Slave fermions, slave bosons, and semions from bosonization of the two-dimensional t-J model

J. Fröhlich
Theoretical Physics, Eidgenössische Technische Hochschule Hönggerberg, CH-8093 Zürich, Switzerland

P. A. Marchetti
Dipartimento di Fisica, "G. Galilei," Istituto Nazionale di Fisica Nucleare, Via F. Marzolo, 8, I-35131 Padova, Italy
(Received 6 April 1992)

We extend Abelian and non-Abelian bosonization formulas found in a previous paper, in combination with coherent-state methods, to present a systematic derivation of the slave-fermion, slave-boson, and semion representations of the two-dimensional t-J model in path-integral form. The slave-fermion and slave-boson representations are shown to arise from two different gauge fixing constraints introduced in the path-integral representation of the bosonized t-J model. For each representation we discuss the approximations leading to a mean-field theory. In the mean-field theory based on the semion representation, the holons are shown to be "Dirac semions."

I. INTRODUCTION

The cuprate oxides which led to the discovery of high-T_c superconductivity[1] have one structural characteristic which appears to play a key role in the phenomenon of superconductivity: the CuO_2 planes. It is widely argued that the two-dimensional t-J model captures the essential low-energy physics of such planes. Let us briefly review the main ideas (see Ref. 2 for details).

In the undoped materials, the formal Cu valence is $2+$ and the O valence is $2-$; i.e., there is a hole in the cupper $3d$ shell, and the $2p$ shell of the oxygen is completely filled. The hole primarily occupies the highest-energy $3d_{x^2-y^2}$ orbital, where a strong on-site Coulomb repulsion inhibits the presence of two electrons. The spin-$\frac{1}{2}$ moments on the Cu sites are antiferromagnetically ordered at low temperature (see Fig. 1).

Superconductivity appears when the materials are doped in such a way that the formal valence of the cupper ions is raised to $Cu^{2+\delta+}$, $\delta \simeq 0.2$. Because of hybridization between the $3d_{x^2-y^2}$ cupper orbitals and the $2p_x, 2p_y$ orbitals of the oxygen, it appears favorable for the holes introduced by doping to go into a combination of the O orbitals with the same symmetry of the central $3d_{x^2-y^2}$ orbital of the cupper ion and, furthermore, to form a spin singlet with the spin moment of the cupper.[3] A singlet of this sort in one CuO_4 has a sizable overlap with a neighboring CuO_4, since there is one O site in common (see Fig. 2). Therefore it has a relevant nearest-neighbor hopping term. The Hamiltonian of the t-J model describes the motion of this charged spin singlet in a background of antiferromagnetically ordered spin-$\frac{1}{2}$ moments and can be written as

$$H_{t\text{-}J} = P_G \left[\sum_{\langle ij \rangle} \sum_{\alpha} -t\psi^\dagger_{i\alpha}\psi_{j\alpha} + J\mathbf{S}_i \cdot \mathbf{S}_j \right] P_G \ , \tag{1.1}$$

where the sum over i runs over Cu sites, the sum over α runs over spin indices (spin up and spin down), and $\psi^\dagger_{i\alpha}$ is the creation operator for an electron in the $3d_{x^2-y^2}$ at site i. Furthermore, P_G is the Gutzwiller projection, eliminating double occupation, which is introduced to model the strong on-site Coulomb repulsion. The second term in (1.1) is the (antiferromagnetic) Heisenberg term, where

$$\mathbf{S}_i = \sum_{\alpha,\beta} \psi^\dagger_{i\alpha} \left[\frac{\boldsymbol{\sigma}}{2} \right]_{\alpha\beta} \psi_{i\beta} \tag{1.2}$$

(a summation over repeated spin indices will be understood from now on and sums over spin indices frequently omitted).

Anderson[4,5] suggested that the low-energy excitations of the t-J model, at low doping, are not holes (corresponding to the absence of electrons in the $d_{x^2-y^2}$ orbital at some site), carrying charge 1 and spin $\frac{1}{2}$, but *holons,*

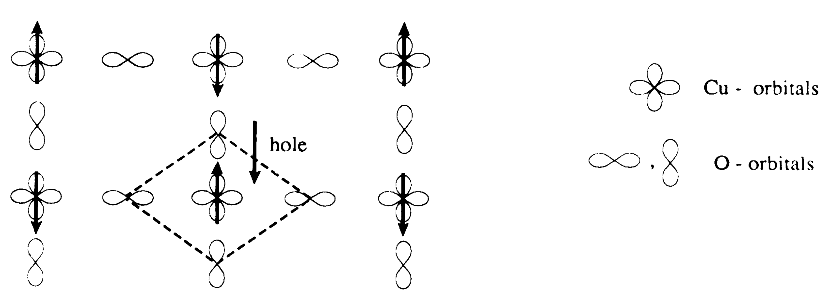

FIG. 1. $3d_{x^2-y^2}$ Cu orbitals and the $2p_x, 2p_y$ O orbitals in CuO_2 planes. The spin-$\frac{1}{2}$ moments of the Cu sites and of a hole introduced by doping are indicated. The dashed lines indicate hybridization.

©1992 The American Physical Society

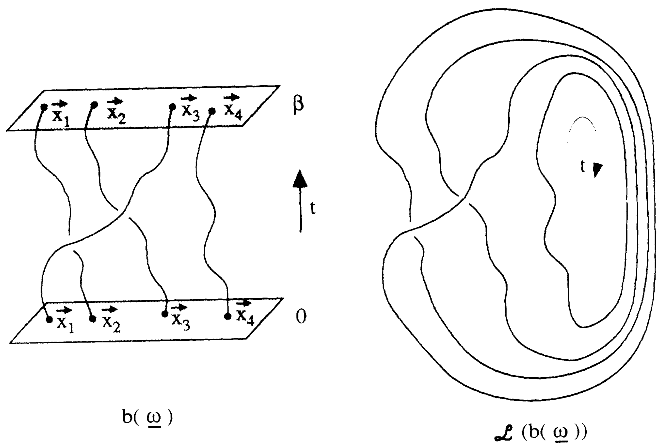

FIG. 2. Heavy lines describe the set of paths $\underline{\omega}$ for $N=4$; $\mathcal{L}(b(\underline{\omega}))$ is obtained identifying the $t=\beta$ and 0 planes.

charged and spinless, and *spinons,* carrying spin $\frac{1}{2}$, but neutral. To implement these ideas of spin-charge separation, one tries to use the following ansatz: One rewrites the hole operator as

$$\psi_{i\alpha}=\tilde{e}_i^{\dagger}s_{i\alpha}\ , \tag{1.3}$$

where $\tilde{e}_i$ is a charged spinless fermion operator representing the holon and $s_{i\alpha}$ is a spin-$\frac{1}{2}$ boson operator, representing the spinon. These operators must obey the constraint

$$\tilde{e}_i^{\dagger}\tilde{e}_i+s_{i\alpha}^{\dagger}s_{i\alpha}=1\ . \tag{1.4}$$

The heuristic idea underlying (1.4) goes as follows: At each site, either there is no d electron ($\tilde{e}_i^{\dagger}\tilde{e}_i=1$) or there is a spin up ($s_{i\uparrow}^{\dagger}s_{i\uparrow}=1$) or there is a spin down ($s_{i\downarrow}^{\dagger}s_{i\downarrow}=1$). This is the so-called *slave-fermion approach.*[6] Alternatively, one tries to decompose

$$\psi_{i\alpha}=e_i^{\dagger}\tilde{s}_{i\alpha}\ , \tag{1.5}$$

where e_i is a hard-core boson operator $[e_i^2=(e_i^{\dagger})^2=0]$, representing the holon, $\tilde{s}_{i\alpha}$ is a spin-$\frac{1}{2}$ neutral fermion operator, representing the spinon, and the constraint

$$e_i^{\dagger}e_i+\tilde{s}_{i\alpha}^{\dagger}\tilde{s}_{i\alpha}=1 \tag{1.6}$$

is imposed. This is the *slave-boson approach.*[5,7]

If one converts these ideas into path-integral language, one observes a mismatch of degrees of freedom: The hole (or electron) and spinon would be described by fields with four degrees of freedom, the holon by fields with two degrees of freedom, and the constraint eliminates only one degree of freedom. The heuristic idea to get rid of this mismatch is the following: Let us use capital letters to denote fields used in path integral and lowercase letters to denote the corresponding field operators. In the decompositions analogous to (1.3) and (1.5),

$$\begin{aligned}\Psi_{i\alpha}&=\tilde{E}_i^{*}S_{i\alpha}\ ,\\ \Psi_{i\alpha}&=E_i^{*}\tilde{S}_{i\alpha}\ ,\end{aligned} \tag{1.7}$$

there is an underlying gauge symmetry:

$$\begin{aligned}&\tilde{E}_i\to e^{i\Theta_i}\tilde{E}_i\ ,\quad S_{i\alpha}\to e^{i\Theta_i}S_{i\alpha}\ ,\\ &E_i\to e^{i\Theta_i}E_i\ ,\quad \tilde{S}_{i\alpha}\to e^{i\Theta_i}\tilde{S}_{i\alpha}\ .\end{aligned} \tag{1.8}$$

In two dimensions an Abelian gauge field has only one degree of freedom. Therefore one can introduce a gauge field, the "RVB-gauge field,"[8,9] coupled to holon and spinon fields, and consider as physical only gauge-invariant quantities. This procedure should eliminate the unwanted degrees of freedom.

In this paper we show how to implement these ideas *in an exact form* (i.e., without approximations) in the t-J model, and we prove that the slave-fermion and slave-boson approaches are related to each other by changing the gauge fixing of the gauge invariance described above. The main tool we use is the Abelian bosonization formula which permits us to rewrite the t-J model as a model of bosons, satisfying hard-core conditions, which are coupled to a statistical U(1)-gauge field with Chern-Simons action.[10,11] Introducing a suitable decomposition of the boson field and using a variant of the Holstein-Primakoff transformation[12] for path integrals of hard-core bosons, we are able to explicitly exhibit holon and spinon fields with the desired statistics.

Laughlin[13,14] has advocated the idea that, at least in certain phases, holons and spinons are neither bosons nor fermions, but *semions,* i.e., particles that, under exchange, acquire $e^{\pm i\pi/2}$ phase factors. Excitations in two-dimensional systems whose statistics is described by phase factors $e^{\pm i2\pi\theta}$, $\theta\notin\mathbb{Z}/2$, are called *anyons*[15] and cannot be created by local-field operators (see Ref. 11 for a simple discussion of this point). The best localization one can hope to achieve is to localize anyon-field operators on strings reaching out to infinity in fixed-time planes. Therefore, in this situation, one cannot have a local decomposition of the hole operator, as (1.3) or (1.5). However, in this paper we show that one can construct holon-field $[\tilde{e}(\gamma_i)]$ and spinon-field $[s_\alpha(\gamma_i)]$ operators localized on strings, γ_i, where i denotes the starting site of the string, such that, in a somewhat formal sense,

$$\psi_{i\alpha}=\tilde{e}^{\dagger}(\gamma_i)s_\alpha(\gamma_i)\ , \tag{1.9}$$

and the fields $\tilde{e}(\gamma_i),s(\gamma_i)$ obey semion statistics, i.e., are anyon-field operators with statistics parameter $\theta=\pm\frac{1}{4}$. The main tool is a non-Abelian bosonization formula[11] converting the t-J model into a system of hard-core bosons coupled to a U(1) and an SU(2) statistical gauge field with Chern-Simons actions.

One possible factorization of the boson field is expressed as a product of a charged boson field, only coupled to the U(1)-statistical-gauge field, and a neutral boson field carrying spin $\frac{1}{2}$, coupled to the SU(2)-statistical-gauge field. In the non-Abelian bosonization formula, the coefficients of the Chern-Simons action must be chosen precisely in such a way that the corresponding string-localized, gauge-invariant fields are *semions* describing the holon and spinon, respectively. This semion picture turns out to be especially adequate to discuss mean-field theory in the "generalized flux phase," as suggested by

Laughlin.[14] The holons turn out to be "Dirac" semions (see Sec. VI).

We end this Introduction with an outline of the contents of our paper. In Sec. II, we review some basic notions in the quantum-statistical mechanics of systems of nonrelativistic, charged fermions and bosons. In particular, we recall the Feynman-Kac formula expressing the partition function in terms of Brownian paths.

In Sec. III we first present the bosonization formulas of two-dimensional partition functions. Then we briefly consider their extension to correlation functions. Finally, we discuss the modifications needed to adapt bosonization to lattice-field theories, including ones equivalent to the t-J model. The basic idea[11] underlying bosonization in two dimensions is to rewrite the *minus signs* in the Feynman-Kac representation of the partition function of a *fermion* system, reflecting the Fermi statistics of the particles, as the expectation values of Wilson loops (traces of path-ordered exponentials of a gauge field along loops) in topological Chern-Simons theory, with the loops identified as the imaginary-time world lines of the particles.

In Sec. IV we present two different choices of coherent states for hard-core bosons, leading to a path-integral representation of hard-core-boson theories which are related to each other by some kind of path-integral version of the Holstein-Primakoff transformation.

In Sec. V we combine the results of Secs. III and IV to show how one can derive a path-integral formulation of the slave-fermion and slave-boson representations for the t-J model, using Abelian bosonization formulas.

In Sec. VI we reexpress the t-J model as a system of interacting semions, using non-Abelian bosonization formulas.

