Depression of the superfluid transition in ^{4}He: Renormalization-group theory

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(Received 6 April 1992)

We study superfluid ^{4}He near T_λ in a homogeneous metastable state where a finite superfluid velocity v_s is present. Neglecting vortices we perform a renormalization-group calculation of the superfluid current $J_s(v_s)$ and determine the critical velocity $v_{sc}(T)$ at which the superfluid state becomes unstable. We apply this result to the situation where the superfluid velocity is induced by a finite heat current Q. A critical heat current $Q_c(T)$ corresponding to $v_{sc}(T)$ is found which implies a transition temperature $T_\lambda(Q)=T_\lambda[1-A_0Q^x]$ that is lower than T_λ. We determine the exact exponent $x=[(d-1)\nu]^{-1}\approx 0.744$ in $d=3$ dimensions and calculate A_0 in one-loop order. Our results for A_0 and x are compared with recent experimental data on the depression $T_\lambda-T_\lambda(Q)$ of the superfluid transition temperature.

I. INTRODUCTION

According to two-fluid hydrodynamics,[1] the flow in superfluid ^{4}He can be characterized by two velocities v_s and v_n of the superfluid and normal component, respectively. The superfluid current J_s at the temperature $T<T_\lambda$ can exist only at velocities v_s smaller than a certain critical velocity $v_{sc}(T)$ (for reviews see, e.g., Refs. 2 and 3). (More precisely, the relevant quantity is the relative velocity v_s-v_n but for simplicity we assume $v_n=0$ in the following.) The state at finite $v_s<v_{sc}(T)$ may be interpreted as a metastable state, since it remains stable over a long period of time due to large potential barriers associated with the creation of vortices.[2-6] If, at constant v_s, the temperature T is raised the superfluid current J_s is destabilized at some (perhaps sharply defined) temperature $T_c(v_s)$ that is lower than the usual transition temperature $T_\lambda=T_c(0)$. In the $T-v_s$ plane this yields a boundary $T_c(v_s)$ that can be identified as the line $v_{sc}(T)$ of critical velocities[3] (Fig. 1). On the basis of phenomenological considerations it has been argued[5] that near T_λ the asymptotic temperature dependence is

$$v_{sc}(T)\approx A_{sc}[(T_\lambda-T)/T_\lambda]^\nu , \tag{1.1}$$

and thus

$$T_\lambda-T_c(v_s)\approx T_\lambda A_{sc}^{-1/\nu}v_s^{1/\nu} , \tag{1.2}$$

with an exponent $\nu\approx\frac{2}{3}$ in good agreement with experiments on persistent superfluid currents.[7] No reliable quantitative estimate is available so far for the amplitude A_{sc}. Clearly the effect of critical fluctuations on $T_c(v_s)$ or $v_{sc}(T)$ is non-negligible and should be treated by means of the renormalization group (RG). It is remarkable that up to now there exists no RG calculation on this interesting problem (as far as we know). In this paper we shall present such a calculation.

Apart from persistent-current methods[7] a superfluid (counterflow) velocity v_s (more precisely v_s-v_n) can also be induced by a heat current Q.[1] Therefore, one expects, in accordance with phenomenological theories,[8-15] that in the presence of a finite heat current Q the superfluid transition occurs at some temperature $T_\lambda(Q)$ below the ordinary transition temperature $T_\lambda(0)\equiv T_\lambda$. In the T-Q plane this yields a line $T_\lambda(Q)$ (Fig. 2), which under ideal circumstances, should be closely related to $T_c(v_s)$. Near $T_\lambda(Q)$ the heat transport depends on Q in a nonlinear fashion. Recently we have studied the nonlinear region (Fig. 2) of normal-fluid ^{4}He.[16,17] The main objective of our present work is to extend this study to the nonlinear region of superfluid ^{4}He. In particular we wish to calculate the depression $T_\lambda-T_\lambda(Q)$ of the superfluid transition temperature.

In early experimental work[18-21] a depression $T_\lambda-T_\lambda(Q)$ of the superfluid transition temperature by a heat current Q was indeed reported but an unambiguous interpretation of these measurements remained difficult.[15,22-24] Recently a depression was measured[25] in a range of sufficiently small Q where perturbing effects

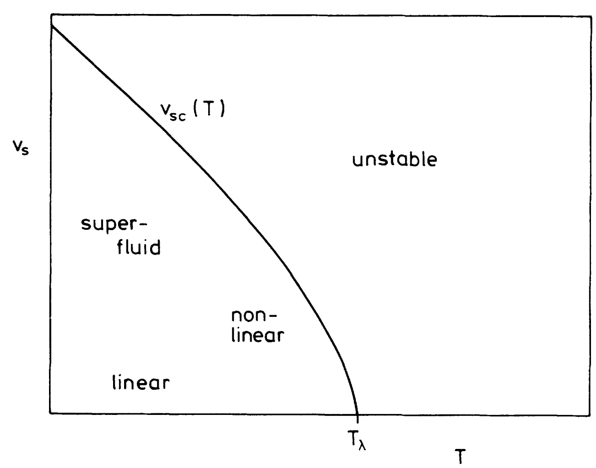

FIG. 1. Schematic plot of the critical superfluid velocity v_{sc} (solid line) as a function of temperature according to (1.1). In the small-v_s region the superfluid current J_s is a linear function of v_s. Near $v_{sc}(T)$, the v_s dependence of J_s becomes nonlinear. Above $v_{sc}(T)$ the superfluid phase is unstable.

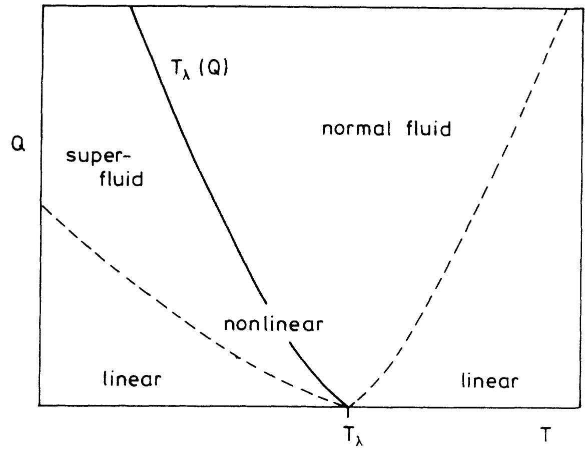

FIG. 2. Schematic plot of the transition temperature $T_\lambda(Q)$ (solid line) in the presence of a heat current Q. Above $T_\lambda(Q)$ the superfluid phase of ^{4}He becomes unstable. The normal-fluid state at finite Q is inhomogeneous with a finite temperature gradient. The dashed lines (not sharply defined) indicate where the small-Q region crosses over to the nonlinear region, compare Fig. 1 of Ref. 16. In the nonlinear region the superfluid density below $T_\lambda(Q)$ and the local thermal conductivity above $T_\lambda(Q)$ become Q dependent (Ref. 17).

due to thermal gradients in the superfluid were negligible. Quantitative data for $T_\lambda - T_\lambda(Q)$ were obtained, which could be represented as[25]

$$[T_\lambda - T_\lambda(Q)]/T_\lambda = A_0 Q^x \tag{1.3}$$

with an effective exponent $x = 0.813 \pm 0.012$. This appeared to be in good agreement with the effective exponent $x \approx 0.80$ obtained[25] from a prediction by Onuki[14] but disagreed[24,26] with a subsequent modification[15,27] $x = 3/4$ of this prediction.

Obviously critical fluctuations play a non-negligible role in the vicinity of the point $T = T_\lambda$, $Q = 0$ in the $Q - T$ plane from where the line $T_\lambda(Q)$ emerges (Fig. 2). While earlier theories[8–13] neglected critical fluctuations altogether, an attempt was made by Onuki[14,15] to anticipate their possible influence on $T_\lambda(Q)$ at the level of mean-field and scaling considerations. Clearly a more fundamental treatment of these fluctuations is desirable in order to see whether the scaling considerations can be justified and what is the quantitative effect of the fluctuations on the depression of the superfluid transition. In this paper we present a renormalization-group treatment of this problem.

