \to e^{if}\phi, \qquad c_\mu \to c_\mu + \partial_\mu f, \tag{8.5}$$

with all the other fields unchanged; here, $f(x) = -\sum_r \theta(x - z_r)$. The constraint condition (7.18a) is not modified but (7.18b) is modified as

$$\phi^\dagger\phi = \frac{1}{2\alpha_\phi}\epsilon_{ij}\partial_i c_j - \frac{\pi}{\alpha_\phi}\sum_r \delta^2(x - z_r). \tag{8.6}$$

Substituting this into (7.18a) we get

$$\psi^\dagger\psi = \frac{1}{2\alpha}\epsilon_{ij}\partial_i a_j - \frac{\pi}{2\alpha\alpha_\phi}\epsilon_{ij}\partial_i c_j + \frac{\pi^2}{\alpha\alpha_\phi}\sum_r \delta^2(x - z_r), \tag{8.7}$$

from which it follows that a local daughter vortex is made of $Q^{(2)}$ of the bosonized electrons, where

$$Q^{(2)} = \frac{\pi^2}{\alpha\alpha_\phi} = \frac{-1}{2pk + 1}. \tag{8.8}$$

The mass $M^{(2)}$ and the electric charge $e^{(2)}$ of the local daughter vortex is given by $M^{(2)} = m|Q^{(2)}|$ and $e^{(2)} = eQ^{(2)}$. We have put index (2) to the quantities associated with the daughter vortices.

Similarly as before, the field theory of the daughter vortices may be constructed by introducing a new field operator $\phi^{(2)}$ annihilating local daughter vortices. The Hamilton density $\mathcal{H}^{(2)\rm vortex}$ is constructed analogously together with a new CS gauge field $c_k^{(2)}$ describing their anyonic character, and their statistics parameter $\alpha^{(2)}$ is given by the generalized reciprocal relation in terms of α_ϕ and a new integer $p^{(2)}$. This process proceeds as much

as we wish.

In general, the Hamiltonian density

$$\mathcal{H}^{(n)}_{\rm FQHE} = \mathcal{H} + \sum_{i=1}^{n} \mathcal{H}^{(i)\rm vortex} \tag{8.9}$$

describes the fractional quantum Hall state at the nth level. Here,

$$\mathcal{H}^{(i)\rm vortex} = \frac{1}{2M^{(i)}} |(D_1^{(i)} - iD_2^{(i)})\phi^{(i)}|^2, \tag{8.10}$$

where

$$iD_\mu^{(i)} = i\partial_\mu + c_\mu^{(i)} + \frac{\pi}{\alpha^{(i-1)}} c_\mu^{(i-1)}, \tag{8.11}$$

and $\alpha^{(0)} \equiv \alpha$, $c_\mu^{(0)} \equiv a_\mu$, $\alpha^{(1)} \equiv \alpha_\phi$, $c_\mu^{(1)} \equiv c_\mu$, $\phi^{(1)} \equiv \phi$. The CS fields $c_\mu^{(i)}$ are determined by the constraint equations

$$\phi^{(i-1)\dagger}\phi^{(i-1)} + \frac{\pi}{\alpha^{(i-1)}} \phi^{(i)\dagger}\phi^{(i)} = \frac{1}{2\alpha^{(i-1)}} \epsilon_{ij}\partial_i c_j^{(i-1)}, \tag{8.12a}$$

for $1 \le i \le n$, and

$$\phi^{(n)\dagger}\phi^{(n)} = \frac{1}{2\alpha^{(n)}} \epsilon_{ij}\partial_i c_j^{(n)}. \tag{8.12b}$$