II. SOME PRELIMINARIES

In this section we recall some basic notions required in Abelian and non-Abelian bosonization of two-dimensional fermion systems. Our bosonization formulas work for general systems of fermions with an arbitrary half-integer spin interacting via instantaneous, spin-independent two-body potentials and in an external electromagnetic field. In non-Abelian bosonization, however, one must assume that Zeeman (and spin-orbit) terms are absent. For a complete treatment of non-Abelian bosonization, we refer the reader to Ref. 11. In the next section we briefly review the main ideas, discussing first spin-$\frac{1}{2}$ fermions in the continuum, where our formulas take a familiar form. Then we indicate the modification necessary to discuss fermions on a (spatial) lattice.

We consider a system of identical, nonrelativistic, spin-$\frac{1}{2}$ quantum-mechanical particles of mass m and charge e, moving in the plane $\mathbb{R}^2$, with the Hamiltonian

$$H^{(N)}(eA)=\sum_{j=1}^{N}\frac{1}{2m}(\nabla^{(j)}_{e\mathbf{A}(\mathbf{x}_j)})^2+eA_0(\mathbf{x}_j)+\sum_{1\le i<j\le N}u(\mathbf{x}_i-\mathbf{x}_j)\ , \tag{2.1}$$

where $(\mathbf{A}, A_0)$ is the electromagnetic-gauge potential,

$$\nabla_{e\mathbf{A}}=\nabla-ie\,\mathbf{A} \tag{2.2}$$

is the covariant derivative, and u is a two-body potential. We use units where $\hbar=c=1$. The partition function at temperature T is defined by

$$Z(N,\beta|eA)=\mathrm{Tr}_{\mathcal{H}^{(N)}}(e^{-\beta H^{(N)}(eA)})\ , \tag{2.3}$$

where $\beta=1/kT$, k is the Boltzmann constant and $\mathrm{Tr}_{\mathcal{H}^{(N)}}$ denotes the trace over the N-particle Hilbert space $\mathcal{H}^{(N)}$, containing *symmetric* wave functions if the particles are *bosons* and *antisymmetric* wave functions if the particles are *fermions.* One can express the partition function (2.3) in terms of Wiener integrals over Brownian paths, as expressed in the Feynman-Kac formula. Let $\omega(\cdot)$ denote a (Brownian) path in $\mathbb{R}^2$, let Σ_N be the group of permutations of N objects, and for $\pi\in\Sigma_N$, let $\sigma(\pi)$ denote the signature of π, i.e., the number of exchanges in $\pi \bmod 2$. In a somewhat imprecise, but suggestive notation, the Feynman-Kac formula is the identity

$$\begin{aligned}Z(N,\beta|eA)=\sum_{\pi\in\Sigma_N}\varepsilon^{\sigma(\pi)}\sum_{\alpha_1,\dots,\alpha_N}\int d^2x_1\cdots d^2x_N\int_{\substack{\omega_j(0)=\mathbf{x}_j\\ \omega_j(\beta)=\mathbf{x}_{\pi(j)}}}\prod_{j=1}^{N}\mathcal{D}\omega_j\\ \times\exp\left\{-\int_0^\beta dt\left[\frac{m}{2}\sum_j\dot\omega_j^2(t)+\sum_{i<j}u(\omega_i(t)-\omega_j(t))\right]\right\}\\ \times\exp\left\{i\sum_j\left[\int A_l(\omega_j(t))d\omega_j^l(t)+A_0(\omega_j(t))dt\right]\right\}\prod_j\delta_{\alpha_j\alpha_{\pi(j)}}\ ,\end{aligned} \tag{2.4}$$

$$\varepsilon=\begin{cases}-1 & \text{for fermions}\ ,\\ +1 & \text{for bosons}\ ,\end{cases} \tag{2.5}$$

α_j are the spin indices, and periodic boundary conditions (BC's) in the time direction are imposed. According to a standard convention in Euclidean field theory, we work with an imaginary scalar potential; i.e., in (2.4) we have replaced A_0 by iA_0. To simplify the notation, given a potential $(\mathbf{A}, A_0)$, we define a one-form by

$$A=\sum_{l=1}^{2}A_l dx^l+A_0 dt\ , \tag{2.6}$$

and for a path ω in $\mathbb{R}^2\times[0,\beta]$, we set

$$\int_\omega A = \int (A_l d\omega^l + A_0 dt) . \quad (2.7)$$

For later purposes one needs a generalization of (2.4). Consider a U(1)-gauge field B and an SU(2)-gauge field $V \equiv V^a \sigma^a/2$ acting on spin space. Define the Hamiltonian $H^{(N)}(eA+B,V)$ as the Hamiltonian obtained from (2.1) by substituting

$$eA \to eA + B + V . \quad (2.8)$$

Then we have a Feynman-Kac formula similar to (2.4), but where we need to take a path-ordered exponential of the non-Abelian gauge field:

$$Z(N,\beta|eA+B,V) = \sum_{\pi\in\Sigma_N} \varepsilon^{\sigma(\pi)} \sum_{\alpha_1,\ldots,\alpha_N} \int d^2x_1 \cdots d^2x_N \int_{\substack{\omega_j(0)=\mathbf{x}_j \\ \omega_j(\beta)=\mathbf{x}_{\pi(j)}}} \prod_{j=1}^N \mathcal{D}\omega_j \times \exp\left\{ -\int_0^\beta dt \left[\sum_j \frac{m}{2}\dot\omega_j^2(t) + \sum_{i<j} u(\omega_i(t)-\omega_j(t)) \right] \right\} \times \prod_j \exp\left[i\int_{\omega_j}(eA+B) \right] \prod_j \left[P\exp\left[i\int_{\omega_j} V \right] \right]_{\alpha_j\alpha_{\pi(j)}} . \quad (2.9)$$

In (2.9), $P(\cdot)$ denotes path ordering, which amounts to the usual time ordering $T(\cdot)$ when "time" is identified with a parameter parametrizing the path. [Our general formula (2.9) fails if a Zeeman term is present; see Ref. 11.]

The paths $\omega_j(\cdot)$ in (2.4) or (2.9) can be thought of as (virtual) trajectories of the quantum-mechanical particles.

Remark 2.1. A mathematically correct version of (2.4) or (2.9) involves interpreting the formal expression

$$\text{const} \times \int_{\substack{\omega(0)=\mathbf{x} \\ \omega(\beta)=\mathbf{y}}} \mathcal{D}\omega \exp\left[-\frac{m}{2}\int_0^\beta dt\,\dot\omega^2(t) \right] \quad (2.10a)$$

as the Wiener measure $dW^{\beta,m}_{\mathbf{xy}}(\omega)$, defined in terms of the heat kernel through

$$(e^{(\beta/2m)\Delta})(\mathbf{x},\mathbf{y}) = \int dW^{\beta,m}_{\mathbf{xy}}(\omega) \quad (2.10b)$$

and confining the system to a *bounded region* $\Lambda \in \mathbb{R}^2$.

Given the partition function $Z(N,\beta|eA)$, the grand-canonical partition function with chemical potential μ is defined by

$$\Xi(\beta,\mu|eA) = \sum_{N=0}^{\infty} \frac{e^{\beta\mu N}}{N!} Z(N,\beta|eA) . \quad (2.11)$$

In the formalism of second quantization, one describes the degrees of freedom of spin-$\frac{1}{2}$ particles by two-component field operators

$$\psi(\mathbf{x}) = \{\psi_\alpha(\mathbf{x}), \alpha=1,2\} ,$$

transforming under the spin-$\frac{1}{2}$ representation of SU(2) and satisfying the equal-time (anti-) commutation relations

$$[\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{y})]_\pm = 0 = [\psi^\dagger_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})]_\pm ,$$
$$[\psi_\alpha(\mathbf{x}),\psi^\dagger_\beta(\mathbf{y})]_\pm = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta} . \quad (2.12)$$

In (2.12) the dagger denotes the adjoint in the Fock space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)} ,$$

and + or − are chosen for fermions or bosons, respectively. The Hamiltonian of the system in this formalism is

$$H(\psi,\psi^\dagger|eA) = \int d^2x \left\{ \psi^\dagger_\alpha(\mathbf{x}) \left[\frac{1}{2m}(\nabla_{e\mathbf{A}})^2 - \mu + eA_0 \right] \psi_\alpha(\mathbf{x}) + \int d^2y\, \psi^\dagger_\alpha(\mathbf{x})\psi^\dagger_\beta(\mathbf{y}) u(\mathbf{x}-\mathbf{y})\psi_\beta(\mathbf{y})\psi_\alpha(\mathbf{x}) \right\} . \quad (2.13)$$

The grand-canonical partition function is given by

$$\Xi(\beta,\mu|eA) = \mathrm{Tr}_{\mathcal{F}(\mathcal{H})}(e^{-\beta H(\psi,\psi^\dagger|eA)}) . \quad (2.14)$$

Using coherent states for bosons or fermions, one can convert Ξ into a path integral over complex or Grassmann (anticommuting) fields $\Psi = \{\Psi_\alpha(\mathbf{x},t), \alpha=1,2\}$, respectively. For fixed $\mathbf{x},t,\alpha$, these fields are standard c-number variables for bosons and Grassmann anticommuting variables for fermions.[16]

In terms of these fields, one can rewrite the grand-partition function as

$$\Xi(\beta,\mu|eA) = \int \mathcal{D}\Psi\,\mathcal{D}\Psi^*\, e^{-S(\Psi,\Psi^*,eA)} , \quad (2.15)$$

where

$$S(\Psi,\Psi^*,eA) = \int_0^\beta dt\, \Psi^*_\alpha(\mathbf{x},t)\frac{\partial}{\partial t}\Psi_\alpha(\mathbf{x},t) + \int_0^\beta dt\, H(\Psi^*(t),\Psi(t),eA(t)) , \quad (2.16)$$

where $H(\Psi^*(t),\Psi(t),eA(t))$ denotes the functional obtained from (2.13) by replacing $\psi(\mathbf{x})$ by $\Psi(\mathbf{x},t)$ and $\psi^\dagger(\mathbf{x})$ by $\Psi^*(\mathbf{x},t)$. In (2.15) we impose antiperiodic (periodic) B.C.'s in time on the Ψ,Ψ^* fields for fermions (bosons).

III. BOSONIZATION IN TWO-DIMENSIONAL SYSTEMS

Here we present the basic idea underlying bosonization in two-dimensional nonrelativistic systems. We have seen that, in the Feynman-Kac representation of the partition function, the only difference between fermions and bosons is found in the factor $\varepsilon^{\sigma(\pi)}$, where $\varepsilon=+1$ for bosons and $\varepsilon=-1$ for fermions. We therefore unfold the problem of bosonization from this starting point, and we first take a closer look at the Feynman-Kac formulas (2.4) and (2.9).

It follows from (2.4) and (2.9) that the effect of coupling our system of particles to gauge fields, such as $eA+B$, or V, is simply to multiply the original formula by phase factors of the form $\exp(i\int_{\omega}eA+B)$ or $[P\exp(i\int_{\omega}V)]_{\alpha\beta}$. In gauge-theory jargon these (path-ordered) exponentials are called Wilson lines. Let us consider the collection of paths $\underline{\omega}=\{\omega_j, j=1,\dots,N\}$ in (2.4) and (2.9): They start at the points $\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$ and end in the points $\{\mathbf{x}_{\pi(1)},\dots,\mathbf{x}_{\pi(N)}\}$ and are always directed forward in (imaginary) time. In two space dimensions, these paths form what is called a *geometric braid,* denoted by $b\equiv b(\underline{\omega})$, provided they do not intersect each other (see Fig. 2). We denote by $\pi(b)$ the permutation π corresponding to the braid b. In two space dimensions, the set of "world lines" $\{(t,\omega_j(t))\}_{j=1}^{N}$, with $\omega_i(0)\neq\omega_j(0)$, for $i\neq j$, which have nontrivial intersections, has zero measure with respect to the Wiener measure

$$\text{const}\times\prod_j \mathcal{D}\omega_j \exp\left[-\sum_j \frac{m}{2}\int_0^\beta dt\,\dot\omega_j^2(t)\right].$$

Intersections of distinct world lines can therefore be neglected.

Because of periodic boundary conditions in time, the 0 and β planes are identified, and a braid $b(\underline{\omega})$ is actually closed to a *link* $\mathcal{L}(b(\omega))$, i.e., to a union of disjoint oriented loops (see Fig. 2). For a set of paths $\underline{\omega}$, the Wilson lines in (2.9), summed over spin indices, are given by

$$2^{|\mathcal{L}(b(\underline{\omega}))|}\exp\left[i\int_{\mathcal{L}(b(\underline{\omega}))} eA+B\right]\times\text{tr}\left[P\exp\left[i\int_{\mathcal{L}(b(\underline{\omega}))}V\right]\right], \tag{3.1}$$

where $|\mathcal{L}|$ denotes the number of disconnected components of $\mathcal{L}$, and the trace (tr) is normalized and taken over each component of the link. In gauge-theory jargon, the traces of path-ordered exponentials appearing in (3.1) are called *Wilson loops.*

To simplify our notation, we define

$$\begin{aligned}\exp\left[i\int_{\mathcal{L}}B\right]&=W(\mathcal{L}|B),\\ \text{Tr}P\exp\left[i\int_{\mathcal{L}}V\right]&=W(\mathcal{L}|V).\end{aligned}\tag{3.2}$$

Suppose we can find a measure on configurations of the fields B and V, whose expectation value we denote by $\langle(\cdot)\rangle$, such that

$$\langle W(\mathcal{L}(b)|B)W(\mathcal{L}(b)|V)\rangle=(-1)^{\sigma(\pi(b))}. \tag{3.3}$$

Then we can rewrite the partition function for *two-dimensional fermions* as

$$Z(N,\beta|eA)=\Big\langle\sum_{\pi\in\Sigma_N}\int d^2x_1\cdots d^2x_N\int_{\substack{\omega_j(0)=\mathbf{x}_j\\ \omega_j(\beta)=\mathbf{x}_{\pi(j)}}}\prod_j\mathcal{D}\omega_j \times\exp\left\{-\int_0^\beta dt\left[\sum_j\frac{m}{2}\dot\omega_j^2(t)+\sum_{i<j}u(\omega_i(t)-\omega_j(t))\right]\right\}\exp\left[i\int_{\mathcal{L}(b(\underline{\omega}))}eA\right]W(\mathcal{L}(b(\underline{\omega}))|B)W(\mathcal{L}(b(\underline{\omega}))|V)\Big\rangle. \tag{3.4}$$

The sign $(-1)^{\sigma(\pi)}$ has disappeared from the right-hand side of the Feynman-Kac formula (3.4), and a comparison with (2.9) shows that the right-hand side of (3.4) is the partition function of a system of N bosons coupled to gauge fields $eA+B$ and V.