In the following we give a brief outline of our approach. Consider the equation for heat conduction in the stationary state of liquid ^{4}He (Refs. 16, 17, and 28)

$$\lambda_0 \frac{\partial T(z,Q)}{\partial z} + g_0 J_s + Q/k_B T_\lambda = 0 , \tag{1.4}$$

$$J_s = \mathrm{Im}\left\langle \psi^*(\mathbf{x}) \frac{\partial}{\partial z} \psi(\mathbf{x}) \right\rangle , \tag{1.5}$$

where g_0 is the dynamic coupling of model F.[29,30] Equations (1.4) and (1.5) determine the temperature profile $T(z,Q)$ in the presence of a finite stationary heat current Q (in the z direction). The average in (1.5) is defined by the dynamic statistical weight of model F, where the complex field $\psi(\mathbf{x})$ is a (conveniently normalized) effective wave function[8] of the Bose condensate.[31] We shall neglect the effect of vortices, which implies $\partial T/\partial z = 0$ below $T_\lambda(Q)$, hence we obtain the relation

$$Q = -g_0 k_B T_\lambda J_s . \tag{1.6}$$

Our strategy is to perform a RG calculation of $J_s = J_s(v_s, T)$ at given v_s and T and to determine a critical velocity $v_{sc}(T)$ beyond which a homogeneous current $J_s(v_s, T)$ becomes unstable. The corresponding critical current $J_{sc}(T) = J_s(v_{sc}, T)$ determines a critical heat current $Q_c = Q_c(T)$ according to (1.6),

$$Q_c(T) = -g_0 k_B T_\lambda J_{sc}(T) , \tag{1.7}$$

which can then be inverted to obtain $T_\lambda(Q)$.

In Sec. II the statistical model and the mean-field results for J_s and v_{sc} are presented. The free energy of the metastable state is defined in Sec. III, and the contributions of the fluctuations to the order parameter and the superfluid current are calculated up to one-loop order. In Sec. IV the field-theoretic RG approach[32–36] is employed to determine the effect of the critical fluctuations in d dimensions. Our results can be expressed in terms of the effective parameters[37] that are known from critical statics of ^{4}He at $Q = 0$. In Sec. V we discuss our results with respect to the instability at the critical velocity $v_{sc}(T)$ and present a quantitative prediction on the corresponding depression of the superfluid transition temperature $T_\lambda(Q)$. As an exact result for the exponent x in (1.3) in d dimensions we find

$$x = \frac{1}{\nu(d-1)} , \tag{1.8}$$

where ν is the correlation-length exponent. We also compare our results with previous measurements[7,25] of $v_{sc}(T)$ and $T_\lambda(Q)$ and predict the temperature dependence of the superfluid density near $v_{sc}(T)$.

II. STATISTICAL MODEL AND MEAN-FIELD THEORY

We consider superfluid ^{4}He at a given temperature $T < T_\lambda$ in a homogeneous state at a given superfluid velocity $\mathbf{v}_s$. As noted above, this homogeneous state constitutes a metastable state that can be considered as a quasiequilibrium state if vortex configurations are neglected. A statistical description of this state has been suggested,[5,38] in analogy to the equilibrium state at $\mathbf{v}_s = 0$, in terms of the probability distribution

$$p\{\psi(\mathbf{x})\} \sim \exp - H\{\psi(\mathbf{x})\} , \tag{2.1}$$

for a (conveniently normalized) effective wave function[8] $\psi(\mathbf{x})$ of the Bose condensate.[31] The fluctuations are assumed to be restricted to some small neighborhood of a locally stable minimum of $H\{\psi\}$ (see also Sec. III). Near T_λ the functional $H\{\psi\}$ is identified with the free energy (divided by $k_B T$) of the phenomenological Ginzburg-Landau-Pitaevskii theory,[8,9,39,40]

$$H\{\psi(\mathbf{x})\}=\int d^d x\left[\frac{r_0}{2}|\psi|^2+\frac{1}{2}|\nabla\psi|^2+u_0|\psi|^4\right], \tag{2.2}$$

where $r_0(T)$ depends linearly on $T_\lambda - T$. The main quantity of interest is the superfluid current density, which is proportional to

$$\mathbf{J}_s = \mathrm{Im}\langle \psi(\mathbf{x})^* \nabla\psi(\mathbf{x})\rangle . \tag{2.3}$$

For $\mathbf{v}_s \neq 0$ this theory was previously[3,8-15] treated in the mean-field approximation where only the configuration $\psi_{\mathrm{mf}}(\mathbf{x})$ at a local minimum of H was considered,

$$\left.\frac{\delta H}{\delta\psi}\right|_{\psi=\psi_{\mathrm{mf}}(\mathbf{x})}=0 , \tag{2.4}$$

with the solution

$$\psi_{\mathrm{mf}}(\mathbf{x})=\eta_{\mathrm{mf}}(r_0,k)\exp i\mathbf{kx} , \tag{2.5}$$

$$\eta_{\mathrm{mf}}(r_0,k)^2=-\frac{r_0+k^2}{4u_0} , \tag{2.6}$$

$$\mathbf{v}_s=\frac{\hbar}{m}\mathbf{k} . \tag{2.7}$$

In this approximation the superfluid current density is given by

$$\begin{aligned}\mathbf{J}_s^{\mathrm{mf}}(r_0,\mathbf{k})&=\mathrm{Im}[\psi_{\mathrm{mf}}(\mathbf{x})^*\nabla\psi_{\mathrm{mf}}(\mathbf{x})]\\ &=\eta_{\mathrm{mf}}(r_0,k)^2\mathbf{k} , &(2.8)\\ &=V^{-1}\partial H\{\psi_{\{}\mathrm{mf}\}/\partial\mathbf{k} , &(2.9)\end{aligned}$$

where V is the volume of the system. The mean-field solution is thermodynamically stable only if the free energy $H\{\psi_{\mathrm{mf}}\}$ satisfies

$$\partial^2 H\{\psi_{\mathrm{mf}}\}/\partial k^2>0 . \tag{2.10}$$

Because of (2.9) this implies

$$\frac{\partial}{\partial k}|\mathbf{J}_s^{\mathrm{mf}}|=-\frac{r_0+3k^2}{4u_0}>0 . \tag{2.11}$$

Thus an instability arises at the critical wave number[3,8-15]

$$k_c^{\mathrm{mf}}=\frac{1}{\sqrt{6}}(-2r_0)^{1/2} \tag{2.12}$$

corresponding to the critical velocity $v_{sc}^{\mathrm{mf}}=(\hbar/m)k_c^{\mathrm{mf}}$. We note that the mean-field functions (2.6) and (2.8) do not show any singular behavior at $k=k_c^{\mathrm{mf}}$. It will be interesting to see (in Sec. V A) how the fluctuations affect the critical wave number and the analytic structure of $\langle\psi\rangle$ and J_s near the instability.

III. FREE ENERGY AND BARE PERTURBATION THEORY

In order to take critical fluctuations into account we shall perform a perturbation expansion around the mean-field solution (2.5)–(2.7) at given r_0 and $\mathbf{k}$ (or $\mathbf{v}_s$). For this purpose it will be convenient to express the model (2.1), (2.2) in terms of the two-component vector

$$\underline{\varphi}(\mathbf{x})=\begin{bmatrix}\varphi_1(\mathbf{x})\\ \varphi_2(\mathbf{x})\end{bmatrix}, \tag{3.1}$$

where the real fields φ_1 and φ_2 are defined by

$$\psi(\mathbf{x})=[\varphi_1(\mathbf{x})+i\varphi_2(\mathbf{x})]\exp i\mathbf{kx} \tag{3.2}$$

at given $\mathbf{k}$. Then the Hamiltonian (2.2) and the superfluid current (2.3) can be expressed as

$$\begin{aligned}H\{\underline{\varphi}(\mathbf{x}),\mathbf{k}\}=\int d^d x\{&\tfrac{1}{2}r_0\underline{\varphi}\cdot\underline{\varphi}+\tfrac{1}{2}[(\nabla+\mathbf{Ik})\underline{\varphi}]\cdot[(\nabla+\mathbf{Ik})\underline{\varphi}]\\ &+u_0(\underline{\varphi}\cdot\underline{\varphi})^2\} , &(3.3)\end{aligned}$$

$$\mathbf{J}_s(r_0,\mathbf{k})=\langle\underline{\varphi}(\mathbf{x})[(\mathbf{k}-\mathbf{I}\nabla)\underline{\varphi}(\mathbf{x})]\rangle , \tag{3.4}$$

with the antisymmetric matrix

$$I=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{3.5}$$

In (3.3) the differential operator ∇ acts on $\underline{\varphi}(\mathbf{x})$ only within a square bracket.