The vortex solitons are described by field operators $\phi^{(i)}$ together with the statistics parameter $\alpha^{(i)}$, which is determined by the generalized reciprocal relation in terms of $\alpha^{(i-1)}$ and an integer $p^{(i)}$ as

$$\alpha^{(i)} = -\frac{\pi^2}{\alpha^{(i-1)}} + 2\pi p^{(i)}. \tag{8.13}$$

The mass $M^{(i)}$ and the electric charge $e^{(i)}$ are given by $M^{(i)} = m|Q^{(i)}|$ and $e^{(i)} = eQ^{(i)}$, respectively, with

$$Q^{(i)} = \Big(\prod_{k=0}^{i-1} \frac{\pi}{\alpha^{(k)}}\Big) = \frac{1}{q^{(i)}}. \tag{8.14}$$

Here, $q^{(i)}$ is the odd denominator of the filling factor $\nu^{(i)} = p^{(i)}/q^{(i)}$. These properties follow since the vortex solitons in the problem consist of $Q^{(i)}$ bosonized electrons. These vortex solitons are identified as Laughlin's quasiparticles (or quasiholes) at the corresponding level of the hierarchy.[2]

Finally, we discuss on the ground state of the Hamiltonian $H^{(n)}_{\rm FQHE} = \int d^2x\, \mathcal{H}^{(n)}_{\rm FQHE}$. The ground states are nonuniform with a zero-energy mode for general values of the filling factor. However, the ground state exists uniquely and is represented by a constant solution when and only when the filling factor ν takes a particular value, i.e., $\nu = \nu^{(n)}$ with

$$\begin{aligned}\nu^{(n)} &= \frac{\pi}{\alpha}\Big[1 + \frac{\pi^2}{\alpha\alpha^{(1)}}\Big[1 + \frac{\pi^2}{\alpha^{(1)}\alpha^{(2)}}\Big[1 + \cdots \frac{\pi^2}{\alpha^{(n-1)}\alpha^{(n)}}\Big]\cdots\Big] \\ &= \cfrac{1}{k + \cfrac{1}{2p^{(1)} + \cfrac{1}{2p^{(2)} + \cfrac{1}{\ddots \cfrac{1}{2p^{(n)}}}}}}.\end{aligned} \tag{8.15}$$

We interpret the ground state as a condensed phase of all ψ and $\phi^{(i)}$ quanta. This hierarchy is precisely that of Haldane and Halperin.[2] Thus, our formalism gives effectively a field-theoretical realization of their hierarchy construction.

We also examine the linear response to the external electric field E_i. As we show in Appendix B, we find that the Hall conductance is given by $\sigma_{xy} = \nu^{(n)}(e^2/2\pi)$. Therefore, we may identify the above ground state with the FQH state at $\nu = \nu^{(n)}$.

IX. DISCUSSIONS

In this paper we have presented a field-theoretical formulation of the FQHE and its hierarchy structure. The FQH states are condensed phases of bosonized electrons and vortices which appear at appropriate filling factors. We have analyzed condensations of vortices based on an effective field theory of local vortex solitons.

Strictly speaking, our basic Hamiltonian (3.1) with the extra $:(\widehat{\psi}^\dagger\widehat{\psi})^2:$ term is correct up to the first order of α/π, since it has been derived up to this order. Nevertheless, our results are physically quite reasonable for any values of α/π, and in particular they will be applicable to the case of electrons where α/π is an odd integer.

We have shown that the electron system has the phase of quantum Hall liquid with the Coulomb energy (4.14) at $\nu = \pi/\alpha$. (The liquid becomes a superfluid at $\alpha = 0$ or $eB = 0$.) In our mean-field approximation the ground state is given by the constant solution. Then, by taking account of small fluctuations around it, we have derived the Laughlin wave function as the ground-state wave function in the presence of the Coulomb interaction.

On the other hand, in the vicinity of $\nu = \pi/\alpha$, the ground state contains vortex solitons. They are ex-

pected to form a Wigner crystal in order to minimize the Coulomb energy as far as the soliton density is small. The detailed analysis of such a possibility is given in a separate paper,[14] where the wave function of the Wigner crystal is found to coincide with that of the Laughlin quasiholes.