Expressed in terms of path integrals over fields, this means that if $\Phi=\{\Phi_\alpha(\mathbf{x},t),\alpha=1,2\}$ are complex fields and $\Psi=\{\Psi_\alpha(\mathbf{x},t),\alpha=1,2\}$ are Grassmann fields, then the *grand canonical partition function* is given by

$$\begin{aligned}\Xi(\beta,\mu|eA)&=\int\mathcal{D}\Psi\mathcal{D}\Psi^*\,e^{-S(\Psi^*,\Psi,eA)}\\ &=\Big\langle\int\mathcal{D}\Phi\mathcal{D}\Phi^*\,e^{-S(\Phi,\Phi^*,eA+B,V)}\Big\rangle.\end{aligned}\tag{3.5}$$

To get an idea how an expectation value satisfying (3.3) is found, we remark that such an expectation value clearly remains unchanged if we locally deform the link without changing its topology; i.e., it provides topological invariants. As Witten showed in Ref. 17, expectation values of Wilson loops in (topological) Chern-Simons theories are topological invariants. For the U(1)-gauge field B and SU(2)-gauge field V, the Chern-Simons actions are given by

$$\begin{aligned}S_{\text{CS}}(B)&=\frac{1}{4\pi i}\int\varepsilon_{\mu\nu\rho}B^\mu\partial^\nu B^\rho,\\ S_{\text{CS}}(V)&=\frac{1}{4\pi i}\int\varepsilon_{\mu\nu\rho}\text{Tr}(V^\mu\partial^\nu V^\rho+\tfrac{2}{3}V^\mu V^\nu V^\rho).\end{aligned}\tag{3.6}$$

The expectation values of Wilson loops are defined by

$$\langle W(\mathcal{L}|Z)\rangle_G^k=\frac{\int\mathcal{D}Ze^{-kS_{\text{CS}}(Z)}W(\mathcal{L}|Z)}{\int\mathcal{D}Ze^{-kS_{\text{CS}}(Z)}}, \tag{3.7}$$

where $Z=B,V$, $G=\text{U}(1),\text{SU}(2)$, respectively, and k is an arbitrary real number for B and an integer for V.

Remark 3.1. In order to give an unambiguous meaning to (3.7), one must choose a "framing" of $\mathcal{L}$.[17] For further details, we refer to Ref. 11.

The *key formulas* are

$$\langle W(\mathcal{L}(b)|B)\rangle^{1/(2l+1)}=(-1)^{\sigma(\pi(b))}, \tag{3.8}$$

$$\langle W(\mathcal{L}(b)|B)W(\mathcal{L}(b)|V)\rangle^{2/(2l+1),(-1)^l}_{U(1),SU(2)} := \langle W(\mathcal{L}(b)|B)\rangle^{2/(2l+1)}_{U(1)}\langle W(\mathcal{L}(b)|V)\rangle^{(-1)^l}_{SU(2)} = (-1)^{\sigma(\pi(b))}, \tag{3.9}$$

for $l=0,1,2,\ldots$.

Inserting (3.7) and (3.8) (with $V=0$) in (3.4), one obtains the Abelian bosonization formula discussed in Ref. 10. Equations (3.7) and (3.9), inserted in (3.4), yield the non-Abelian bosonization formulas of Ref. 11.

To summarize, Eq. (3.5) combines the following *two bosonization formulas:*

$$\int \mathcal{D}\Psi\,\mathcal{D}\Psi^*\, e^{-S(\Psi^*,\Psi,eA)} = \frac{\int \mathcal{D}B \int \mathcal{D}\Phi\,\mathcal{D}\Phi^* \exp(-\{S(\Phi,\Phi^*,eA+B)+[1/(2l+1)]S_{CS}(B)\})}{\int \mathcal{D}B\, \exp\{-[1/(2l+1)]S_{CS}(B)\}} = \frac{\int \mathcal{D}B\,\mathcal{D}V \int \mathcal{D}\Phi\,\mathcal{D}\Phi^* \exp(-\{S(\Phi,\Phi^*,eA+B,V)+[2/(2l+1)]S_{CS}(B)+(-1)^l S_{CS}(V)\})}{\int \mathcal{D}B\,\Phi V \exp(-\{[2/(2l+1)]S_{CS}(B)+(-1)^l S_{CS}(V)\})}, \tag{3.10}$$

for $l=0,1,2,\ldots$.

Remark 3.2. To be more precise, one must add in (3.10) a "neutralizing current at infinity" (see Ref. 11).

Next, we describe bosonization formulas for the correlation functions of fermions. (Proofs are sketched in the Appendix.)

Let $\langle(\cdot)\rangle^{\Psi}$ denote an expectation value in the functional integral

$$\Xi(\beta,\mu|eA)^{-1}\int \mathcal{D}\Psi\,\mathcal{D}\Psi^*\, e^{-S(\Psi,\Psi^*,eA)}(\cdot), \tag{3.11}$$

where S is the Euclidean action defined in (2.16). The imaginary-time (Matsubara) Green functions are the expectation values

$$\langle \Psi^{\#}_{\alpha_1}(x_1)\cdots\Psi^{\#}_{\alpha_n}(x_n)\rangle^{\Psi},$$

where $x_i=(\mathbf{x}_i,x_i^0)$ and $\Psi^{\#}_{\alpha}=\Psi_\alpha,\Psi^*_\alpha$.

Let us define bosonic field variables

$$\Phi_\alpha(\gamma_x|B)=\Phi_\alpha(x)\exp\left[i\int_{\gamma_x}B\right],\quad \Phi^*_\alpha(\gamma_x|B)=\Phi^*_\alpha(x)\exp\left[-i\int_{\gamma_x}B\right],$$
$$\Phi_\alpha(\gamma_x|B,V)=\exp\left[i\int_{\gamma_x}B\right]\left[P\exp\left[i\int_{\gamma_x}V\right]\right]_{\alpha\beta}\Phi_{\beta(x)},\tag{3.12}$$
$$\Phi^*_\alpha(\gamma_x|B,V)=\Phi^*_\beta(x)\left[P\exp\left[-i\int_{\gamma_x}V\right]\right]_{\beta\alpha}\exp\left[-i\int_{\gamma_x}B\right],$$

where γ_x is a line in the plane at time x^0 starting at $\mathbf{x}$ and ending in the world line of the neutralizing current at infinity (see remark 3.2). Let $T(\cdot)$ indicate time ordering. Then, using the generalization of the Feynman-Kac formula discussed in the Appendix, a variant of the arguments yielding our bosonization formulas (3.4) and (3.10) can be used to establish the identities

$$\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}(x_n)\Psi_{\delta_1}(y_1)\cdots\Psi_{\delta_n}(y_n))\rangle^{\Psi} = \langle\langle T(\Phi^*_{\alpha_1}(\gamma_{x_1}|B)\cdots\Phi^*_{\alpha_n}(\gamma_{x_n}|B)\Phi_{\delta_1}(\gamma_{y_1}|B)\cdots\Phi_{\delta_n}(\gamma_{y_n}|B))\rangle^{\Phi,B}\rangle^{1/(2l+1)}_{U(1)} \tag{3.13}$$

and, for spin-singlet correlation functions,

$$\sum_{\alpha_1\cdots\alpha_n}\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}(x_n)\Psi_{\alpha_1}(y_1)\cdots\Psi_{\alpha_n}(y_n))\rangle^{\Psi} = \sum_{\alpha_1\cdots\alpha_n}\langle\langle T(\Phi^*_{\alpha_1}(\gamma_{x_1}|B,V)\cdots\Phi^*_{\alpha_n}(\gamma_{x_n}|B,V)\Phi_{\alpha_1}(\gamma_{y_1}|B,V)\cdots\Phi_{\alpha_n}(\gamma_{y_n}|B,V))\rangle^{\Phi,B,V}\rangle^{2/(2l+1),(-1)^l}_{U(1),SU(2)}, \tag{3.14}$$

where $\langle(\cdot)\rangle^{\Phi,B}$ $[\langle(\cdot)\rangle^{\Phi,B,V}]$ is the expectation value corresponding to the Euclidean action $S(\Phi,\Phi^*,eA+B)$ $[S(\Phi,\Phi^*,eA+B,V)]$.

Sketches of the proofs of (3.13) and (3.14) are given in the Appendix.

Finally, we consider the modifications needed to bosonize two-dimensional lattice fermion models. Our formulas are applicable to the two-dimensional t-J model, as discussed in Secs. V and VI.

Lattice gauge fields are maps from links (nearest-neighbor pairs) $\langle ij \rangle$ of the lattice to a gauge group G. In particular, given a continuum U(1)-gauge field B and a continuum SU(2)-gauge field V, one can define corresponding U(1)- and SU(2)-lattice-gauge fields by associating to a link $\langle ij \rangle$ the variables

$$\exp\left[i\int_{\langle ij\rangle} B\right] \in \mathrm{U}(1), \quad P\exp\left[i\int_{\langle ij\rangle} V\right] \in \mathrm{SU}(2) . \tag{3.15}$$

With a slight abuse of language, a map

$$X:\langle ij\rangle \to X_{\langle ij\rangle} \in \mathbb{C} \tag{3.16}$$

is called a lattice-gauge field, too, in spite of the fact that the complex numbers $\mathbb{C}$ do not form a multiplicative group (unless 0 is deleted). Let $\Phi=\{\Phi_\alpha(i),\alpha=1,2\}$ denote a two-component, complex lattice scalar field, defined as a mapping from the sites of the lattice, i, to $\mathbb{C}^2$. The lattice covariant derivative associated with the gauge fields $X_{\langle ij\rangle}$, $\exp(i\int_{\langle ij\rangle}B)$, and $P\exp(\int_{\langle ij\rangle}V)$ is denoted by $\nabla^d_{X,B,V}$ and is defined by setting

$$(\nabla^d_{X,B,V}\Phi)_{\alpha\langle ij\rangle}=X_{\langle ij\rangle}\exp\left[i\int_{\langle ij\rangle}B\right]\left[P\exp\left[i\int_{\langle ij\rangle}V\right]\right]_{\alpha\beta}\Phi_{j\beta}-\Phi_{i\alpha} . \tag{3.17}$$

With these conventions the following lattice Feynman-Kac formula holds. Let $H_d^{(N)}(X,B,V)$ be the Hamiltonian for a system of N spin-$\frac{1}{2}$ particles on the lattice $\mathbb{Z}^2$, given, in first quantized notation, by the equation

$$H_d^{(N)}(X,B,V)=\sum_{l=1}^{N}[(\nabla^{d(l)}_{X,B,V})^2+B_0(j_l)+V_0(j_l)]+\sum_{1\le l<m\le N}u(j_l,j_m) , \tag{3.18}$$

where l and n label the particles and each j_l is an arbitrary lattice site. Then

$$\begin{aligned}Z^d(N,\beta|X,B,V)&=\mathrm{Tr}_{\mathcal{H}_d^{(N)}}(e^{-\beta H_d^{(N)}(X,B,V)})\\&=\sum_{\pi\in\Sigma_N}\varepsilon^{\sigma(\pi)}\sum_{\alpha_1,\ldots,\alpha_N}\sum_{j_1,\ldots,j_N}\int_{\substack{\omega_l(0)=j_l\\ \omega_l(\beta)=j_{\pi(l)}}}\prod_{l=1}^{N}d\mu^\beta(\omega_l)\\&\quad\times\exp\left[-\sum_{l<m}\int_0^\beta dt\,u(\omega_l(t),\omega_m(t))\right]\prod_l\left[\prod_{\langle ij\rangle\in S(\omega_l)}X_{\langle ij\rangle}(t_{\langle ij\rangle})\right]\\&\quad\times\left[\exp\left[i\int_{\omega_l}B\right]\right]\left[P\exp\left[i\int_{\omega_l}V\right]\right]_{\alpha_l\alpha_{\pi(l)}} .\end{aligned} \tag{3.19}$$

In (3.19), $\varepsilon=+1$ for bosons and -1 for fermions; the sum over $\alpha_1,\ldots,\alpha_N$ runs over the spin indices, the sums over $j_1,\ldots,j_N$ range over the sites of (a finite domain in) the lattice, $d\mu(\omega_l)$ is a (Poisson) measure on random walks in the lattice $\mathbb{Z}^2$, parametrized by a continuous time, replacing the Wiener measure

$$\text{const}\times\mathcal{D}\omega\exp\left[-\frac{m}{2}\int_0^\beta\dot{\omega}^2(t)dt\right]$$

used in the continuum, $S(\omega)$ is the set of steps the path ω takes in the lattice $\mathbb{Z}^2$, and $t_{\langle ij\rangle}$ is the random time at which ω makes the step from i to j.

Remark 3.4. More precisely, $d\mu^\beta(\omega)$ is defined in terms of the lattice heat kernel by

$$e^{\beta(\nabla^d)^2}(i,j)=\int_{\substack{\omega(0)=i\\ \omega(\beta)=j}}d\mu^\beta(\omega) . \tag{3.20}$$

For more details, see, e.g., Ref. 18.