In order to calculate averages such as $\langle\underline{\varphi}\rangle$ and $\mathbf{J}_s$, we introduce the free energy

$$F\{r_0,k,\underline{h}(\mathbf{x})\}=-\ln\int D\underline{\varphi}\exp-\left[H\{\underline{\varphi}(\mathbf{x}),\mathbf{k}\}-\int d^d x\underline{\varphi}(\mathbf{x})\underline{h}(\mathbf{x})\right] . \tag{3.6}$$

From (3.6) we obtain

$$\langle\underline{\varphi}\rangle=\begin{bmatrix}\eta(r_0,k)\\ 0\end{bmatrix}=\left.\frac{\delta F}{\delta\underline{h}(\mathbf{x})}\right|_{\underline{h}=0} . \tag{3.7}$$

and one easily verifies

$$\mathbf{J}_s(r_0,\mathbf{k})=V^{-1}\partial F\{r_0,k,0\}/\partial\mathbf{k} , \tag{3.8}$$

where V is the volume of the system. Since we are interested in the description of a metastable state at $\mathbf{k}\neq 0$ the functional integration in (3.6) must be restricted in the sense that $\int D\underline{\varphi}\cdots$ denotes an integration over the space of functions $\underline{\varphi}(\mathbf{x})$ only in some neighborhood around the local minimum (2.4)–(2.6). The allowed fluctuations $\underline{\varphi}(\mathbf{x})-\underline{\varphi}_{\mathrm{mf}}$ at given $\mathbf{k}$ should not pass beyond the barrier between neighboring uniform states with different $\mathbf{k}$ as discussed in detail in Refs. 5 and 38. This excludes vortex configurations of $\psi(\mathbf{x})$. In practice, perturbation theory around $\underline{\varphi}_{\mathrm{mf}}$ will provide an operational definition of the functional integral (3.6). This will lead to an order parameter of plane-wave structure,

$$\langle\psi(\mathbf{x})\rangle=\eta(r_0,k)\exp i\mathbf{kx} , \tag{3.9}$$

corresponding to a uniform current-carrying state with the superfluid velocity $\mathbf{v}_s$ given by (2.7). Equation (3.9) has the same spatial dependence as $\psi_{\rm mf}(\mathbf{x})$, (2.5), but with a different amplitude $\eta \neq \eta_{\rm mf}$ due to the effect of the fluctuations.

For the purpose of a perturbation calculation it will be convenient to turn to a thermodynamic potential that depends on $\langle \underline{\varphi} \rangle$. This is achieved by the Legendre transform

$$\hat{\Gamma}\{\langle \underline{\varphi}(\mathbf{x})\rangle, r_0, k\} = F\{r_0, k, \underline{h}(\mathbf{x})\} + \int d^dx \, \langle \underline{\varphi}(\mathbf{x})\rangle \cdot \underline{h}(\mathbf{x}) \; , \tag{3.10}$$

which is the generating functional of vertex functions. We shall consider (3.10) only in the physical case $\underline{h} \to 0$, where $\langle \underline{\varphi} \rangle$ is of the form (3.7). Using the notation

$$\Gamma(\eta, r_0, k) \equiv \hat{\Gamma}\left\{ \begin{pmatrix} \eta \\ 0 \end{pmatrix}, r_0, k \right\} , \tag{3.11}$$

we obtain the amplitude $\eta(r_0, k)$ of the order parameter (3.9) as the solution of the equation

$$\frac{\partial}{\partial \eta} \Gamma(\eta, r_0, k) = 0 \tag{3.12}$$

because of $\underline{h} = 0$. Substituting (3.10) and (3.11) into (3.8) we obtain

$$\mathbf{J}_s(r_0, \mathbf{k}) = V^{-1} \frac{\partial}{\partial \mathbf{k}} \Gamma(\eta, r_0, k)|_{\eta = \eta(r_0, k)} \tag{3.13}$$

[compare (2.9)]. Thermodynamic stability requires that $\partial^2 \Gamma(\eta, r_0, k)/\partial k^2$ is positive [compare (2.10)]. Together with (3.13) this implies the stability property

$$\frac{\partial}{\partial k} |\mathbf{J}_s(r_0, \mathbf{k})| > 0 \; , \tag{3.14}$$

which is the generalized version of (2.11). The remaining task is a perturbative calculation of the potential $\Gamma(\eta, r_0, k)$ (3.11).

The mean-field part of $\Gamma(\eta, r_0, k)$ corresponds to $H\{\psi_{\rm mf}\}$ and is given by

$$\Gamma_{\rm mf}(\eta, r_0, k) = V[\tfrac{1}{2}(r_0 + k^2)\eta^2 + u_0 \eta^4] \; . \tag{3.15}$$

In Appendix A we calculate the contribution of the fluctuations in one-loop order. The result is

$$\begin{aligned}\Gamma(\eta, r_0, k) = \Gamma_{\rm mf}(\eta, r_0, k) &+ \tfrac{1}{2} V \int_{\mathbf{p}} \ln[(r_0 + p^2 + k^2)^2 - 4(\mathbf{p}\cdot\mathbf{k})^2 \\ &+ 16 u_0 \eta^2 (r_0 + p^2 + k^2) + 48 u_0^2 \eta^4]\end{aligned} \tag{3.16}$$

with $\int_{\mathbf{p}} \equiv \int d^dp/(2\pi)^d$. According to (3.12) and (3.13) we obtain from (3.16) the bare one-loop expressions for the square of the order-parameter amplitude and for the superfluid current as

$$\eta(r_0, k)^2 = \eta_{\rm mf}(r_0, k)^2 - M(c_0, k) + O(u_0) \tag{3.17}$$

with

$$M(c_0, k) = \int_{\mathbf{p}} (4p^2 + c_0^2) \Delta(\mathbf{p})^{-1} \; , \tag{3.18}$$

$$c_0 = [-2(r_0 + k^2)]^{1/2} \; , \tag{3.19}$$

$$\Delta(\mathbf{p}) = p^2(p^2 + c_0^2) - 4(\mathbf{k}\cdot\mathbf{p})^2 \; , \tag{3.20}$$

and

$$\mathbf{J}_s(r_0, \mathbf{k}) = \mathbf{J}_s^{\rm mf}(r_0, \mathbf{k}) - \mathbf{k} L(c_0, k) + O(u_0) \tag{3.21}$$

with

$$L(c_0, k) = \int_{\mathbf{p}} [2p^2 + 4(\mathbf{k}\cdot\mathbf{p})^2/k^2] \Delta(\mathbf{p})^{-1} \; . \tag{3.22}$$

In Appendix B the integrals (3.18) and (3.22) are evaluated in d dimensions using dimensional regularization (at infinite cutoff Λ). The results are

$$M(c_0, k) = -\frac{1}{\varepsilon} A_d c_0^{d-2} \left[4F\left[-\frac{d-2}{2}, \frac{1}{2}; \frac{d}{2}; 4k^2/c_0^2 \right] - F\left[\frac{\varepsilon}{2}, \frac{1}{2}; \frac{d}{2}; 4k^2/c_0^2 \right] \right] , \tag{3.23}$$

$$L(c_0, k) = -\frac{1}{\varepsilon} A_d c_0^{d-2} \left[2F\left[-\frac{d-2}{2}, \frac{1}{2}; \frac{d}{2}; 4k^2/c_0^2 \right] + \frac{4}{d} F\left[-\frac{d-2}{2}, \frac{3}{2}; \frac{d+2}{2}; 4k^2/c_0^2 \right] \right] , \tag{3.24}$$

where $F(a, b; c; z)$ is the hypergeometric function and

$$A_d = \frac{\Gamma(3 - d/2)}{2^{d-2} \pi^{d/2} (d-2)} \tag{3.25}$$

is a geometric factor. The bare perturbative results do of course not yet correctly describe the effect of the fluctuations as the critical point $r_0 = r_{0c}$, $\mathbf{k} = \mathbf{0}$ (or $\mathbf{v}_s = \mathbf{0}$) is approached. In the subsequent section we shall employ the minimal renormalization procedure in d dimensions[34–36] to obtain the correct critical behavior of η and $\mathbf{J}_s$.