A typical feature of our effective theory of topological solitons, which explains the hierarchy of FQH states, is that it involves only operators associated with antivortices. This is so because there are no vortex solitons as classical solutions to the self-dual equation. This gives actually a constraint on the parameter p in the reciprocal relation (7.9), i.e., $p > 0$. Then, it is seen from (8.2) that the possible FQH states are at the filling factors $\frac{1}{3}$, $\frac{2}{7}$, etc. We are unable to find the states with the filling factors $\frac{2}{3}$, $\frac{5}{7}$, etc., in this scheme. However, when we take into account the particle-hole symmetry, these states are possible. Namely, let us regard the ψ field as a hole operator with its charge e opposite to the electron charge $-e$; its Hamiltonian is given by (3.6) with the replacement of $-e$ by e. The resultant system allows vortex solitons, and it is easy to see that condensations of these vortices occur at the hole filling factor $\nu_h = \frac{1}{3}, \frac{2}{7}$, etc. These states are the condensed states of bosonized electrons with the filling factor $\nu = 1 - \nu_h = \frac{2}{3}, \frac{5}{7}$, etc.

As we have just mentioned, there are no vortex solitons as solutions to the self-dual equation. Although there are vortex solutions in our Hamiltonian system, they are collective excitations of electrons not only in the lowest Landau level but also in higher Landau levels. This might imply that a state containing both vortices and antivortices is absent in the lowest Landau level. However, we expect that this is not the case. In this paper we have conjectured that a vortex would appear as a bound state of antivortices and an electron by the Coulomb interaction. For example, a vortex with its charge $-e/3$ may be composed of two antivortices with its charge $e/3$ and an additionally doped electron with its charge $-e$. In this context we should mention recent works due to Jain *et al.*,[30] where a vortex wave function has been constructed, for example, by putting $N-1$ electrons in the lowest Landau level and one electron in the second Landau level, and then by multiplying its wave function by the Jastrow factor $\prod_{i>k}(z_i - z_j)^2$. This suggests that the wave function of our vortex soliton might correspond to Jain's wave function because the vortex soliton inevitably involves electrons in higher Landau levels. It is worthwhile to examine such a possibility to make clearer the issue of vortices in our formalism, which we would like to pursue in a future work.

Once a vortex with $n = 1$ is obtained, it is easy to generalize our effective field theory to include both antivortices ($n = -1$) and vortices ($n = 1$), by introducing two types of operators $\phi_\pm$. The kinetic part of the resultant Lagrangian looks like Eq. (30) of Ref. 10.

Our field-theoretical formalism of anyons can also be applied to a double-layer electron system as in Ref. 31, where an even-denominator FQHE has been observed recently.[32] We believe that our formalism reveals another way of understanding the FQHE and related phenomena.

ACKNOWLEDGMENTS

One of us (A.I.) appreciates the hospitality of the staff members at the Institute for Nuclear Study, University of Tokyo, where part of this work was done.

APPENDIX A

Let us consider a canonical ensemble of anyons at a finite temperature. The partition function is given by

$$\begin{aligned} Z_N &= \mathrm{tr}(e^{-\beta H})_N \\ &= \frac{1}{N!}\sum_{\sigma\epsilon S_N} \langle z_1, z_2, \ldots, z_N| \\ &\quad \times e^{-\int_0^\beta dtL}|z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(N)}\rangle, \end{aligned} \tag{A1}$$

with the Lagrangian L being defined by (2.1). Here, S_N is the permutation group, and the summation runs over all possible permutations, implying that the world lines are *braided* in all possible ways. The initial state and the final state consist of the same set of anyons; the kth anyon starts from the position z_k and ends at $z_{\sigma(k)}$. This requirement is crucial in what follows. This is the reason why we formulate the theory at a finite temperature. Now,

$$\int_0^\beta dtL = \frac{m}{2}\sum\int_0^\beta dt\Big(\frac{dz_r^k}{dt}\Big)^2 + i\frac{\alpha}{\pi}\sum_{s<r}\Delta\theta_{rs}, \tag{A2}$$

where $\Delta\theta_{rs}$ is the change of the azimuthal angle between the rth and sth anyons:

$$\Delta\theta_{rs} = \theta(z_{\sigma(r)} - z_{\sigma(s)}) - \theta(z_r - z_s). \tag{A3}$$