Using (3.19), one can bosonize a system of lattice fermions with the Hamiltonian $H_d^{(N)}(X)$ by following the arguments given for continuum systems. One still uses Chern-Simons gauge fields B,V in the continuum, defining the associated lattice covariant derivatives through (3.17). [For the U(1) theory, one can alternatively use lattice-gauge fields as discussed in Ref. 18.] The proof of the bosonization formulas will go through unchanged, with an *additional condition* on the two-body potential u: In the continuum models we used the fact that the probability that two Brownian paths intersect at a fixed time is zero in two dimension [see the comment before Eq. (3.1)]. This is not true for random walks in $\mathbb{Z}^2$. In order to be able to associate geometric braids with the paths $\omega_1,\ldots,\omega_N$ in (3.21), we must assume that *u contain a hard-core term,* so that random walks in (3.21) never intersect each other.

IV. COHERENT STATES FOR HARD-CORE BOSONS

It is well known that lattice systems of hard-core bosons can equally well be described as spin-$\frac{1}{2}$ quantum spin systems.[19] Consider, for example, a Heisenberg model of the following form: Let $\phi_i,\phi_i^\dagger$ be lattice boson-field operators satisfying the canonical commutation relations

$$[\phi_i,\phi_j]_-=0=[\phi_i^\dagger,\phi_j^\dagger]_- , \quad [\phi_i,\phi_j^\dagger]_-=\delta_{ij} \tag{4.1}$$

and define the Hamiltonian by

$$H_b=\sum_{i,j}(t_{ij}\phi_i^\dagger\phi_j+\mathrm{H.c.})+\sum_i\mu_i\phi_i^\dagger\phi_i+\sum_{i,j}u_{ij}\phi_i^\dagger\phi_j^\dagger\phi_j\phi_i\ , \tag{4.2}$$

where i,j are sites of the lattice and the two-body potential u satisfies the hard-core condition

$$u_{ii}=+\infty\ . \tag{4.3}$$

The hard-core condition (4.3) reduces the Hilbert space of the wave functions of the system to the subspace of functions that vanish whenever two arguments coincide. On each lattice site one can therefore replace $\phi,\phi^\dagger$ by Pauli matrices

$$\phi\to\sigma^-=\begin{bmatrix}0&0\\1&0\end{bmatrix}\ ,\qquad \phi^\dagger\to\sigma^+=\begin{bmatrix}0&1\\0&0\end{bmatrix}\ , \tag{4.4}$$

with the convention that matrices on different sites commute with each other. The hard-core constraint is automatically satisfied, since

$$(\sigma_i^+)^2=(\sigma_i^-)^2=0\ . \tag{4.5}$$

The substitution (4.4) converts the Hamiltonian H_b defined in (4.2) into the spin Hamiltonian

$$H_s=\sum_{i,j}(t_{ij}\sigma_i^+\sigma_j^-+\mathrm{H.c.})+\sum_i\mu_i\sigma_i^+\sigma_i^- +\sum_{i\neq j}u_{ij}\sigma_i^+\sigma_j^+\sigma_j^-\sigma_i^-\ . \tag{4.6}$$

We want to express the partition function of the system in terms of a path integral over fields. In order to reach this goal, one has to introduce coherent states for bosons for the system with Hamiltonian H_b and coherent states for spins if H_s is considered. We briefly recall the coherent-state formalism. Coherent states for lattice bosons are labeled by a sequence of complex numbers $\underline{z}=\{z_j\}$, where j runs over the sites of the lattice. They are defined by

$$|\underline{z}\rangle=\exp\left[-\tfrac{1}{2}\sum_j|z_j|^2\right]\exp\left[\sum_j z_j\phi_j^\dagger\right]|0\rangle\ , \tag{4.7}$$

where $|0\rangle$ is the vacuum vector of the boson Fock space. The following relations hold:[16]

$$\int\prod_j\frac{d\,\mathrm{Im}z_j d\,\mathrm{Re}z_j}{2\pi i}|\underline{z}\rangle\langle\underline{z}|=1\ , \tag{4.8}$$

$$\phi_i|\underline{z}\rangle=z_i|\underline{z}\rangle\ ,\quad \langle\underline{z}|\phi_i^\dagger=z_i^*\langle\underline{z}|\ . \tag{4.9}$$

Coherent states for spin-$\frac{1}{2}$ systems can be labeled by a sequence of complex numbers $\underline{\chi}=\{\chi_j\}$, with $|\chi_j|\le1$, where j runs over the sites of the lattice, and are defined by

$$|\underline{\chi}\rangle=\prod_j[(1-|\chi_j|^2)^{1/2}]\exp\left[\sum_j\frac{\chi_j}{(1-|\chi_j|^2)^{1/2}}\sigma_j^-\right]|0\rangle_s\ , \tag{4.10}$$

where $|0\rangle_s=\otimes_j\binom{1}{0}$ is the state with spin up on all sites. The following relations hold:[20]

$$\int_{|\chi_j|\le1}\prod_j\frac{2d\,\mathrm{Im}\chi_j d\,\mathrm{Re}\chi_j}{\pi}|\underline{\chi}\rangle\langle\underline{\chi}|=1\ , \tag{4.11}$$

$$\langle\underline{\chi}|\sigma_i^+\sigma_i^-|\underline{\chi}\rangle=1-|\chi_i|^2\ ,\quad \langle\underline{\chi}|\sigma_i^+\sigma_k^-|\underline{\chi}\rangle=\chi_i(1-|\chi_i|^2)^{1/2}\chi_k^*(1-|\chi_k|^2)^{1/2}\ ,\quad k\neq i\ . \tag{4.12}$$

Next, we use the trivial identity

$$\mathrm{Tr}(e^{-\beta H_\#})=\lim_{N\to\infty}\mathrm{Tr}(e^{(-\beta/N)H_\#})^N\ ,\quad \#=b,s\ ; \tag{4.13}$$

i.e., we divide the interval $[0,\beta]$ into N intervals of length $\delta=\beta/N$, with starting points at the times $t_n=n\delta$, $0\le n\le N-1$, and insert, at each time t_n, the completeness relations (4.8) and (4.11) for $\#=b,s$, respectively. Using a standard procedure,[16] one obtains that

$$\mathrm{Tr}(e^{-\beta H_b})=\int\mathcal{D}\Phi\,\mathcal{D}\Phi^*\,e^{-S(\Phi,\Phi^*)}\ , \tag{4.14}$$

where

$$S(\Phi,\Phi^*)=\int_0^\beta dt\left[\sum_i\Phi_i^*(t)\frac{\partial}{\partial t}\Phi_i(t)+\sum_{i,j}(t_{ij}\Phi_i^*(t)\Phi_j(t)+\mathrm{H.c.})+\sum_i\mu_i\Phi_i^*(t)\Phi_i(t)+\sum_{i,j}u_{ij}\Phi_i^*(t)\Phi_j^*(t)\Phi_j(t)\Phi_i(t)\right] \tag{4.15}$$

and

$$\mathrm{Tr}(e^{-\beta H_s})=\int_{|\chi|\le1}\mathcal{D}\chi\,\mathcal{D}\chi^*\,e^{-S(\chi,\chi^*)}\ , \tag{4.16}$$

where

$$S(\chi,\chi^*)=\int_0^\beta dt\left[\sum_i \chi_i^*(t)\frac{\partial}{\partial t}\chi_i(t)+\sum_{i,j}(t_{ij}\chi_j^*(t)[1-|\chi_j(t)|^2]^{1/2}\chi_i(t)[1-|\chi_i(t)|^2]^{1/2}+\text{H.c.})\right.$$
$$\left.+\sum_i \mu_i(1-|\chi_i(t)|^2)+\sum_{i\neq j}u_{ij}(1-|\chi_i(t)|^2)(1-|\chi_j(t)|^2)\right] . \tag{4.17}$$

Since H_b and H_s describe the same system, the path integrals (4.14) and (4.16) are equal.

The field operators corresponding to Φ and χ are related by a Holstein-Primakoff transformation

$$\begin{aligned}\phi_i&=\chi_i^\dagger(1-\chi_i^\dagger\chi_i)^{1/2} ,\\ \phi_i^\dagger&=(1-\chi_i^\dagger\chi_i)^{1/2}\chi_i ,\end{aligned} \tag{4.18}$$

and both ϕ and χ satisfy the hard-core condition.

V. ABELIAN BOSONIZATION OF THE t-J MODEL

We now turn to the t-J model. We shall apply Abelian bosonization to the t-J model in order to show how one can derive the slave-fermion and slave-boson formalism from a clean path-integral treatment using the coherent states of Sec. IV. We recall that the Hamiltonian of the t-J model is given by

$$H=P_G\left[-t\sum_{\langle ij\rangle}(\psi_{j\alpha}^\dagger\psi_{i\alpha}+\text{H.c.})+J\sum_{\langle ij\rangle}\left[\psi_{i\alpha}^\dagger\frac{(\sigma)_{\alpha\beta}}{2}\psi_{i\beta}\right]\cdot\left[\psi_{j\gamma}^\dagger\frac{(\sigma)_{\gamma\delta}}{2}\psi_{j\delta}\right]+\mu\sum_i\psi_{i\alpha}^\dagger\psi_{i\alpha}\right]P_G , \tag{5.1}$$

where P_G is the Gutzwiller projection, the operators $\{\psi_{i\alpha}^{\#}\}$ are the electron operators, and μ, the chemical potential, is chosen such that, in the ground state,

$$\langle\psi_{i\alpha}^\dagger\psi_{i\alpha}\rangle=1-\delta ,\quad 0\le\delta\ll 1 , \tag{5.2}$$

where δ is a measure for the doping, with $\delta=0$ corresponding to undoped materials.

We rewrite the partition function of the model in a form convenient to apply the lattice version of our bosonization formulas discussed at the end of Sec. III. The Gutzwiller projection P_G can be made superfluous by adding to the Hamiltonian a hard-core exclusion potential

$$\sum_{i,j}u_{ij}\psi_{i\alpha}^\dagger\psi_{j\beta}^\dagger\psi_{j\beta}\psi_{i\alpha} , \tag{5.3}$$

where

$$u_{ij}=\begin{cases}0 , & i\neq j\\ \infty , & i=j .\end{cases} \tag{5.4}$$

Next, we apply a Hubbard-Stratonovich transformation to rewrite the J term. We introduce an auxiliary complex lattice-gauge field $X_{\langle ij\rangle}$ and rewrite the (grand-canonical) partition function of the t-J model at temperature $T=(\beta k)^{-1}$ as[21,9]

$$\Xi(\beta,\mu)=\int\mathcal{D}X\,\mathcal{D}X^*\,\mathcal{D}\Psi\,\mathcal{D}\Psi^*\,e^{-S(X,X^*,\Psi,\Psi^*)} , \tag{5.5}$$

where

$$S(X,X^*,\Psi,\Psi^*)=\int_0^\beta d\tau\left\{\frac{2}{J}\sum_{\langle ij\rangle}X_{\langle ij\rangle}^*(\tau)X_{\langle ij\rangle}(\tau)+\sum_i\Psi_i^*(\tau)\frac{\partial}{\partial\tau}\Psi_i(\tau)\right.$$
$$\left.+\sum_{\langle ij\rangle}[(-t+X_{\langle ij\rangle}(\tau))(\Psi_j^*(\tau)\Psi_i(\tau))+\text{H.c.}]+U(\Psi^*(\tau)\Psi(\tau))\right\} , \tag{5.6}$$

where

$$U(\Psi^*(\tau)\Psi(\tau))=(\mu-4J)\sum_i\Psi_i^*(\tau)\Psi_i(\tau)+\sum_{i,j}\tilde{u}_{ij}\Psi_{i\alpha}^*(\tau)\Psi_{j\beta}^*(\tau)\Psi_{j\beta}(\tau)\Psi_{i\alpha}(\tau) , \tag{5.7}$$

with

$$\tilde{u}_{ij}=\begin{cases}\infty & \text{for } i=j ,\\ -J & \text{if } i,j \text{ are nearest neighbors} ,\\ 0 & \text{otherwise} ,\end{cases} \tag{5.8}$$

and Ψ,Ψ^* are Grassmann fields.

Using a lattice version of our Abelian bosonization formula (see end of Sec. III), applicable because of the hard-core condition (5.4), one succeeds in bosonizing expressions (5.5) and (5.6). We obtain the identity

$$\Xi(\beta,\mu)=\left\langle \int \mathcal{D}X\,\mathcal{D}X^*\,\mathcal{D}\Phi\,\mathcal{D}\Phi^*\, e^{-S(X,X^*,\Phi,\Phi^*,B)}\right\rangle^1_{\mathrm{U}(1)}$$
$$=\left\langle \int \mathcal{D}\Phi\,\mathcal{D}\Phi^*\, e^{-S(\Phi,\Phi^*,B)}\right\rangle^1_{\mathrm{U}(1)}, \tag{5.9}$$

where $\langle(\cdot)\rangle^1_{\mathrm{U}(1)}$ has been defined in (3.7) (here we set $k\equiv 1/2l+1=1$, i.e., $l=0$), Φ,Φ^* are commuting fields, and

$$S(\Phi,\Phi^*,B)=\int_0^\beta d\tau\left\{\sum_j \Phi_j^*(\tau)\left[\frac{\partial}{\partial\tau}+iB_0(j,\tau)\right]\Phi_j(\tau)+\sum_{\langle ij\rangle}\left[-t\Phi_i^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]\Phi_j(\tau)+\text{H.c.}\right]\right.$$
$$\left.+\frac{J}{4}\left|\Phi_i^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]\Phi_j(\tau)\right|^2+U(\Phi^*(\tau)\Phi(\tau))\right\}. \tag{5.10}$$

These formulas will help us understand how a separation of spin and charge might arise, i.e., how spinons and holons might appear in this formalism.