IV. RENORMALIZATION OF $\langle \psi \rangle$ AND J_s

The field-theoretic renormalization of the model (2.1), (2.2) at $\mathbf{k} = \mathbf{0}$ is well established.[32,33] Obviously, no new ultraviolet divergencies arise at $\mathbf{k} \neq \mathbf{0}$, hence it is not surprising that the Z factors of the standard renormalizations[33–35]

$$\psi = Z_\varphi^{1/2} \psi^R \; , \tag{4.1}$$

$$r_0 - r_{0c} = Z_r r, \quad r = a(T - T_\lambda)/T_\lambda \; , \tag{4.2}$$

$$u_0=\mu^\varepsilon A_d^{-1}Z_\varphi^{-2}Z_u u \ , \tag{4.3}$$

will suffice to renormalize the bare results

$$\mathbf{J}_s=\mathbf{J}_s(r_0-r_{0c},u_0,\mathbf{k}) \tag{4.4}$$

and

$$\eta=\eta(r_0-r_{0c},u_0,k) \tag{4.5}$$

presented in the preceding section. We note that r_{0c} is of $O(u^{2/\varepsilon})$,[35] thus we may replace r_0 by r_0-r_{0c} in all one-loop expressions (3.16)–(3.24). The renormalized counterpart of (4.5) is

$$\eta^R(r,u,\mu,k)=Z_\varphi^{-1/2}\eta(Z_r r,\mu^\varepsilon A_d^{-1}Z_\varphi^{-2}Z_u u,k) \ , \tag{4.6}$$

thus in one-loop order

$$\eta^R(r,u,\mu,k)=\eta(Z_r r,\mu^\varepsilon A_d^{-1}Z_u u,k)+O(u) \ , \tag{4.7}$$

since $Z_\varphi(u)=1+O(u^2)$. Together with[34]

$$Z_r(u)=1+16u/\varepsilon+O(u^2), \quad Z_u(u)=1+40u/\varepsilon+O(u^2) \ , \tag{4.8}$$

this yields

$$[\eta^R(r,u,\mu,k)]^2=A_d\mu^{d-2}\left\{\left[\frac{1}{8u}-\frac{5}{\varepsilon}\right]\frac{c^2}{\mu^2}+\frac{2}{\varepsilon}(-2r/\mu^2)+\frac{1}{\varepsilon}\left[\frac{c}{\mu}\right]^{d-2}\left[4F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]-F\left[\frac{\varepsilon}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]\right]+O(u)\right\} \tag{4.9}$$

with

$$c^2=-2(r+k^2) \ . \tag{4.10}$$

To identify the renormalization of $\mathbf{J}_s$ we invoke the fact[17] that the dynamic coupling g_0 of model F and the heat current Q are multiplicatively renormalized by the same (static) Z factor Z_m. These normalizations read[17,30,41]

$$g_0=(\mu^\varepsilon/A_d)^{1/2}(\chi_0 Z_m)^{1/2}g \ , \tag{4.11}$$

$$Q=(\chi_0 Z_m)^{1/2}Q^R \ , \tag{4.12}$$

with Z_m being related to the additive renormalization of the specific heat.[34,36] Thus Z_m is cancelled in (1.6) if (1.6) is rewritten in terms of the renormalized quantities g and Q^R. This implies, as an exact result valid to all orders, that the superfluid current $\mathbf{J}_s$ (2.3), is renormalized according to

$$\mathbf{J}_s^R(r,u,\mu,\mathbf{k})=\mathbf{J}_s(Z_r r,\mu^\varepsilon A_d^{-1}Z_\varphi^{-2}Z_u u,\mathbf{k}) \ , \tag{4.13}$$

i.e., without multiplication by a Z factor. Combining (4.13) with (3.21) and (3.24) yields in one-loop order

$$\mathbf{J}_s^R(r,u,\mu,\mathbf{k})=\mathbf{k}\mu^{d-2}A_d\left\{\left[\frac{1}{8u}-\frac{5}{\varepsilon}\right]\frac{c^2}{\mu^2}+\frac{2}{\varepsilon}(-2r/\mu^2)+\frac{1}{\varepsilon}\left[\frac{c}{\mu}\right]^{d-2}\left[2F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]+\frac{4}{d}F\left[-\frac{d-2}{2},\frac{3}{2};\frac{d+2}{2};4k^2/c^2\right]\right]+O(u)\right\} \tag{4.14}$$

with c given by (4.10). Equations (4.6) and (4.13) imply

$$\eta^R(r,u,\mu,k)=\left[\exp\int_l^1\zeta_\varphi(u(l'))dl'/l'\right]\eta^R(r(l),u(l),\mu l,k) \tag{4.15}$$

and

$$\mathbf{J}_s^R(r,u,\mu,\mathbf{k})=\mathbf{J}_s^R(r(l),u(l),\mu l,\mathbf{k}) \ , \tag{4.16}$$

where $r(l)$ and $u(l)$ are the standard effective parameters (see, e.g., Ref. 37) determined by

$$r(l)=r\exp\int_1^l\zeta_r(u(l'))dl'/l' \ . \tag{4.17}$$

$$l\frac{d}{dl}u(l)=\beta_u(u(l)) \tag{4.18}$$

with $u(1)=u$. So far we have not yet specified the flow parameter l. If the above results were applicable to the entire k-T plane (or v_s-T plane) including the region $k^2 \gg -r(l)$, a k-dependent choice of the flow parameter l would be necessary, for example,

$$c(l)^2=-2[r(l)+k^2]=\mu^2 l^2 \ .$$

From (4.9) and (4.14)–(4.16) we find, however, that the one-loop terms of η and $\mathbf{J}_s$ exhibit an imaginary part if $4k^2/c(l)^2>1$ or $k^2>[-2r(l)]/6$, thus a static theory for a (stable) order parameter and superfluid current exists only in the small-k region $k^2<[-2r(l)]/6$. This corresponds to the onset of the instability at $k=k_c^{\mathrm{mf}}$ (2.12), obtained already in mean-field theory. This instability will be further discussed in Sec. V. It is well known that in the "hydrodynamic" region $k^2 \ll -2r(l)$ the standard choice of the flow parameter below T_λ (Ref. 37)

$$\frac{-2r(l)}{\mu^2 l^2}=1 \tag{4.19}$$

is appropriate. We employ this k-independent choice also up to $k \lesssim k_c^{\mathrm{mf}}$ as will be justified in Sec. V.

Since the canonical dimension of the order parameter is $\Lambda^{(d-2)/2}$ we obtain from (4.15), (4.16), and (4.19)

$$\eta^R(r,u,\mu,k)^2=\left[\exp\int_l^1 \zeta_\varphi \frac{dl'}{l'}\right](l\mu)^{d-2} f_\eta\{u(l),k/\mu l\} \tag{4.20}$$

and

$$\mathbf{J}_s^R(r,u,\mu,\mathbf{k})=(l\mu)^{d-1}\mathbf{f}_J\{u(l),\mathbf{k}/\mu l\} , \tag{4.21}$$

with the dimensionless amplitude functions

$$f_\eta\{u,k/\mu\}\equiv\eta^R(-\tfrac{1}{2},u,1,k/\mu)^2 , \tag{4.22}$$

$$\mathbf{f}_J\{u,\mathbf{k}/\mu\}\equiv\mathbf{J}_s^R(-\tfrac{1}{2},u,1,\mathbf{k}/\mu) . \tag{4.23}$$

These functions are given in one-loop order by (4.9) and (4.14). Equation (4.19) is equivalent to[35,36]

$$\mu l=\xi(-2t)^{-1}[1+O(u^2)], \quad t<0 , \tag{4.24}$$

with

$$t=(T-T_\lambda)/T_\lambda , \tag{4.25}$$

where

$$\xi(t)=\xi_0 t^{-\nu}[1+a_\xi t^\Delta+\dots] \quad t>0 \tag{4.26}$$

is the correlation length above T_λ. The parameters in (4.26) are[42] $\nu=0.672$, $\Delta=0.50$, and $\xi_0=1.43\times10^{-8}$ cm, $a_\xi=0.0363$ at saturated vapor pressure (SVP).

V. RESULTS AND DISCUSSION

We summarize our results for the order parameter and the superfluid current in d dimensions as a function of the reduced temperature $t<0$ and the wave vector $\mathbf{k}=(m/\hbar)\mathbf{v}_s$:

$$\eta^2=Z_\varphi(u)\left[\exp\int_{u[-2t]}^{u}\frac{\zeta_\varphi(u')}{\beta_u(u')}du'\right]\xi(-2t)^{2-d}f_\eta\{u[-2t],k\xi(-2t)\} , \tag{5.1}$$

$$\mathbf{J}_s=\xi(-2t)^{1-d}(\mathbf{k}/k)f_J\{u[-2t],k\xi(-2t)\} . \tag{5.2}$$

The dimensionless amplitude functions f_η and f_J read in one-loop order

$$f_\eta\{u,\kappa\}=A_d\left\{(8u)^{-1}(1-2\kappa^2)+\frac{1}{\varepsilon}(-3+10\kappa^2)\right.$$
$$\left.+\frac{1}{\varepsilon}(1-2\kappa^2)^{(d-2)/2}\left[4F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};z\right]-F\left[\frac{\varepsilon}{2},\frac{1}{2};\frac{d}{2};z\right]\right]+O(u)\right\} . \tag{5.3}$$

$$f_J\{u,\kappa\}=\kappa A_d\left\{(8u)^{-1}(1-2\kappa^2)+\frac{1}{\varepsilon}(-3+10\kappa^2)\right.$$
$$\left.+\frac{1}{\varepsilon}(1-2\kappa^2)^{(d-2)/2}\left[2F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};z\right]+\frac{4}{d}F\left[-\frac{d-2}{2},\frac{3}{2};\frac{d+2}{2};z\right]\right]+O(u)\right\} , \tag{5.4}$$

where

$$z=4\kappa^2/(1-2\kappa^2) . \tag{5.5}$$

For $d=3$ the hypergeometric functions $F(a,b;c;z)$ can be expressed in terms of elementary functions (see Appendix C). For $\xi(-2t)$ and $u[-2t]$ see (4.26) and Refs. 37 and 43, respectively.