For instance, if the rth and sth anyons simply exchange their positions, i.e., $\sigma(r) = s$ and $\sigma(s) = r$, then we get $\Delta\theta_{rs} = \pi\mathrm{mod}2\pi$. In general, it is easy to see for any kind of permutations we find that $\sum_{r<s}\Delta\theta_{rs} = 0$ or $\pi\mathrm{mod}2\pi$.

Therefore, if $\alpha = 0\mathrm{mod}2\pi$, the α dependence entirely disappears from the partition function. Namely, two theories with α and $\alpha + 2\pi p$ with an integer p define the same physics.

When we derive the statistics term G by performing a singular gauge transformation (7.1), such a term as $\theta(z_r - z_r)$ appears formally. This term is not well defined, and we need a regularization. Let us recall[10] that when a singular gauge transformation creates closed loops of fluxes, representing creation of vortex-antivortex pairs and annihilation of the same pairs in a relativistic setting, the statistics term G defines nothing but the Gauss linking number. Here, the same problem is encountered; it is known as the problem of self-linking. A well-established method[33] for its regularization is provided by a *framing*, i.e., a "line-splitting" regularization. Instead of a single string labeled r, we replace it with a set of two strings labeled r and r' with an infinitesimal separation ϵ. In our present problem, it is always possible to make such a framing that a new string is defined by $z_{r'}^k = z_r^k + \epsilon^k$ with a constant vector ϵ for any anyon

r, and define that $\theta(z_r - z_r) = \lim_{\epsilon\to 0}\theta(z_{r'} - z_r)$. Then, $(\alpha/2\pi)\sum_r \theta(z_r - z_r)$ is just a constant independent of trajectories. Such a term is irrelevant, and we may set $\theta(z_r - z_r) = 0$ without loss of generality.

In this way, the periodicity in the statistics parameter α is rather obvious in particle mechanics. However, this is not the case in the second-quantized formalism, i.e., in Eq. (2.17). It will only be realized nonperturbatively. In the semiclassical approximation we are using for a description of the FQHE, we have to choose a particular period which suits the problem. For instance, the FQH state is described simply by a classical ground state (3.12) when we choose a period such that $\nu = \alpha/\pi$ is satisfied. If we choose another period, the same physics must follow but in a very complicated manner. At the moment, however, we cannot prove this periodicity explicitly in the second-quantized formalism.

APPENDIX B

In this appendix, examining the linear response to the external electric field E_j, we derive the Hall current on the FQH state at $\nu = \nu^{(n)}$ in a functional integral formalism. The induced current is given by

$$\begin{aligned}\langle J_j\rangle &= Z^{-1}\int D\mu \frac{\delta}{\delta A_j}\exp\Big(i\int d^3x\,\mathcal{L}^{(n)}_{\rm FQHE}\Big)\\ &= -\frac{i}{2m}\langle\psi^\dagger \overleftrightarrow{\rm D}_j\,\psi\rangle,\end{aligned}\tag{B1}$$

where $D\mu$ is an appropriate integration measure of fields ψ, $\phi^{(i)}$, etc.; the external gauge potential A_0 has been introduced by the minimal coupling only with the bosonized electron field ψ. Our gauge choice is such that $E_i = -\partial_i A_0$ and $\partial_1 A_2 - \partial_2 A_1 = B$. The $\mathcal{L}^{(n)}_{\rm FQHE}$ is defined by

$$\mathcal{L}^{(n)}_{\rm FQHE} = \mathcal{L} + \sum_{i=1}^{n}\mathcal{L}^{(i)\rm vortex},\tag{B2}$$