Next, we make a polar decomposition of the $\mathbb{C}^2$-valued field Φ:

$$\Phi_{i\alpha}(\tau)=R_i(\tau)Z_{i\alpha}(\tau), \tag{5.11}$$

with

$$R_i(\tau):=[\Phi_{i\alpha}^*(\tau)\Phi_{i\alpha}(\tau)]^{1/2}, \tag{5.12}$$

where a summation over spin indices is understood and

$$Z_{i\alpha}^*(\tau)Z_{i\alpha}(\tau)=1. \tag{5.13}$$

The field R is non-negative, and each $Z_i=\{Z_{i\alpha}\}_{\alpha=1}^2$ is S^3 valued.

Since $R\geq 0$, the field operator r corresponding to the Euclidean field variable R cannot satisfy standard commutation relations. To avoid problems arising from this fact, we introduce a scalar lattice field Θ with values in $[-\pi,\pi)$, by inserting one of the following identities into (5.9):

$$\text{(a)}\quad \int \mathcal{D}\Theta\prod_{l,\tau}\delta\left[\prod_{\langle ij\rangle:l\in\langle ij\rangle}\left\{e^{i(\Theta_i(\tau)-\Theta_j(\tau))}\frac{Z_{i\alpha}^*(\tau)Z_{j\alpha}(\tau)}{|Z_{i\alpha}^*(\tau)Z_{j\alpha}(\tau)|}\right\}-1\right]=1 \tag{5.14}$$

or

$$\text{(b)}\quad \int \mathcal{D}\Theta\prod_{l,\tau}\delta\left[\prod_{\langle ij\rangle:l\in\langle ij\rangle}\left\{e^{i(\Theta_i(\tau)-\Theta_j(\tau))}\frac{Z_{i\alpha}^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]Z_{j\alpha}(\tau)}{|Z_{i\alpha}^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]Z_{j\alpha}(\tau)|}\right\}-1\right]=1. \tag{5.15}$$

We define

$$H_j(\tau)=R_j(\tau)e^{i\Theta_j(\tau)},\quad \Sigma_{j\alpha}(\tau)=Z_{j\alpha}(\tau)e^{-i\Theta_j(\tau)}, \tag{5.16}$$

and make the following change of variables:

$$R,\Theta,Z_\alpha,Z_\alpha^*\to H,H^*,\Sigma_\alpha,\Sigma_\alpha^*.$$

The bosonized action for the t-J model then becomes

$$S(H,H^*,\Sigma,\Sigma^*,M,B)=\int_0^\beta d\tau\left\{\sum_j\left[H_j^*(\tau)\left[\frac{\partial}{\partial\tau}+iB_0(j,\tau)\right]H_j(\tau)+H_j^*(\tau)H_j(\tau)\Sigma_j^*(\tau)\frac{\partial}{\partial\tau}\Sigma_j(\tau)\right]\right.$$
$$+\sum_{\langle ij\rangle}\left[\left[-tH_i^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]H_j(\tau)\Sigma_i^*(\tau)\Sigma_j(\tau)+\text{H.c.}\right]\right.$$
$$\left.+\frac{J}{4}H_i^*(\tau)H_i(\tau)H_j^*(\tau)H_j(\tau)|\Sigma_{i\alpha}^*(\tau)\Sigma_{j\alpha}(\tau)|^2\right]$$
$$\left.+\sum_j iM_j(\tau)[\Sigma_j^*(\tau)\Sigma_j(\tau)-1]+\tilde{U}(H^*(\tau)H(\tau))\right\}, \tag{5.17}$$

where $M_i(\tau)$ is a Lagrange multiplier enforcing the constraint

$$\Sigma_{i\alpha}^{*}(\tau)\Sigma_{i\alpha}(\tau)=1\ , \qquad (5.18)$$

derived from (5.13), and

$$\tilde{U}(H^{*}(\tau)H(\tau))=U(H^{*}(\tau)H(\tau))-\sum_j \ln H_j^{*}(\tau)H_j(\tau)\ . \qquad (5.19)$$

From (5.17) one derives that H gives rise to a field operator, h, satisfying canonical commutation relations. To (5.17) one has to add the constraints

$$\text{(a)}\quad \prod_{\langle ij\rangle : l\in\langle ij\rangle} \frac{\Sigma_{i\alpha}^{*}(\tau)\Sigma_{j\alpha}(\tau)}{|\Sigma_{i\alpha}^{*}(\tau)\Sigma_{j\alpha}(\tau)|}=1 \qquad (5.20)$$

or

$$\text{(b)}\quad \prod_{\langle ij\rangle : l\in\langle ij\rangle} \frac{\Sigma_{i\alpha}^{*}\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]\Sigma_{j\alpha}(\tau)}{|\Sigma_{i\alpha}^{*}\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]\Sigma_{j\alpha}(\tau)|}=1\ , \qquad (5.21)$$

for all l,τ.

We note that the action (5.17), supplemented by the constraint (a) of (5.20), is invariant under the U(1)-gauge transformations

$$\begin{aligned} &B_\mu(\mathbf{x},\tau)\to B_\mu(\mathbf{x},\tau)-\partial_\mu\Lambda(\mathbf{x},\tau)\ ,\quad \mu=0,1,2\ ,\\ &H_j(\tau)\to e^{i\Lambda(j,\tau)}H_j(\tau)\ ,\\ &\Sigma_{j\alpha}(\tau)\to\Sigma_{j\alpha}(\tau)\ ,\\ &M_j(\tau)\to M_j(\tau)\ . \end{aligned} \qquad (5.22)$$

This shows that the gauge field B couples to the complex field H. The action (5.17) supplemented by the constraint (b) [see (5.21)] is invariant under the U(1)-gauge transformations

$$\begin{aligned} &B_\mu(\mathbf{x},\tau)\to B_\mu(\mathbf{x},\tau)-\partial_\mu\Lambda(\mathbf{x},\tau)\ ,\\ &H_j(\tau)\to H_j(\tau)\ ,\\ &\Sigma_{j\alpha}(\tau)\to e^{i\Lambda(j,\tau)}\Sigma_{j\alpha}(\tau)\ ,\\ &M_j(\tau)\to M_j(\tau)\ . \end{aligned} \qquad (5.23)$$

Therefore, with constraint (b), the gauge field B couples to the S^3-valued field Σ.

Both constraints (a) of (5.20) and (b) of (5.21) can be viewed as *different gauge fixings* of the following local gauge invariance exhibited by the action (5.17):

$$\begin{aligned} &H_j(\tau)\to H_j(\tau)e^{i\Lambda_j(\tau)}\ ,\\ &\Sigma_{j\alpha}(\tau)\to\Sigma_{j\alpha}(\tau)e^{-i\Lambda_j(\tau)}\ . \end{aligned} \qquad (5.24)$$

Because of the hard-core condition on H inherited from (5.4), one can apply to (5.17) the "Holstein-Primakoff" transformation, discussed in Sec. IV, which leads from (4.15) to (4.17). For this purpose we introduce a complex field E, with $|E|\le 1$, and apply the transformation from (4.15) to (4.17) (Sec. IV) to expression (5.17), identifying the fields Φ and χ of Sec. IV with H and E, respectively. We then define a field S by setting

$$S_{j\alpha}(\tau)=[1-|E_j(\tau)|^2]^{1/2}\Sigma_{j\alpha}(\tau)\ . \qquad (5.25)$$

As a result of these transformations, the action, expressed in terms of E and S, is given by

$$\begin{aligned} S(E,E^{*},S,S^{*},M,B) &= \int_0^\beta d\tau\Bigg\{\sum_j\left[E_j^{*}(\tau)\frac{\partial}{\partial\tau}E_j(\tau)+iB_0(j,\tau)(1-E_j^{*}(\tau)E_j(\tau))+S_j^{*}(\tau)\frac{\partial}{\partial\tau}S_j(\tau)\right]\\ &\quad+\sum_{\langle ij\rangle}\left[\left\lbrace -tE_j^{*}(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]E_i(\tau)S_i^{*}(\tau)S_j(\tau)+\text{H.c.}\right\rbrace+\frac{J}{4}\left|S_{i\alpha}^{*}(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]S_{j\alpha}(\tau)\right|^2\right]\\ &\quad+\sum_j iM_j(\tau)(S_j^{*}(\tau)S_j(\tau)+E_j^{*}(\tau)E_j(\tau)-1)+U(1-E^{*}(\tau)E(\tau))\Bigg\}\ , \end{aligned} \qquad (5.26)$$

and in U the hard-core condition must be omitted.

If we supplement (5.26) by the constraint (a),

$$\prod_{\langle ij\rangle : l\in\langle ij\rangle}\frac{S_{j\alpha}^{*}(\tau)S_{i\alpha}(\tau)}{|S_{j\alpha}^{*}(\tau)S_{i\alpha}(\tau)|}=1\ , \qquad (5.27)$$

for all $l\in\mathbb{Z}^2$ and $\tau\in[0,\beta]$, then we can use the Abelian bosonization formula in reverse order: By integrating out the gauge field B, the complex (bosonic) field E is converted to a (fermionic) Grassmann field $\tilde{E}$. In order to eliminate the unwanted term iB_0 in (5.26), we first rewrite it as

$$\sum_j iB_0(j,\tau)=\frac{i}{2\pi}\sum_j\int d^2x\, B_0(\mathbf{x},\tau)2\pi\delta(\mathbf{x}-j)\ . \qquad (5.28)$$

Then we define $B_\mu^{\text{cl}}(\mathbf{x})$, $\mu=1,2$ such that

$$\varepsilon^{\mu\nu}\partial_\mu B_\nu^{\text{cl}}(\mathbf{x})=2\pi\sum_j\delta(\mathbf{x}-j)\ . \qquad (5.29)$$

By a translation

$$B_\mu(\mathbf{x},\tau)\to B_\mu(\mathbf{x},\tau)+B_\mu^{\text{cl}}(\mathbf{x})\ , \qquad (5.30)$$

the term (5.28) can be absorbed in $S_{\rm CS}(B)$.

Because of (5.29), we have that, for every plaquette p in the lattice,

$$\prod_{\langle ij\rangle\in\partial p}\exp\left[i\int_{\langle ij\rangle}B^{\rm cl}\right]=\exp\left[i\int_p d^2x\,\varepsilon^{\mu\nu}\partial_\mu B^{\rm cl}_\nu(\mathbf{x})\right]=1\ ,$$

and, hence as the lattice $\mathbb{Z}^2$ has trivial topology, there exists a scalar U(1) lattice field Λ such that

$$\exp\left[i\int_{\langle ij\rangle}B^{\rm cl}\right]=e^{i(\Lambda_j-\Lambda_i)}\ .$$

A change of variables,

$$E_j(\tau)\to E_j(\tau)e^{i\Lambda_j}\ ,$$

finally proves that the term iB_0 in (5.26) can be eliminated.

The action, as a functional of the anticommuting field $\tilde{E}$ and commuting fields $\{S_\alpha\}$, is given by

$$S(\tilde{E},\tilde{E}^*,S,S^*,M)=\int_0^\beta d\tau\left\{\sum_j\left[\tilde{E}_j^*(\tau)\frac{\partial}{\partial\tau}\tilde{E}_j(\tau)+S_j^*(\tau)\frac{\partial}{\partial\tau}S_j(\tau)\right]\right.$$
$$+\sum_{\langle ij\rangle}\left[(-t\tilde{E}_j^*(\tau)\tilde{E}_i(\tau)S_i^*(\tau)S_j(\tau)+\text{H.c.})+\frac{J}{4}|S_{j\alpha}^*(\tau)S_{i\alpha}(\tau)|^2\right]$$
$$\left.+\sum_j iM_j(\tau)(S_j^*(\tau)S_j(\tau)+\tilde{E}_j^*(\tau)\tilde{E}_j(\tau)-1)+U(1-\tilde{E}^*(\tau)\tilde{E}(\tau))\right\}\ . \tag{5.31}$$

Note that, using the constraint enforced by the Lagrange multiplier field M, one can rewrite the sum of the quartic S term and the last term in (5.31) as

$$\sum_{\langle ij\rangle}J\left[S_{i\alpha}^*(\tau)\left[\frac{\sigma}{2}\right]_{\alpha\beta}S_{i\beta}(\tau)\right]\cdot\left[S_{j\gamma}^*(\tau)\left[\frac{\sigma}{2}\right]_{\gamma\delta}S_{j\delta}(\tau)\right]+\sum_j\mu(1-\tilde{E}_j^*(\tau)\tilde{E}_j(\tau))\ . \tag{5.32}$$

Neglecting the constraint (5.27), which is a gauge fixing for the gauge invariance

$$\tilde{E}_j(\tau)\to e^{i\Lambda_j(\tau)}\tilde{E}_j(\tau)\ ,\quad S_{j\alpha}(\tau)\to e^{i\Lambda_j(\tau)}S_{j\alpha}(\tau)\ , \tag{5.33}$$

the action (5.31) and (5.32) can be derived from a Hamiltonian

$$H=\sum_{\langle ij\rangle}\left\{(-t\tilde{e}_j^\dagger\tilde{e}_i s_{i\alpha}^\dagger s_{j\alpha}+\text{H.c.})+J\left[s_i^\dagger\frac{\sigma}{2}s_i\right]\cdot\left[s_j^\dagger\frac{\sigma}{2}s_j\right]\right\}+\sum_j\mu(1-\tilde{e}_j^\dagger\tilde{e}_j)\ , \tag{5.34}$$

with the constraint

$$s_{j\alpha}^\dagger s_{j\alpha}+\tilde{e}_j^\dagger\tilde{e}_j=1\ , \tag{5.35}$$

where $\tilde{e}$ is a fermion field satisfying canonical anticommutation relations and $s_{j\alpha}$ is a spin-$\frac{1}{2}$ Bose field satisfying canonical commutation relations. This is exactly the approximation corresponding to the *slave-fermion ansatz,* where $\tilde{e}$ are the holon-field and s_α the spinon-field operators.