A. Determination of the instability

In Figs. 3 and 4 we have plotted the functions (5.3) and (5.4) for $d=3$ and $u=u^*=0.0362$ The zero-loop terms $\sim O(u^{-1})$ corresponding to the mean-field approximation yield the dashed lines. At $\kappa=\kappa_c^{\mathrm{mf}}=1/\sqrt{6}$ corresponding to

$$k_c^{\mathrm{mf}}=\xi(-2t)^{-1}\frac{1}{\sqrt{6}} , \tag{5.6}$$

the zero-loop term of f_J has a maximum. We see that for $\kappa\le\kappa_c^{\mathrm{mf}}$ the one-loop contributions are rather small, which justifies the choice (4.19) for the flow parameter. The one-loop contributions contain nonanalytic terms with algebraic singularities at $\kappa=\kappa_c^{\mathrm{mf}}=1/\sqrt{6}$. The leading nonanalytic terms are (see Appendix D)

$$\frac{1}{\pi}2^{-3/2}6^{1/4}(\kappa_c^{\mathrm{mf}}-\kappa)^{1/2} \tag{5.7}$$

and

$$-\frac{1}{\pi}2^{-3/2}6^{5/4}(\kappa_c^{\mathrm{mf}}-\kappa)^{3/2} \tag{5.8}$$

for f_η and f_J, respectively, in $d=3$ dimensions. (We note that the exponents depend on the dimension d, see Appendix D.) The algebraic singularities are clearly seen in Figs. 3 and 4. For $\kappa > \kappa_c^{\rm mf}$ these terms develop (unphysical) imaginary parts that are shown as dotted lines in Figs. 3 and 4. We interpret these findings as follows.

Within mean-field theory the range of stability of a thermodynamic state is identified on the basis of a phenomenological stability criterion such as (2.10). By contrast, no separate requirement is needed in an exact statistical theory. In fact, the appearance of imaginary parts at $k=k_c^{\rm mf}$ in our one-loop result shows that the mean-field instability comes out of the calculation in a direct way. Thus it is not necessary to impose the stability requirement (3.14) on the superfluid current, since (3.14) is automatically satisfied as a result of the statistical treatment. We are of course interested in the correction $k_c - k_c^{\rm mf}$ of the critical wave number due to fluctuations. One way to determine this fluctuation effect in leading order would be to calculate f_η and f_J in two-loop order and to identify the shift $k_c - k_c^{\rm mf}$ from the onset of imaginary parts of the *two-loop* terms. An alternative way is to invoke the stability property (3.14) and to look for the maximum of the *one-loop* function f_J (5.4), i.e., for the maximum of the solid line in Fig. 4. We see that the shift of the position of the maximum due to the one-loop term is very small. The numerical value of k_c is (for $d=3$ and $u=u^*$)

$$k_c(T)=0.972k_c^{\rm mf}=0.397\xi(-2t)^{-1} \,. \tag{5.9}$$

We believe that the smallness of the one-loop correction indicates that (5.9) is close to the exact result for k_c (apart from the effect of vortices). A comparable situation appears to exist for the order-parameter function f_η at $\kappa=0$ where the one-loop correction vanishes[36] and the (Borel) sum of all higher-loop terms turns out to be negligibly small.[44]

We note that it would be misleading to perform a strict expansion of k_c with respect to the coupling u. This expansion, as obtained from the solution $\kappa=\kappa_c(u)$ of the equation $\partial f_J\{u,\kappa\}/\partial\kappa=0$, would yield

$$k_c=k_c^{\rm mf}[1-a_c u+O(u^2)] \tag{5.10}$$

with the large correction amplitude ($d=3$)

$$a_c=\pi\sqrt{24}-8=7.39 \,, \tag{5.11}$$

thus (5.10) would be applicable only to the uninteresting range $u \ll u^*$. We see that the expansion of k_c with respect to u is considerably less useful than performing a loop expansion only for f_J with k_c being determined directly from the maximum of f_J.

From the superfluid current $J_s=\rho_s v_s$, one obtains the superfluid density at finite superfluid velocity as J_s/v_s. Thus, apart from a constant prefactor, the superfluid density at finite k is given by

$$\begin{aligned}\rho_s(k)&=\text{const.}J_s/k\\ &=\text{const}\xi(-2t)^{1-d}k^{-1}f_J\{u(-2t),k\xi(-2t)\} \,,\end{aligned} \tag{5.12}$$

which for $k\to 0$ is reduced to the known one-loop expression[36,45]

$$\rho_s(0)=\text{const}\,A_d\xi(-2t)^{2-d}\left[\frac{1}{8u[-2t]}+\frac{1}{d}\right] . \tag{5.13}$$

The ratio $\rho_s(k)/\rho_s(0)$ is a function of $\kappa=k\xi(-2t)$, which is plotted in Fig. 5 for $d=3$ and $u=u^*$. The κ dependence is rather weak, with a finite value at κ_c and with an algebraic singularity of the type (5.8).

From Fig. 4 we see that the superfluid current is a linear function of k in the small-k region $k \lesssim 0.5k_c$, where $\rho_s(k)/\rho_s(0)$ is approximately constant (Fig. 5). Thus J_s and v_s are linearly related in this "linear regime" (Fig. 1). Similarly there is a linear relationship between the heat current $Q(t,k)$ (see subsection C below) and the

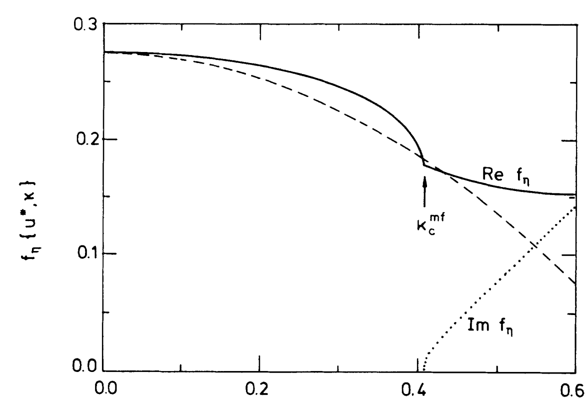

FIG. 3. Amplitude function $f_\eta\{u^*,\kappa\}$ (solid line) of the square of the order parameter as a function of $\kappa=k\xi$ in $d=3$ dimensions, as given by (5.3) and (C10). At $\kappa_c^{\rm mf}=1/\sqrt{6}$, f_η is singular according to (5.7). In the unstable region ($\kappa>\kappa_c^{\rm mf}$), f_η becomes complex (solid line: real part; dotted line: imaginary part). The dashed line represents the mean-field (zero-loop) result, which is regular at $\kappa_c^{\rm mf}$

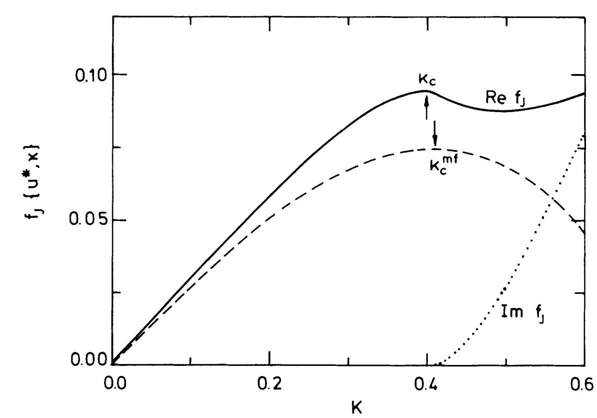

FIG. 4. Amplitude function $f_J\{u^*,\kappa\}$ (solid line) of the superfluid current J_s as a function of $\kappa=k\xi$ in $d=3$ dimensions, as given by (5.4) and (C11). At $\kappa_c^{\rm mf}=1/\sqrt{6}$, f_J is singular according to (5.8). For $\kappa>\kappa_c^{\rm mf}$, f_J is complex (solid line: real part; dotted line: imaginary part). The dashed line represents the mean-field (zero-loop) result with a maximum at $\kappa_c^{\rm mf}$. The instability of J_s sets in at $\kappa_c<\kappa_c^{\rm mf}$, (5.9), where the solid line has its maximum.