where $\mathcal{L}^{(i)\rm vortex}$ is constructed from the Hamiltonian (8.10) and the constraint conditions (8.12a) and (8.12b) as

$$\begin{aligned}\mathcal{L}^{(i)\rm vortex} &= \phi^{(i)\dagger} iD_0^{(i)}\phi^{(i)} - \frac{1}{2M^{(i)}}|(D_1^{(i)} - iD_2^{(i)})\phi^{(i)}|^2\\ &\quad -\frac{1}{4\alpha^{(i)}}\epsilon^{\mu\nu\lambda}c_\mu^{(i)}\partial_\nu c_\lambda^{(i)}.\end{aligned}\tag{B3}$$

The following relations hold, corresponding to the Euler-Lagrange equations of (B2):

$$\begin{aligned}-\frac{i}{2m}\langle\psi^\dagger \overleftrightarrow{\rm D}_j\,\psi\rangle + \frac{\pi}{\alpha}F_j^{(1)} &= \frac{1}{2\alpha}\epsilon_{jk}\langle c_{k0}^{(0)}\rangle,\\ -F_j^{(1)} + \frac{\pi}{\alpha^{(1)}}F_j^{(2)} &= \frac{1}{2\alpha^{(1)}}\epsilon_{jk}\langle c_{k0}^{(1)}\rangle,\\ -F_j^{(2)} + \frac{\pi}{\alpha^{(2)}}F_j^{(3)} &= \frac{1}{2\alpha^{(2)}}\epsilon_{jk}\langle c_{k0}^{(2)}\rangle,\end{aligned}\tag{B4}$$

and so on, where $c_{k0}^{(i)} = \partial_k c_0^{(i)} - \partial_0 c_k^{(i)}$ and

$$F_j^{(i)} = \frac{i}{2M^{(i)}}\langle\phi^{(i)\dagger}\overleftrightarrow{D}_j^{(i)}\,\phi^{(i)}\rangle.\tag{B5}$$

From these equations it follows that

$$\langle J_j\rangle = \frac{1}{2\pi}\epsilon_{jk}\sum_{i=0}^{n}\frac{\pi}{\alpha^{(0)}}\cdots\frac{\pi}{\alpha^{(i)}}\langle c_{k0}^{(i)}\rangle,\tag{B6}$$

where

$$\langle c_{k0}^{(i)}\rangle = Z^{-1}\int D\mu\, c_{k0}^{(i)}\exp\Big(i\int d^3x\,\mathcal{L}^{(n)}_{\rm FQHE}\Big).\tag{B7}$$

Making changes of variables,

$$c_0^{(i)} \to c_0^{(i)} + \frac{\pi}{\alpha^{(0)}}\cdots\frac{\pi}{\alpha^{(i)}}eA_0,\tag{B8}$$

we find that

$$\langle c_{k0}^{(i)}\rangle = \frac{\pi}{\alpha^{(0)}}\cdots\frac{\pi}{\alpha^{(i-1)}}eA_{k0} + \langle c_{k0}^{(i)}\rangle',\tag{B9}$$

where the average $\langle\cdots\rangle'$ is taken together with the transformed Lagrangian by way of (B8). It is easy to see that the constant terms and all the terms linear in A_0 vanish in $\langle c_{k0}^{(i)}\rangle'$. Consequently, we find that

$$\begin{aligned}\langle J_j\rangle &= \frac{e}{2\pi}\epsilon_{jk}\sum_{i=0}^{n}\Big(\frac{\pi}{\alpha^{(0)}}\cdots\frac{\pi}{\alpha^{(i-1)}}\Big)^2\frac{\pi}{\alpha^{(i)}}E_k,\\ &= \frac{e}{2\pi}\nu^{(n)}\epsilon_{jk}E_k,\end{aligned}\tag{B10}$$

where $\nu^{(n)}$ has been defined in (8.15).

[1] See, e.g., *The Quantum Hall Effect*, 2nd ed., edited by S. Girvin and R. Prange (Springer-Verlag, New York, 1990), for a review.

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