Furthermore, from the arguments in Sec. III, it follows that the correlation functions of the Grassmann field $\Psi_{j\alpha}(\tau)$ with action (5.6) are equal to the correlation functions of $\tilde{E}_j^*(\tau)S_{j\alpha}(\tau)$ with action (5.31) and constraint (5.27), i.e.,

$$\Psi_{j\alpha}(\tau)\sim\tilde{E}_j^*(\tau)S_{j\alpha}(\tau)\ , \tag{5.36}$$

so that the hole (or electron) field of the t-J model can be rewritten *exactly* as the product of a charged fermionic holon field and a spin-$\frac{1}{2}$ bosonic spinon field.

Finally, we note that one can rewrite the quartic spinon term (5.31) by using a Hubbard-Stratonovich transformation, introducing a complex RVB-gauge field Δ,

$$\frac{J}{4}|S_{j\alpha}^*(\tau)S_{i\alpha}(\tau)|^2\to\frac{2}{J}\Delta^*_{\langle ij\rangle}(\tau)\Delta_{\langle ij\rangle}(\tau)+(\Delta_{\langle ij\rangle}(\tau)S_{j\alpha}^*(\tau)S_{i\alpha}(\tau)+\text{H.c.})\ . \tag{5.37}$$

If we supplement (5.26) with constraint (b), we can use our bosonization formulas in reverse order to fermionize S after integrating over B. To carry out the fermionization of S, we require the following two preliminary steps.

(i) Taking into account the constraint enforced by the field M, we rewrite

$$iB_0(j,\tau)(1-E_j^*(\tau)E_j(\tau))=iB_0(j,\tau)S_{j\alpha}^*(\tau)S_{j\alpha}(\tau)\ . \tag{5.38}$$

(ii) As in (5.37), we introduce an auxiliary complex RVB-gauge field $\tilde{\Delta}$ to decouple the quartic S term in (5.26); i.e., we replace

$$\frac{J}{4}\left|S^*_{i\alpha}(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]S_{j\alpha}(\tau)\right|^2\to\frac{2}{J}\tilde\Delta^*_{\langle ij\rangle}(\tau)\tilde\Delta_{\langle ij\rangle}(\tau)+\left[\tilde\Delta_{\langle ij\rangle}(\tau)S^*_{i\alpha}(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]S_{j\alpha}(\tau)+\text{H.c.}\right], \quad (5.39)$$

and we introduce an auxiliary complex gauge field, C, rewriting the constraint (b) as

$$\int\mathcal{D}C\,\mathcal{D}C^*\prod_{\langle ij\rangle,\tau}\left\{\delta\left[C_{\langle ij\rangle}(\tau)-S^*_i(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]S_j(\tau)\right]\right.$$
$$\left.\times\delta\left[C^*_{\langle ij\rangle}(\tau)-S^*_j(\tau)\exp\left[-i\int_{\langle ij\rangle}B(\tau)\right]S_i(\tau)\right]\right\}\prod_{l,\tau}\delta\left[\prod_{\langle ij\rangle:l\in\langle ij\rangle}e^{i\arg C_{\langle ij\rangle}(\tau)}-1\right]. \quad (5.40)$$

After these two steps, our bosonization formulas apply in a straightforward way. Let $\tilde S_\alpha$ denote the Grassmann field obtained from S_α after integration over B. Then the action in terms of E, $\tilde S$, and $\tilde\Delta$ is simply obtained from (5.31) and (5.37) by the substitution $\tilde E\to E$, $S\to\tilde S$, $\Delta\to\tilde\Delta$. Neglecting constraint (b) [Eq. (5.40)] and integrating out $\tilde\Delta$, one obtains an action that can be derived from a Hamiltonian and constraint obtained from (5.34) and (5.35), respectively, after the substitutions

$$\tilde e_j\to e_j\ ,$$
$$s_{j\alpha}\to\tilde s_{j\alpha}\ ,$$

where e is a hard-core bosonic operator satisfying canonical commutation relations and $\tilde s$ is a spin-$\frac{1}{2}$ fermonic operator satisfying canonical anticommutation relations. This is the approximation known as the *slave-boson ansatz,* where $\tilde s_\alpha$ are the spinon-field and e are the holon-field operators.

Furthermore, we have an analog of (5.36), i.e.,

$$\Psi_{j\alpha}(\tau)\sim E^*_j(\tau)\tilde S_{j\alpha}(\tau)\ , \quad (5.41)$$

expressing *exactly* the electron-field operator of the t-J model as the product of a hard-core bosonic holon field and a spin-$\frac{1}{2}$ fermionic spinon field.

Let us briefly comment on some features of a mean-field theory following Refs. 5 and 7. In essence, one is making the following approximations.

One restricts the integration over the fields $\tilde\Delta$ and M to a domain of small fluctuations around a *mean-field value* $\tilde\Delta^{(0)},M^{(0)}$ self-consistently determined; furthermore, one neglects fluctuations in the *amplitude* of $\tilde\Delta$. [This is justified in a large-N expansion.[21,22]] Let us denote the phase fluctuations of $\tilde\Delta$ by Θ and the fluctuations of M by Λ.

One assumes that fluctuations of the link variables $E^*_i(\tau)E_j(\tau)$ and $S^*_{i\alpha}(\tau)S_{j\alpha}(\tau)$ around their expectation values in the external fields $\tilde\Delta,M$ are small when $\tilde\Delta$ and M are close to their mean-field values. This permits one to use a Hartree-Fock factorization.

One uses the *Gorkov approximation* to evaluate expectation values of the above link variables in the external fields $\tilde\Delta^{(0)}e^{i\Theta},M^{(0)}+\Lambda$, obtaining

$$\langle\tilde S^*_{i\alpha}(\tau)\tilde S_{j\alpha}(\tau)\rangle_{\tilde\Delta^{(0)}e^{i\Theta},M^{(0)}+\Lambda}\approx\langle\tilde S^*_{i\alpha}(\tau)\tilde S_{j\alpha}(\tau)\rangle_{\tilde\Delta^{(0)},M^{(0)}}e^{i\Theta_{\langle ij\rangle}(\tau)}=J\tilde\Delta^{(0)}_{\langle ij\rangle}e^{i\Theta_{\langle ij\rangle}(\tau)}$$

and, preserving the gauge invariance (5.33),

$$\langle E^*_i(\tau)E_j(\tau)\rangle_{\tilde\Delta^{(0)}e^{i\Theta},M^{(0)}+\Lambda}\approx\langle E^*_i(\tau)E_j(\tau)\rangle_{\tilde\Delta^{(0)},M^{(0)}}e^{i\Theta_{\langle ij\rangle}(\tau)}=\varepsilon_{\langle ij\rangle}e^{i\Theta_{\langle ij\rangle}(\tau)}\ . \quad (5.42)$$

One neglects higher-order terms in the holon density (plausibly justifiable, since the doping δ is assumed to be small).

These approximations yield the following mean-field partition function:

$$\Xi_{\text{MF}}(\beta,\mu)=\int\mathcal{D}E\,\mathcal{D}E^*\,\mathcal{D}\tilde S\,\mathcal{D}\tilde S^*\,\mathcal{D}\Theta\,\mathcal{D}\Lambda e^{-S_{\text{MF}}(E,E^*,\tilde S,\tilde S^*,\Theta,\Lambda)}\prod_{l,\tau}\delta\left[\exp\left[i\sum_{\langle ij\rangle\ni l}(\arg\tilde\Delta^{(0)}_{\langle ij\rangle}+\Theta_{\langle ij\rangle}(\tau))\right]-1\right], \quad (5.43)$$

where

$$S_{\text{MF}}(E,E^*,\tilde S,\tilde S^*,\Theta,\Lambda)=\int_0^\beta d\tau\left\{\sum_j\left[E^*_j(\tau)\frac{\partial}{\partial\tau}E_j(\tau)+\tilde S^*_j(\tau)\frac{\partial}{\partial\tau}\tilde S_j(\tau)\right]\right.$$
$$+\sum_{\langle ij\rangle}[(-t\varepsilon_{\langle ij\rangle}+J\tilde\Delta^{(0)}_{\langle ij\rangle})e^{i\Theta_{\langle ij\rangle}(\tau)}\tilde S^*_i(\tau)\tilde S_j(\tau)-t\tilde\Delta^{(0)*}_{\langle ij\rangle}e^{-i\Theta_{\langle ij\rangle}(\tau)}E^*_i(\tau)E_j(\tau)]$$
$$\left.+\sum_j[(iM^{(0)}_j+i\Lambda_j+\tilde\mu)(E^*_j(\tau)E_j(\tau))+(iM^{(0)}_j+i\Lambda_j)\tilde S^*_j(\tau)\tilde S_j(\tau)]\right\}. \quad (5.44)$$

Note that in (5.44) constraint (b) simply becomes the Coulomb gauge condition for $\arg\Delta = \arg\Delta^{(0)} + \Theta$.

If we are in the *uniform phase* of the slave-boson approach,[21,9] then $\tilde{\Delta}^{(0)}_{\langle ij \rangle} = \tilde{\Delta}^{(0)}$, where $\tilde{\Delta}^{(0)}$ is a real constant. In this case the mean-field action (5.44) describes hard-core bosons (holons) and fermions (spinons) coupled by an Abelian gauge field $(\Theta_{\langle ij \rangle}, \Lambda_j)$. This theory has been argued *not* to have a "permanent confinement" (see Ref. 9). Interesting results for this mean-field model have been obtained in Ref. 23, showing agreement with experimental data of high-T_c materials in the "strange-metal" phase.

Approximations analogous to those leading to (5.43) and (5.44) applied to (5.31) yield a mean-field partition function which is obtained from (5.43) and (5.44) by making the substitutions $E \to \tilde{E}$, $\tilde{S} \to S$, and $\tilde{\Delta} \to \Delta$. This mean-field theory has been discussed in Ref. 24. An interesting result in this approximate theory is that it describes the Néel order in the ground state ($T=0$) of the two-dimensional Heisenberg model, which corresponds to the t-J model with vanishing doping, $\delta = 0$. At zero temperature there is Bose condensation of the spinons in the slave-fermion mean-field theory, as suggested in Ref. 25. Different decouplings of the quartic spinon term, or taking into account different Néel sublattices, provide the mean-field solutions considered in Refs. 6 and 26.

VI. NON-ABELIAN BOSONIZATION OF THE t-J MODEL

We now turn to the non-Abelian bosonization of the t-J model already sketched in Ref. 11. If one starts again from (5.5) and (5.6), using non-Abelian bosonization formulas for $l=0$, instead of (5.9), one obtains

$$\Xi(\beta,\mu) = \left\langle \int \mathcal{D}\Phi\, \mathcal{D}\Phi^*\, e^{-S(\Phi,\Phi^*,B,V)} \right\rangle^{2,1}_{U(1),SU(2)}, \tag{6.1}$$

where the expectation $\langle (\cdot) \rangle^{2,1}_{U(1),SU(2)}$, has been defined in (3.9), (3.7), and (3.6), and $S(\Phi,\Phi^*,B,V)$ is obtained from (5.10) by making the substitutions

$$B_0 \to B_0 + V_0\ , \quad \exp\left[i\int_{\langle ij \rangle} B\right] \to \exp\left[i\int_{\langle ij \rangle} B\right] P\exp\left[i\int_{\langle ij \rangle} V\right], \tag{6.2}$$

where $V = V^a\sigma^a/2$ is an SU(2)-gauge field as in Sec. III.