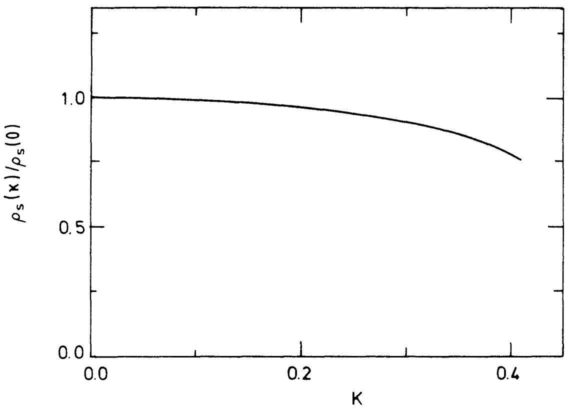

FIG. 5. Superfluid density $\rho_s(k)$ (5.12), at finite superfluid velocity $v_s=(\hbar/m)k$ divided by $\rho_s(0)$ (5.13), as a function of $\kappa=k\xi$ in $d=3$ dimensions. At $k=k_c^{\mathrm{mf}}$, $\rho_s(k)$ is singular but finite.

superfluid velocity $v_s \sim k$ in the linear regime below the dashed line on the left-hand side of Fig. 2. This line is, of course, not sharply defined. It rather indicates a crossover region between the linear and nonlinear regimes, similar to the dashed line above T_λ in Fig. 2 as discussed recently.[16,17]

B. Critical superfluid velocity

Asymptotically, (5.9) yields a critical velocity (1.1),

$$v_{sc}(T)=(\hbar/m)k_c(T)=A_{sc}[(T_\lambda-T)/T_\lambda]^\nu , \quad (5.14)$$

with the amplitude

$$A_{sc}=(\hbar/m)\xi_0^{-1}2^\nu\left[\frac{1}{\sqrt{6}}-0.0112\right] , \quad (5.15)$$

$$=7.03\times10^3 \text{ cm/sec} . \quad (5.16)$$

Since the order parameter is nonzero at $v_{sc}(T)$ the corresponding borderline $T_c(v_s)$ in the v_s-T plane (Fig. 1) should of course not be interpreted as a λ line of critical points [except for $T_c(0)=T_\lambda$] but rather as an analog to a spinodal line of a first-order-like transition where a homogeneous metastable state with a finite order parameter becomes unstable. Within our perturbation theory this transition is not caused by the creation of vortices. Nevertheless local vortex generation may be the dominant mechanism for the actual decay of superflow in experiments using persistent-current methods.[7] The experimental critical velocity[7] has an amplitude $A_{sc}\approx3.8\times10^2$ cm/sec, thus it is about 20 times smaller than the theoretical value (5.16). In the experiment,[7] the helium liquid was flowing through a material with small pores, which our theory does not take into account. This may be the reason for the discrepancy. A considerable discrepancy exists also with the phenomenological estimate $A_{sc}\approx1.5\times10^3$ cm/sec by Langer and Fisher.[5] Clearly, further theoretical work is necessary to explain these discrepancies but also new experiments would be desirable where the influence of vortices and of geometry effects is less dominant. This appears to be realizable in an experiment[25] where a sufficiently small heat current Q induces a superfluid counterflow. In the following subsection we shall apply our results to this situation.

C. Critical heat current and $T_\lambda(Q)$

From (1.6) and (5.2) we obtain the relation between the heat current Q and the superfluid velocity $v_s=(\hbar/m)k$ (apart from a minus sign)

$$Q(t,k)=g_0k_BT_\lambda\xi(-2t)^{1-d}f_J\{u[-2t],k\xi(-2t)\} . \quad (5.17)$$

At SVP, nonasymptotic static effects are rather small,[37,43] therefore we confine the following discussion to the asymptotic region $|t|\lesssim10^{-3}$, where $\xi(-2t)\approx\xi_0 2^{-\nu}|t|^{-\nu}$ and $u[-2t]\approx u^*$. In this region, $Q(t,k)$ has the scaling form

$$Q(t,k)=A_Q|t|^{(d-1)\nu}F[k\xi(-2t)] \quad (5.18)$$

with the universal scaling function (5.4) (see Fig. 4)

$$F[\kappa]=f_J\{u^*,\kappa\} . \quad (5.19)$$

The nonuniversal amplitude A_Q in (5.18) is

$$A_Q=g_0k_BT_\lambda\xi_0^{1-d}2^{(d-1)\nu} , \quad (5.20)$$

$$=8.03\times10^4 \text{ W cm}^2 \text{ at } d=3 . \quad (5.21)$$

The critical value $\kappa_c=0.397$ (5.9), corresponds to the critical heat current (1.7),

$$Q_c(T)=A_Q|t|^{(d-1)\nu}f_J\{u^*,0.397\} \quad (5.22)$$

with

$$f_J\{u^*,0.397\}=0.0944 \quad (5.23)$$

for $d=3$. Inverting (5.22) yields the line

$$T_\lambda(Q)=T_\lambda[1-A_0Q^x] \quad (5.24)$$

in the Q-T plane (Fig. 2) with the exponent

$$x=[(d-1)\nu]^{-1} , \quad (5.25)$$

$$=(2\nu)^{-1}=0.744 \quad \text{at } d=3 \quad (5.26)$$

and the amplitude

$$A_0=(A_Qf_J\{u^*,0.0397\})^{-x} . \quad (5.27)$$

Equation (5.26) confirms Onuki's (corrected) result[15] for x. From his scaling arguments within model F, however, it is not clear whether dynamic-transient effects are cancelled exactly. In our approach the purely static nature of x is a simple consequence of the fact that the ratio g_0/Q of the dynamic quantities g_0 and Q is not renormalized [see (4.11) and (4.12)], thus (5.25) is seen to be an exact result. The earlier expression[14] $x=1/(1+\nu-x_\lambda)$ employed in the data analysis[25] is incorrect and misleading, since it suggests to interpret x_λ as the *effective* dynamic exponent of the thermal conductivity. In view of the stability[37] of the weak-scaling fixed point[41] in $d=3$ dimensions the exponent $(1+\nu-x_\lambda)^{-1}$ would become nonuniversal even asymptotically, in contrast to the correct universal result $x=(2\nu)^{-1}$ (5.26). Now, however, instead of the apparent agreement,[25] we are faced with a discrepancy between our theoretical value (5.26) and the experimental value $x^{\text{expt}}=0.8113\pm0.012$,[25] as noted already in Refs. 16, 24 and 26. In Fig. 6 the original data[25] are shown. The dashed line represents our theoretical result, with A_0 being adjusted. The larger experimental exponent $x^{\text{expt}}>0.744$ is clearly reflected in the larger slope of the data compared to the slope of the dashed line. So far this discrepancy regarding the exponent x is unexplained.

Furthermore, there is a disagreement between our theoretical one-loop value A_0 of the amplitude, (5.27), and the measured[25] value A_0^{exp} by a factor of about two, $A_0^{\text{exp}}/A_0\approx 2$. In Fig. 6 the solid line represents our theoretical result with A_0 given by (5.27). We suspect that part of this discrepancy is not due to our one-loop approximation but due to the fact that our theory does not take vortices into account. Although the experimental temperature gradient in the superfluid has been shown to be negligibly small,[25] it is nevertheless conceivable that vortices may be part of the reason for the discrepancies between the theoretical and measured values of both x and A_0.

It should also be mentioned that in the experiment[25] an *inhomogenous* situation was studied in the presence of an interface between superfluid and normal-fluid helium. Deeply in the superfluid phase the temperature profile approaches the asymptotic value $T_\infty(Q)$ (see Fig. 1 of Ref. 17). It is this quantity that has been measured[25] but it is not identical with $T_\lambda(Q)$, since the latter quantity is defined in an ideal *homogeneous* superfluid state in the absence of an interface. We consider $T_\lambda(Q)$ only as an upper bound for $T_\infty(Q)$ beyond which the homogeneous superfluid becomes unstable. On the other hand, a theoretical study of the interface problem below T_λ (in

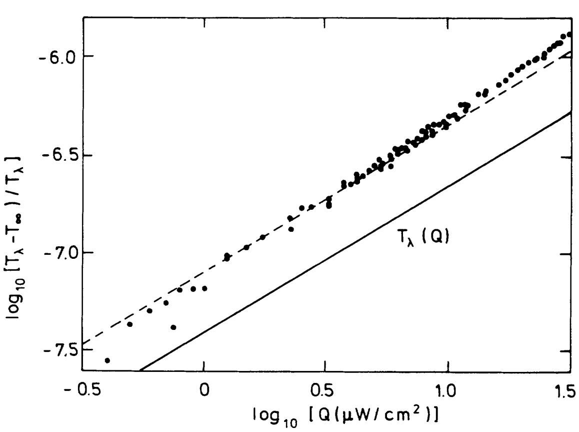

FIG. 6. Depression of the superfluid transition temperature vs the heat current Q on logarithmic scales. The data correspond to $T_\infty(Q)$ as measured in Ref. 25. The solid line represents $[T_\lambda - T_\lambda(Q)]/T_\lambda$, (5.24), with $x=0.744$ and with the one-loop result (5.27) for A_0. The dashed line represents (5.24) with $x=0.744$ but with A_0 being adjusted to the data. The effective exponent of the data is $x^{\text{expt}}=0.81$ (Ref. 25).

lowest order of renormalized perturbation theory) (Ref. 46) indicates that the difference between $T_\infty(Q)$ and $T_\lambda(Q)$ is presumably of $O(10\%)$ or less, which is smaller than the expected inaccuracy of our one-loop result for A_0. Thus it seems to be justified to compare our calculated $T_\lambda(Q)$ with the measured $T_\infty(Q)$.

Finally we attempt to estimate the expected range of validity of model F with respect to the nonlinear Q dependence near T_λ. Good agreement between theory and experiment in the linear region is verified in the temperature range $|(T-T_\lambda)/T_\lambda|\lesssim 10^{-3}$.[37] A rough estimate of the corresponding Q range is obtained by the replacement $T\to T_\lambda(Q)$, where $T_\lambda(Q)$ is given by (5.24). This yields

$$Q\lesssim (A_0\times 10^3)^{-1/x}\approx 0.7\ \text{W/cm}^2\ . \tag{5.28}$$

The previous experiments on the depression of the λ transition have been performed at considerably smaller heat currents of the order of 1 μW/cm^2 (Refs. 24 and 25) or 1 mW/cm^2 (Refs. 20 and 21), i.e., well within the expected range (5.28) of applicability of model F.

APPENDIX A: THERMODYNAMIC POTENTIAL

In this appendix we derive the one-loop expression (3.16) for the generating functional of vertex functions (3.10). We decompose the vector $\underline{\varphi}(\mathbf{x})$ (3.1), as

$$\underline{\varphi}(\mathbf{x})=\langle\underline{\varphi}\rangle+\delta\underline{\varphi}(\mathbf{x})\ , \tag{A1}$$

where $\langle\underline{\varphi}\rangle$ is the exact average of $\underline{\varphi}$. Expanding the Hamiltonian (3.3) with respect to $\delta\underline{\varphi}$ yields

$$\begin{aligned}H\{\underline{\varphi},\mathbf{k}\}=H\{\langle\underline{\varphi}\rangle,\mathbf{k}\}&+\int d^dx\left.\frac{\delta H\{\underline{\varphi},\mathbf{k}\}}{\delta\underline{\varphi}(\mathbf{x})}\right|_{\underline{\varphi}=\langle\underline{\varphi}\rangle}\delta\underline{\varphi}(\mathbf{x})\\&+\tfrac{1}{2}\int_p\delta\hat{\underline{\varphi}}(-\mathbf{p})\underline{\underline{K}}(\mathbf{p})\delta\hat{\underline{\varphi}}(\mathbf{p})\\&+\int d^dx\,[4u_0(\langle\underline{\varphi}\rangle\delta\underline{\varphi})(\delta\underline{\varphi}\delta\underline{\varphi})+u_0(\delta\underline{\varphi}\delta\underline{\varphi})^2]\ .\end{aligned} \tag{A2}$$

Here we have written the bilinear term in terms of the Fourier amplitude $\delta\hat{\underline{\varphi}}(\mathbf{p})=\int d^dx\,\delta\underline{\varphi}(\mathbf{x})\exp-i\mathbf{px}$. The 2×2 matrix

$$\underline{\underline{K}}(\mathbf{p})=\begin{bmatrix}\tilde{r}_0+8u_0\langle\varphi_1\rangle^2 & 2i\mathbf{kp}+8u_0\langle\varphi_1\rangle\langle\varphi_2\rangle\\ -2i\mathbf{kp}+8u_0\langle\varphi_1\rangle\langle\varphi_2\rangle & \tilde{r}_0+8u_0\langle\varphi_2\rangle^2\end{bmatrix} \tag{A3}$$

with

$$\tilde{r}_0=r_0+k^2+p^2+4u_0(\langle\varphi_1\rangle^2+\langle\varphi_2\rangle^2) \tag{A4}$$

defines the propagator $\underline{\underline{G}}(\mathbf{p})=\underline{\underline{K}}(\mathbf{p})^{-1}$ of the perturbation theory. The vertices are determined by the third- and fourth-order terms of (A2). The generating functional of vertex functions (3.10) is given as usual by[33,47]

$$\hat{\Gamma}\{\langle\underline{\varphi}\rangle,r_0,k\}=H\{\langle\underline{\varphi}\rangle,\mathbf{k}\}+\sum_{l=1}^{\infty}\hat{\Gamma}_l\{\langle\underline{\varphi}\rangle,r_0,k\} \tag{A5}$$

with

$$\hat{\Gamma}_1\{\langle\underline{\varphi}\rangle,r_0,k\}=\tfrac{1}{2}V\int_{\mathbf{p}}\ln[\det\underline{\underline{K}}(\mathbf{p})]\ , \tag{A6}$$

where V is the volume of the system. For $l\geq 2$ the terms $\hat{\Gamma}_l$ are given by the negative sum of all one-particle irreducible vacuum diagrams with l loops. Here, we confine ourselves to the one-loop term (A6) for the special case

$$\langle\underline{\varphi}\rangle=\begin{bmatrix}\eta\\0\end{bmatrix}, \tag{A7}$$

where η does not depend on $\mathbf{x}$. In this case we obtain from (A3)

$$\det\underline{\underline{K}}(\mathbf{p})=(r_0+k^2+p^2)^2-4(\mathbf{k}\cdot\mathbf{p})^2+16u_0\eta^2(r_0+k^2+p^2)+48u_0^2\eta^4\ . \tag{A8}$$

Together with (A5)–(A7) this leads to the one-loop result (3.16).

APPENDIX B: ONE-LOOP INTEGRALS

In the following we calculate the integrals (3.18) and (3.22) in d dimensions using dimensional regularization. These integrals are not standard because of the anisotropy of the integrands due to the finite wave vector $\mathbf{k}$. Defining Θ by $\mathbf{k}\cdot\mathbf{p}=kp\cos\Theta$, we rewrite (3.18) and (3.22) as

$$M(c,k)=\int_{\mathbf{p}}\frac{4p^2+c^2}{p^2(p^2+c^2-4k^2\cos^2\Theta)}\ , \tag{B1}$$

$$L(c,k)=\int_{\mathbf{p}}\frac{2+4\cos^2\Theta}{p^2+c^2-4k^2\cos^2\Theta}\ . \tag{B2}$$

These integrals are of the type

$$I=\int_{\mathbf{p}}f(p^2,\cos^2\Theta) \tag{B3}$$

$$=S_d\int_0^\infty dp\,p^{d-1}N_d^{-1}\int_0^\pi d\Theta(\sin\Theta)^{d-2}f(p^2,\cos^2\Theta) \tag{B4}$$

$$=S_d\int_0^\infty dp\,p^{d-1}N_d^{-1}\int_0^1 dt\,t^{-1/2}(1-t)^{(d-3)/2}f(p^2,t)\ , \tag{B5}$$

with $S_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$ and

$$N_d=\int_0^\pi d\Theta(\sin\Theta)^{d-2} \tag{B6}$$

$$=\int_0^1 dt\,t^{-1/2}(1-t)^{(d-3)/2}\ . \tag{B7}$$

In (B5) and (B7) we have substituted $t=\cos^2\Theta$. In terms of Euler's beta function[48,49]

$$B(a,b)=\int_0^1 dt\,t^{a-1}(1-t)^{b-1}=\Gamma(a)\Gamma(b)/\Gamma(a+b)\ , \tag{B8}$$

we have

$$N_d=B\left[\frac{1}{2},\frac{d-1}{2}\right]. \tag{B9}$$

The t integration can now be performed using the integral representation of the hypergeometric function[48,49]

$$F(a,b;c;z)=B(b,c-b)^{-1}\int_0^1 dt\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} \tag{B10}$$

for $\mathrm{Re}c>\mathrm{Re}b>0$. This yields

$$M(c,k)=S_d\int_0^\infty dp p^{d-1}w^{-2}(4+c^2/p^2)F\left[1,\frac{1}{2};\frac{d}{2};4k^2/w^2\right], \tag{B11}$$

$$L(c,k)=S_d\int_0^\infty dp\,p^{d-1}w^{-2}\left[2F\left[1,\frac{1}{2};\frac{d}{2};4k^2/w^2\right]+\frac{4}{d}F\left[1,\frac{3}{2};\frac{d+2}{2};4k^2/w^2\right]\right], \tag{B12}$$

with

$$w^2=p^2+c^2\ , \tag{B13}$$

In the limit $p\to\infty$ the hypergeometric functions in (B11) and (B12) become $F(a,b;c;0)=1$, hence the ultraviolet behavior of the integrands is determined by $p^{d-1}w^{-2}\sim p^{d-3}$. This implies that both integrals are convergent for $d<2$ and (ultraviolet) divergent for $d\geq 2$ According to the procedure of dimensional regularization we first perform the integrations at $d<2$ and then extend the results to $d>2$ by analytic continuation in d. Substituting $x=c^2/w^2$ we obtain

$$M(c,k)=S_d\tfrac{1}{2}c^{d-2}\int_0^1 dx\,(1-x)^{(d-2)/2}x^{-d/2}[4+x/(1-x)]F\left[1,\frac{1}{2};\frac{d}{2};4xk^2/c^2\right], \tag{B14}$$

$$L(c,k)=S_d\tfrac{1}{2}c^{d-2}\int_0^1 dx\,(1-x)^{(d-2)/2}x^{-d/2}\left[2F\left[1,\frac{1}{2};\frac{d}{2};4xk^2/c^2\right]+\frac{4}{d}F\left[1,\frac{3}{2};\frac{d+2}{2};4xk^2/c^2\right]\right]. \tag{B15}$$