To exhibit holons and spinons, one again uses a polar decomposition of Φ [see (5.11)] and introduces an auxiliary U(1) lattice scalar field Θ. In Eq. (6.1) one inserts one of the identities

$$\text{(a}'\text{)}\quad \int \mathcal{D}\Theta \prod_{l,\tau} \delta \left[\prod_{\langle ij \rangle : l \in \langle ij \rangle} \left\{ e^{i(\Theta_i(\tau) - \Theta_j(\tau))} \frac{Z^*_{i\alpha}(\tau) \left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} Z_{j\beta}(\tau)}{\left| Z^*_{i\alpha}(\tau) \left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} Z_{j\beta}(\tau)\right|} \right\} - 1 \right] = 1 \tag{6.3}$$

or

$$\text{(b}'\text{)}\quad \int \mathcal{D}\Theta \prod_{l,\tau} \delta \left[\prod_{\langle ij \rangle : l \in \langle ij \rangle} \left\{ e^{i(\Theta_i(\tau) - \Theta_j(\tau))} \frac{Z^*_{i\alpha}(\tau)\exp\left[i\int_{\langle ij \rangle} B(\tau)\right] \left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} Z_{j\beta}(\tau)}{\left| Z^*_{i\alpha}(\tau)\exp\left[i\int_{\langle ij \rangle} B(\tau)\right] \left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} Z_{j\beta}(\tau)\right|} \right\} - 1 \right] = 1\ . \tag{6.4}$$

Following the procedure explained in the previous section, one obtains a formula for the bosonized action, similar to (5.26), in terms of fields E and S:

$$\begin{aligned}
S(E,E^*,S,S^*,M,B,V) &= \int_0^\beta d\tau \Bigg\{ \sum_j \left[E^*_j(\tau)\left[\frac{\partial}{\partial\tau} - iB_0(j,\tau)\right] E_j(\tau) + iB_0(j,\tau) + S^*_{j\alpha}(\tau)\left[\frac{\partial}{\partial\tau} + iV_0(j,\tau)\right]_{\alpha\beta} S_{j\beta}(\tau) \right] \\
&\quad + \sum_{\langle ij \rangle} \Bigg[-tE^*_j(\tau)\exp\left[i\int_{\langle ij \rangle} B(\tau)\right] E_i(\tau) S^*_{i\alpha}(\tau)\left[P\exp\left[\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} S_{j\beta}(\tau) + \text{H.c.} \\
&\qquad + \frac{J}{4}\left| S^*_{i\alpha}(\tau)\left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} S_{j\beta}(\tau)\right|^2 \Bigg] \\
&\qquad + \sum_j iM_j(\tau)(S^*_{j\alpha}(\tau)S_{j\alpha}(\tau) + E^*_j(\tau)E_j(\tau) - 1) + U(1 - E^*(\tau)E(\tau)) \Bigg\}\ .
\end{aligned} \tag{6.5}$$

Let us supplement (6.5) by constraint (a′), i.e.,

$$\prod_{\langle ij \rangle : l \in \langle ij \rangle} \left\{ \frac{S^*_{i\alpha}(\tau)\left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} S_{j\beta}(\tau)}{\left| S^*_{i\alpha}(\tau)\left[P\exp\left[i\int_{\langle ij \rangle} V(\tau)\right]\right]_{\alpha\beta} S_{j\beta}(\tau)\right|} \right\} = 1 \quad \text{for all } l,\tau\ . \tag{6.6}$$

In order to gain a first idea of what this model describes, one tries to again eliminate the term iB_0 in (6.5) by performing a translation similar to (5.30) in the B field. However, this translation is not as innocent as the one made in the previous section, because the coefficient of the U(1) Chern-Simons action is now equal to $1/\pi$ (instead of $1/2\pi$), so that the classical field $B^{\rm cl}$ used in the translation (5.30) has a flux of π per plaquette. More, explicitly, we have that

$$\varepsilon^{\mu\nu}\partial_\mu B^{\rm cl}_\nu(\mathbf{x})=\pi\sum_j \delta(\mathbf{x}-j)$$

and hence

$$\prod_{\langle ij\rangle\in\partial p}\exp\left[i\int_{\langle ij\rangle}B\right]=\exp\left[i\int_p d^2x\,\varepsilon^{\mu\nu}\partial_\mu B_\nu(\mathbf{x})\right]=e^{i\pi}\,. \tag{6.7}$$

As a consequence, the result of the translation (5.30) *cannot* be absorbed in a redefinition of E, as was possible in the previous section. To see the effect of the translation (5.30), it is convenient to partition the sites of the lattice into two sublattices: The first sublattice $L^{(1)}$ consists of all sites whose x coordinate is even; in the second one $L^{(2)}$, the x coordinate of the sites is odd. We choose a gauge for $B^{\rm cl}$ such that

$$\exp\left[i\int_{\langle ij\rangle}B^{\rm cl}\right]=\begin{cases}-1 & \text{if } i,j\in L^{(2)}\\ 1 & \text{otherwise}\,.\end{cases} \tag{6.8}$$

After translating B by $B^{\rm cl}$, as in (5.30) the term iB_0 disappears and the t term in (6.5) can be rewritten as

$$\sum_{\langle ij\rangle}\left[-t_{\langle ij\rangle}E^*_j(\tau)E_i(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)\right]S^*_{i\alpha}\times P\left[\exp\left[i\int_{\langle ij\rangle}V(\tau)\right]\right]_{\alpha\beta}S_{j\beta}+\text{H.c.}\right]\,, \tag{6.9}$$

where

$$t_{\langle ij\rangle}=t\exp\left[i\int_{\langle ij\rangle}B^{\rm cl}\right]\,. \tag{6.10}$$

The field E is coupled to the statistical gauge field B corresponding to a U(1) Chern-Simons action with coefficient $k=2$ (but does not couple to V) and S couples to the statistical gauge field V with SU(2) Chern-Simons action at level $k=1$ (but does not couple to B). Therefore the gauge-invariant nonlocal fields $E_j(\tau)\exp[i\int_{\gamma_j}B(\tau)]$, $P\exp[i\int_{\gamma_j}V(\tau)]$, $S_j(\tau)$ where γ_j is a path starting at j and reaching out to infinity (see Sec. III) are *semion fields*. One may thus interpret the action (6.5) and (6.9) as describing a theory of charged semions, the holons, interacting with spin-$\frac{1}{2}$ semions, the spinons, and interpret the electron as a "composite" of the two. In fact, following the arguments in the Appendix and in Sec. III [see Eq. (3.16)], one finds that (in spin-singlet correlation functions)

$$\Psi_{j\alpha}(\tau)\sim E^*_j(\tau)\exp\left[-i\int_{\gamma_j}B(\tau)\right]\times\left[P\exp\left[i\int_{\gamma_j}V(\tau)\right]\right]_{\alpha\beta}S_{j\beta}(\tau)\,. \tag{6.11}$$

The above ideas on the particle content of the model appear to be realized in the mean-field approximation corresponding to the "generalized flux phase." Introducing an RVB-gauge field Δ to decouple the quartic S term in (6.5) and following the arguments given in the previous sections, one obtains a mean-field action that can be conveniently written introducing the following notation: We denote a link $\langle ij\rangle$ directed in the $\mu=1,2$ direction by $\langle i,\mu\rangle$ and site j, which is a nearest neighbor of i in the μ direction, by $i+\hat{\mu}$. We denote by $E^{(a)}$ the restriction of the field E to the sublattice $L^{(a)}$ for $a=1,2$ [introduced before Eq. (6.8)], and we set

$$\begin{aligned}&\mathbf{E}_i=\begin{bmatrix}E^{(1)}_i\\ E^{(2)}_{i+1}\end{bmatrix}\,,\\ &\mathbf{E}^*_i=(E^{(1)*}_i E^{(2)*}_{i+1})\,,\quad i\in L^{(1)}\,,\\ &\mathbf{E}_i=\begin{bmatrix}E^{(1)}_{i+1}\\ E^{(2)}_i\end{bmatrix}\,,\\ &\mathbf{E}^*_i=(E^{(1)*}_{i+1}E^{(2)*}_i)\,,\quad i\in L^{(2)}\,.\end{aligned} \tag{6.12}$$

Furthermore, we define

$$e^{i\mathbf{B}_{\langle i,\mu\rangle}(\tau)}=\begin{bmatrix}\exp\left[i\int_{\langle i,\mu\rangle}B(\tau)\right] & 0\\ 0 & \exp\left[i\int_{\langle i+\hat{1},\mu\rangle}B(\tau)\right]\end{bmatrix}\,, \tag{6.13}$$

$$\mathbf{B}_0(j,\tau)=\begin{bmatrix}B_0(j,\tau) & 0\\ 0 & B_0(j+\hat{1},\tau)\end{bmatrix}\,, \tag{6.14}$$

and use similar definitions for $e^{i\boldsymbol{\Theta}_{\langle i,\mu\rangle}(\tau)}$, $\boldsymbol{\Lambda}_j(\tau)$. Finally, we introduce the "Dirac" matrices

$$\alpha_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\,,\quad \alpha_2=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\,. \tag{6.15}$$

Using the new notation just introduced, the mean-field action is rewritten as

$$S_{\mathrm{MF}}(\mathbf{E},\mathbf{E}^*,S,S^*,\Theta,\Lambda,B,V)$$
$$=\int_0^\beta d\tau\left\{\sum_{j\in L^{(1)}}\left[\mathbf{E}_j^*(\tau)\left[\frac{\partial}{\partial\tau}-i\mathbf{B}_0(j,\tau)\right]\mathbf{E}_j(\tau)+\mathbf{E}_j^*(\tau)(iM^{(0)}+i\Lambda_j(\tau)+\tilde{\mu})\mathbf{E}_j(\tau)\right]\right.$$
$$+\sum_{\mu=1,2}[-t\Delta^{(0)}_{\langle j,\mu\rangle}\mathbf{E}^*_{j+\mu}(\tau)\alpha_\mu e^{i\mathbf{B}_{\langle j,\mu\rangle}(\tau)}e^{i\Theta_{\langle j,\mu\rangle}(\tau)}\mathbf{E}_j(\tau)+\text{H.c.}]$$
$$+\sum_j\left[S^*_{j\alpha}(\tau)\left[\frac{\partial}{\partial\tau}+iV_0(j,\tau)\right]_{\alpha\beta}S_{j\beta}(\tau)+(iM^{(0)}+i\Lambda_j(\tau))(S^*_j(\tau)S_j(\tau))\right]$$
$$\left.+\sum_{\langle ij\rangle}\left[(-t\varepsilon_{\langle ij\rangle}+J\Delta^{(0)}_{\langle ij\rangle})e^{i\Theta_{\langle ij\rangle}(r)}\left[S^*_{i\alpha}(\tau)\left[P\exp\left[i\int_{\langle ij\rangle}V(\tau)\right]\right]_{\alpha\beta}S_{j\beta}(\tau)\right]+\text{H.c.}\right]\right\}+2S_{\mathrm{CS}}(B)+S_{\mathrm{CS}}(V)\ , \tag{6.16}$$

where

$$\varepsilon_{\langle ij\rangle}=\left\langle E_i^*(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)+B^{\mathrm{cl}}\right]E_j(\tau)\right\rangle_{\Delta^{(0)},M^{(0)}}\ , \tag{6.17}$$

and the remaining notations are as in (5.45), except for some obvious changes.

Suppose that $\Delta^{(0)}_{\langle i,\mu\rangle}$ in (6.16) is a real constant. (As we shall see, this corresponds to a "generalized flux phase."[22,27]) If the gauge field (Θ,Λ) is not confining, as argued in Ref. 8, the particlelike excitations described by the mean-field theory (6.12) are "Dirac" holons with semion statistics and semionic spinons, coupled to each other by the Abelian gauge field (Θ,Λ).

Remark 6.1. The *real-time* field equation for the field $\mathbf{E}$ obtained from the action (6.16) by varying with respect to $\mathbf{E}^*$ is given by

$$\left[i\frac{\partial}{\partial\tau}-\mathbf{A}_0(j,\tau)\right]E_j(\tau)=-t\Delta^{(0)}\sum_{\mu=1}^{2}\alpha_\mu\{e^{i\mathbf{A}_{\langle j,\mu\rangle}(\tau)}\mathbf{E}_{j-\mu}(\tau)+e^{-i\mathbf{A}_{\langle j,\mu\rangle}(\tau)}\mathbf{E}_{j+\mu}(\tau)\}+(M^{(0)}+\tilde{\mu})\mathbf{E}_j(\tau)\ ,$$

where

$$\mathbf{A}_0(j,\tau)=\mathbf{B}_0(j,\tau)+\mathbf{\Lambda}_j(\tau)$$

and

$$\mathbf{A}_{\langle j,\mu\rangle}(\tau)=\mathbf{B}_{\langle j,\mu\rangle}(\tau)+\mathbf{\Theta}_{\langle j,\mu\rangle}(\tau)\ .$$

The last term just shifts all eigenvalues by an amount $M^{(0)}+\tilde{\mu}$ and hence can be omitted. The resulting equation is reminiscent of a discretized, covariant Dirac equation. When the gauge field $\mathbf{A}$ is turned off it describes particlelike excitations with the following dispersion law:

$$\varepsilon(q)=\pm2t\Delta^{(0)}[(\cos q_1)^2+(\cos q_2)^2]^{1/2}\ ;$$

i.e., ε vanishes "linearly" at $q_1=\pm\pi/2$, $q_2=\pm\pi/2$. In other words, E describes four "Dirac particles."

If we choose to supplement (6.5) with constraint (b') [see (6.4)], then, by arguments similar to those used for constraint (b) in the Abelian bosonization of the previous section, one obtains an action expressed in terms of a field E and a field S, which is now coupled to both gauge field V and B. Now one can use the non-Abelian bosonization formula in reverse order, integrating over B and V, to convert S to a Grassmann field $\tilde{S}$. The action one obtains is exactly the same as the one obtained with constraint (b) in the Abelian bosonization formalism of Sec. V. It is useful to note that the RVB-gauge field Δ introduced when constraint (a′) is used (semion representation) is related to the RVB-gauge field $\tilde{\Delta}$ used in connection with constraint (b′) through the equation

$$\tilde{\Delta}_{\langle ij\rangle}(\tau)=\Delta_{\langle ij\rangle}(\tau)\exp\left[i\int_{\langle ij\rangle}B(\tau)+B^{\mathrm{cl}}\right]\ . \tag{6.18}$$

In the generalized flux phase of the slave-boson formalism, the mean field $\tilde{\Delta}^{(0)}$ has a phase satisfying[27]

$$\sum_{\langle ij\rangle\in\partial p}\arg\tilde{\Delta}^{(0)}_{\langle ij\rangle}=\pi(1-\delta)\ , \tag{6.19}$$

while, in the mean-field theory based on constraint (a') (semion picture),

$$\sum_{\langle ij\rangle\in\partial p}\int_{\langle ij\rangle}(B+B^{\mathrm{cl}})=\pi(1-\delta)\ . \tag{6.20}$$

Therefore the mean-field solution $\tilde{\Delta}^{(0)}$ of the generalized flux phase in the slave-boson formalism corresponds in the semionic picture to a mean-field solution where $\Delta^{(0)}$ is a real constant.

Since $\arg\tilde{\Delta}^{(0)}$ exhibits frustration, whereas $\arg\Delta^{(0)}$ does not, one expects that the particle content of the theory in a generalized flux phase is well described by the semionic picture, as argued by Laughlin.[14]

In this paper we have discussed a systematic derivation of the slave-fermion, the slave-boson, and the semion representations of the two-dimensional t-J model, using Abelian and non-Abelian bosonization of interacting fermion systems and coherent-state methods. Since bosonization is limited to one and two space dimensions, our derivation does not generalize to the three- or higher-dimensional t-J model. But we regard this rather as a virtue than as shortcoming of our derivation, because it makes clear what is special about the t-J model in one or two dimensions.

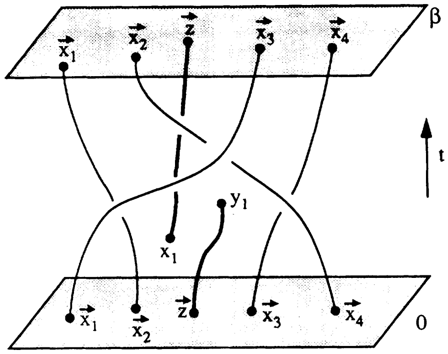

FIG. 3. $\omega,\tilde{\omega}$ for $n=1$, $N=4$, $l_1=1$.

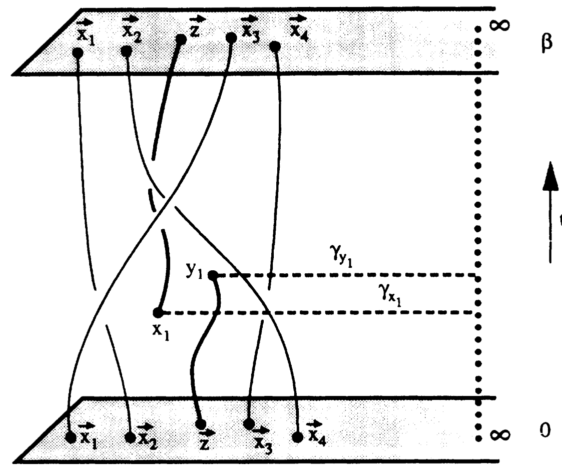

FIG. 4. Dotted line denotes the compensating current at infinity; the $t=\beta$ and 0 planes are identified.

In the literature the slave-fermion, slave-boson, and semion representations have been used as starting points for mean-field approximations to the t-J model, which one hopes describe the main features of different phases of this model. The status of these mean-field approximations, in particular their stability against fluctuations, remains to be understood more clearly.

ACKNOWLEDGMENTS

P.-A. Marchetti thanks N. Nagaosa, F. Toigo, and R. Hlubina for useful discussions.

APPENDIX

In this appendix we present a generalization of the Feynman-Kac formula (2.4), due to Ginibre and Gruber.[28] This formula permits us to prove identities (3.15) and (3.16). Let $T(\cdot)$ denote time ordering. Then, for $x_i^0<x_{i+1}^0, y_i^0<y_{i+1}^0$, $i=1,\dots,n-1$,

$$\begin{aligned}&\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}(x_n)\Psi_{\delta_1}(y_1)\cdots\Psi_{\delta_n}(y_n))\rangle^{\Psi}\\ &=\Xi(\beta,\mu|eA)^{-1}\Bigg\{\sum_{\pi\in\Sigma_n}\varepsilon^{\sigma(\pi)}\prod_{k=1}^{n}\Bigg[\sum_{l_k=0,1,\dots}\Theta(y^0_{\pi(k)}+l_k\beta-x^0_k)\Bigg]\cdot\\ &\qquad\times\int_{\substack{\omega_k(x_k^0)=\mathbf{x}_k\\ \omega_k(y^0_{\pi(k)}+l_k\beta)=\mathbf{y}_{\pi(k)}}}\prod_{k=1}^{n}\Bigg[\mathcal{D}\omega_k\exp\Bigg[-\int_{x_k^0}^{y^0_{\pi(k)}+l_k\beta}dt\,\frac{m}{2}\dot{\omega}_k^2(t)\Bigg]\Bigg]\\ &\qquad\times\prod_k\Bigg[\exp\Bigg[ie\int_{\omega_k}A\Bigg]e^{\mu(y^0_{\pi(k)}+l_k\beta-x^0_k)}\Bigg]G(\omega_1,\dots,\omega_n|eA)\prod_k\delta_{\alpha_k\alpha_{\pi(k)}}\Bigg\}\,,\end{aligned}\tag{A1}$$

where

$$\begin{aligned}G(\omega_1,\dots,\omega_n|eA)=&\sum_{N=0}^{\infty}\frac{1}{N!}e^{\beta\mu N}\sum_{\tilde{\pi}\in\Sigma_N}\varepsilon^{\sigma(\tilde{\pi})}\int d^2\tilde{x}_1\cdots d^2\tilde{x}_N\sum_{\tilde{\alpha}_1,\dots,\tilde{\alpha}_N}\int_{\substack{\tilde{\omega}_j(0)=\tilde{\mathbf{x}}_j\\ \tilde{\omega}_j(\beta)=\tilde{\mathbf{x}}_{\tilde{\pi}(j)}}}\prod_{j=1}^{N}\mathcal{D}\tilde{\omega}_j\\ &\times\exp\Bigg\{-\int_0^{\beta}dt\Bigg[\frac{m}{2}\sum_j\dot{\tilde{\omega}}_j^2(t)+U(\underline{\tilde{\omega}},\omega_1,\dots,\omega_n)(t)\Bigg]\Bigg\}\prod_j\exp\Bigg[i\int_{\tilde{\omega}_j}A\Bigg]\delta_{\tilde{\alpha}_j\tilde{\alpha}_{\tilde{\pi}(j)}}\,.\end{aligned}\tag{A2}$$

In (A1) and (A2), $\varepsilon=+1$ for bosons and $\varepsilon=-1$ for fermions, periodic boundary conditions are imposed in the time direction; Θ in (A1) is the Heaviside step function; $U(\underline{\tilde{\omega}},\omega_2\cdots\omega_n)(t)$ in (A2) is the classical potential energy corresponding to a two-body potential u of a configuration of point particles, obtained by intersecting the paths $\tilde{\omega}_1,\dots,\tilde{\omega}_N,\omega_1,\dots,\omega_n$ with the plane at time t. Note that the paths $\omega_1,\dots,\omega_n$ may intersect this plane several times, in which case one associates a point particle with *every* intersection point. The number of such intersection points increases (decreases) by 1 whenever t passes through one of the times $x_k^0(y_k^0)$ (see Fig. 3).

The key difference with respect to (2.4) in the somewhat lengthy formulas (A1) and (A2) is the appearance of n new paths ω_k, $k=1,\dots,n$, starting at times $x_k^0\in[0,\beta]$ at the points $\mathbf{x}_k$ and ending, after wrapping $l_k=0,1,\dots$ times around the circle of circumference β in the time direction, in the point $\mathbf{y}_k$ at time $y_k^0 \in [0,\beta]$. If we apply formulas (A1) and (A2) to the left-hand side of (3.15), and (3.16), the lines $\gamma_{x_k} \cup \omega_k \cup \gamma_{y_{\pi(k)}}$ together with the world line of the current at infinity form a loop [more precisely a "ribbon graph" (see Fig. 4)]. This permits us to apply the bosonization arguments of Ref. 11.

[1]J. Bednorz and K. A. Müller, Z. Phys. B **64**, 189 (1986).

[2]T. M. Rice, F. Mila, and F. C. Zhang, Philos. Trans. R. Soc. London A **334**, 459 (1991), and references therein.

[3]F. C. Zhang and T. M. Rice, Phys. Rev. **37**, 3754 (1988).

[4]P. W. Anderson, Science **235**, 1196 (1987).

[5]P. W. Anderson, in *Frontiers and Borderlines in Many-Particle Physics,* edited by R. A. Broglia *et al.* (North-Holland, Amsterdam, 1988).

[6]D. Yoshioka, J. Phys. Soc. Jpn. **58**, 1516 (1989), and references therein.

[7]Z. Zou and P. W. Anderson, Phys. Rev. B **37**, 5594 (1988).

[8]G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. **63**, 973 (1987).

[9]L. B. Ioffe and A. I. Larkin, Phys. Rev. B **39**, 8988 (1990).

[10]G. W. Semenoff, Phys. Rev. Lett. **61**, 517 (1988); A. M. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988); M. Lüscher, Nucl. Phys. **B326**, 557 (1989); E. Fradkin, Phys. Rev. B **42**, 570 (1990); see also J. Fröhlich and P. A. Marchetti, Lett. Math. Phys. **16**, 347 (1988).

[11]J. Fröhlich, T. Kerler, and P. A. Marchetti, Nucl. Phys. B **374**, 511 (1992).

[12]T. Holstein and H. Primakoff, Phys. Rev. **38**, 1098 (1940).

[13]R. B. Laughlin, Science **242**, 525 (1988).

[14]R. B. Laughlin, Int. J. Mod. Phys. B **5**, 1507 (1991).

[15]*Fractional Statistics and Anyon Superconductivity,* edited by F. Wilczek (World Scientific, Singapore, 1990).

[16]See, e.g., J. W. Negele and H. Orland, *Quantum Many Particle Systems* (Addison-Wesley, New York, 1988).

[17]E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[18]See the last two references in Ref. 10.

[19]T. Matsubara and H. Matsuda, Prog. Theor. Phys. **16**, 659 (1956); J. Ginibre, Commun. Math. Phys. **10**, 140 (1968).

[20]J. R. Klauder, Ann. Phys. (N.Y.) **11**, 123 (1960); Phys. Rev. D **19**, 2349 (1979).

[21]I. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).

[22]P. B. Wiegmann, Phys. Rev. Lett. **65**, 2070 (1990).

[23]N. Nagaosa and P. A. Lee, Phys. Rev. Lett. **64**, 2450 (1990); Phys. Rev. B **43**, 1233 (1991); L. B. Ioffe and G. Kotliar, *ibid.* **42**, 10 348 (1990).

[24]L. B. Ioffe, and P. B. Wiegmann, Phys. Rev. Lett. **65**, 653 (1990).

[25]D. Yoshioka, J. Phys. Soc. Jpn. **58**, 32 (1989).

[26]B. I. Schraiman and E. D. Siggia, Phys. Rev. B **40**, 9162 (1990).

[27]J. P. Rodriguez and B. Doucot, Europhys. Lett. **11**, 451 (1990); see also P. W. Anderson, B. S. Shastry, and D. Hristopoulos, Phys. Rev. B **40**, 8939 (1989); P. Lederer, D. Poilblanc, and T. M. Rice, Phys. Rev. **63**, 1519 (1989); Y. Hasegawa, O. Narikiyo, K. Kuboki, and H. Fukuyama, J. Phys. Soc. Jpn. **59**, 822 (1990).

[28]J. Ginibre, Commun. Math. Phys. **10**, 140 (1968); J. Ginibre and C. Gruber, *ibid.* **11**, 198 (1968).

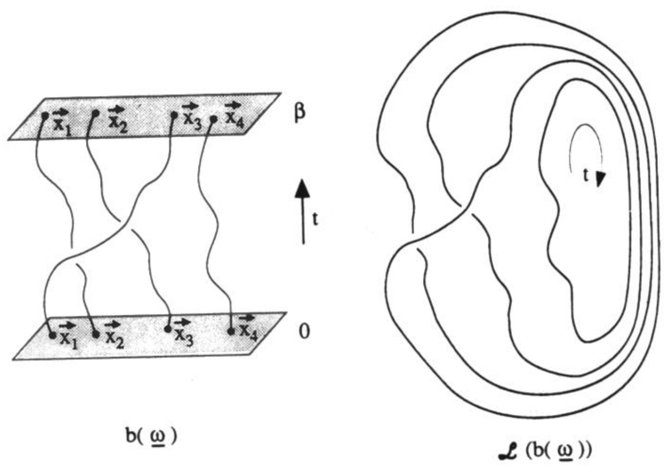

FIG. 2. Heavy lines describe the set of paths $\underline{\omega}$ for $N=4$; $\mathcal{L}(b(\underline{\omega}))$ is obtained identifying the $t=\beta$ and 0 planes.

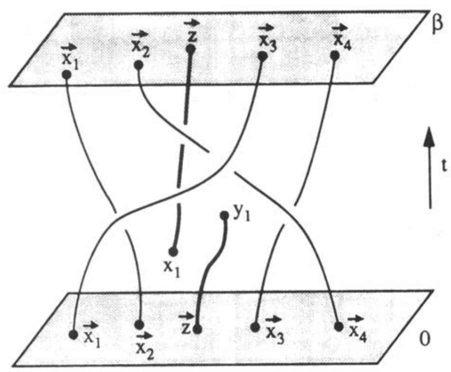

FIG. 3. $\underline{\omega}, \underline{\tilde{\omega}}$ for $n=1$, $N=4$, $l_1=1$.

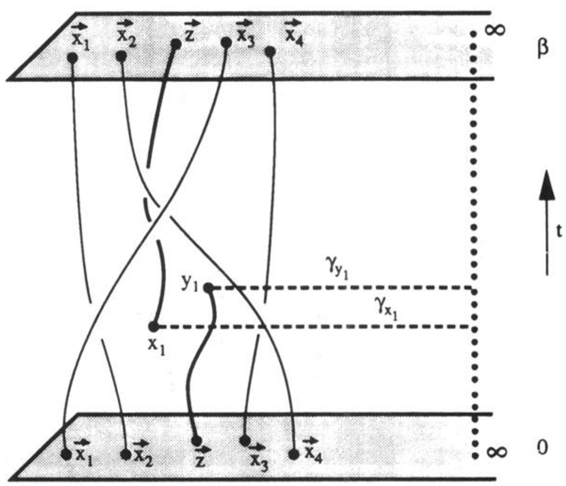

FIG. 4. Dotted line denotes the compensating current at infinity; the $t=\beta$ and 0 planes are identified.