By means of the hypergeometric series for $F(a,b;c;z)$ [see (D1) below] and (B8) one can show

$$\int_0^1 dx\,(1-x)^{\lambda-1}x^{-\lambda}F(1,b;c;xz)=\Gamma(\lambda)\Gamma(1-\lambda)F(1-\lambda,b;c;z)\ . \tag{B16}$$

This integral is convergent only if $0<\lambda<1$. This condition is satisfied for $d<2$ if we identify $\lambda=d/2$ and $\lambda=(d-2)/2$ in (B14) and $\lambda=d/2$ in (B15). Thus we finally obtain

$$M(c,k)=-\frac{1}{\varepsilon}A_d c^{d-2}\left[4F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]-F\left[\frac{\varepsilon}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]\right], \tag{B17}$$

$$L(c,k)=-\frac{1}{\varepsilon}A_d c^{d-2}\left[2F\left[-\frac{d-2}{2},\frac{1}{2};\frac{d}{2};4k^2/c^2\right]+\frac{4}{d}F\left[-\frac{d-2}{2},\frac{3}{2};\frac{d+2}{2};4k^2/c^2\right]\right], \tag{B18}$$

with the geometric factor[34]

$$A_d=S_d\Gamma\left[1-\frac{\varepsilon}{2}\right]\Gamma\left[1+\frac{\varepsilon}{2}\right]=-S_d\frac{\varepsilon}{2}\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{2-d}{2}\right]. \tag{B19}$$

At $k=0$, Eqs. (B17) and (B18) yield

$$M(c,0)=-\frac{3}{\varepsilon}A_d c^{d-2}, \tag{B20}$$

$$L(c,0)=-\frac{1}{\varepsilon}A_d c^{d-2}(2+4/d) \tag{B21}$$

because of $F(a,b;c;0)=1$.

APPENDIX C: AMPLITUDE FUNCTIONS IN THREE DIMENSIONS

In $d=3$ dimensions the amplitude functions (5.3) and (5.4) read

$$f_\eta\{u,\kappa\}=(4\pi)^{-1}\{(8u)^{-1}(1-2\kappa^2)-3+10\kappa^2+(1-2\kappa^2)^{1/2}[4F(-\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)-F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)]+O(u)\}, \tag{C1}$$

$$f_J\{u,\kappa\}=(\kappa/4\pi)\{(8u)^{-1}(1-2\kappa^2)-3+10\kappa^2+(1-2\kappa^2)^{1/2}[2F(-\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)+\tfrac{4}{3}F(-\tfrac{1}{2},\tfrac{3}{2};\tfrac{5}{2};y^2)]+O(u)\}, \tag{C2}$$

where

$$y=z^{1/2}=2\kappa(1-2\kappa^2)^{-1/2}. \tag{C3}$$

Using $F(a,b;c;z)=F(b,a;c;z)$ and[48]

$$(c-b-1)F(a,b;c;z)=(c-1)F(a,b;c-1;z)-bF(a,b+1,c;z), \tag{C4}$$

$$\frac{d}{dz}F(a,b;c;z)=\frac{ab}{c}F(a+1,b+1;c+1;z), \tag{C5}$$

we rewrite the amplitude functions as

$$f_\eta\{u,\kappa\}=(4\pi)^{-1}\{(8u)^{-1}(1-2\kappa^2)-3+10\kappa^2+(1-2\kappa^2)^{1/2}[2F(-\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2};y^2)+F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)]+O(u)\}, \tag{C6}$$

$$\begin{aligned}f_J\{u,\kappa\}=(\kappa/4\pi)\Big\{&(8u)^{-1}(1-2\kappa^2)-3+10\kappa^2+(1-2\kappa^2)^{1/2}\\&\times\Big[F(-\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2};y^2)+F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)+F(-\tfrac{1}{2},\tfrac{3}{2};\tfrac{3}{2};y^2)-\frac{d}{d(y^2)}[F(-\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2};y^2)+F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)]\Big]+0(u)\Big\}.\end{aligned} \tag{C7}$$

We note that[48]

$$F(a,b;b;z)=(1-z)^{-a} \tag{C8}$$

for arbitrary b, and

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};y^2)=y^{-1}\arcsin y. \tag{C9}$$

Thus the amplitude functions at $d=3$ can be expressed in terms of elementary functions:

$$f_\eta\{u,\kappa\}=(4\pi)^{-1}\{(8u)^{-1}(1-2\kappa^2)-3+10\kappa^2+(1-2\kappa^2)^{1/2}[2(1-y^2)^{1/2}+y^{-1}\arcsin y]+O(u)\}, \tag{C10}$$

$$f_J\{u,\kappa\}=\kappa f_\eta\{u,\kappa\}+(\kappa/8\pi)(1-2\kappa^2)^{1/2}y^{-2}[y^{-1}\arcsin y-(1-y^2)^{1/2}]+\kappa O(u). \tag{C11}$$

These functions are plotted in Figs. 3 and 4 for $u=u^*=0.0362$.

APPENDIX D: AMPLITUDE FUNCTIONS NEAR $\kappa_c^{\rm mf}$

The hypergeometric series

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b+n)}{\Gamma(b)}\frac{\Gamma(c)}{\Gamma(c+n)}\frac{z^n}{n!} \tag{D1}$$

is convergent for $|z|<1$. At $z=1$, $F(a,b;c;z)$ has an algebraic singularity that is explicitly seen in the formula[49]

$$\begin{aligned}F(a,b;c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)\\&+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z)\ .\end{aligned} \tag{D2}$$

Replacing the hypergeometric functions on the right-hand side by the series (D1) we see that the leading non-analytic term of $F(a,b;c;z)$ for $z\rightarrow 1$ is

$$\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\ . \tag{D3}$$

The amplitude functions $f_\eta\{u,\kappa\}$ and $f_J\{u,\kappa\}$ are expressed in terms of hypergeometric functions with the argument

$$z=4\kappa^2/(1-2\kappa^2) \tag{D4}$$

[compare (5.3)–(5.5)]. The critical wave number $\kappa_c^{\rm mf}=1/\sqrt{6}$ corresponds to $z=1$. Thus for $\kappa\rightarrow\kappa_c^{\rm mf}$ one obtains to leading order

$$(1-z)=\tfrac{1}{2}6^{3/2}(\kappa_c^{\rm mf}-\kappa)\ . \tag{D5}$$

Applying (D2)–(D5) to the amplitude functions (5.3) and (5.4), we obtain the leading nonanalytic terms

$$-\frac{1}{\varepsilon}A_d 2^{1/2}6^{d-11/4}\frac{\Gamma(d/2)\Gamma(5/2-d)}{\Gamma(\varepsilon/2)\Gamma(1/2)}(\kappa_c^{\rm mf}-\kappa)^{d-5/2}\ , \tag{D6}$$

$$\frac{1}{\varepsilon}A_d 2^{-1/2}6^{d-3/4}\frac{\Gamma(d/2)\Gamma(3/2-d)}{\Gamma(1-d/2)\Gamma(1/2)}(\kappa_c^{\rm mf}-\kappa)^{d-3/2}\ , \tag{D7}$$

for $f_\eta\{u,\kappa\}$ and $f_J\{u,\kappa\}$, respectively. For $d=3$ the leading nonanalytic terms are

$$\frac{1}{\pi}2^{-3/2}6^{1/4}(\kappa_c^{\rm mf}-\kappa)^{1/2}\ , \tag{D8}$$

$$-\frac{1}{\pi}2^{-3/2}6^{5/4}(\kappa_c^{\rm mf}-\kappa)^{3/2}\ , \tag{D9}$$

respectively.

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