Renormalization group of generalized Fibonacci lattices

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We investigate the renormalization group of the generalized Fibonacci lattices associated with the aperiodic sequences as constructed by the inflation rule: $\{A,B\}\rightarrow\{A^nB^m,A\}$, in which m and n are positive integers. The derived renormalization group consists of $2n(n+m-1)+1$ basic renormalization-group transformations. By suitable combinations of these basic transformations, local Green's function and local density of states at any sites can be calculated for electrons on the generalized Fibonacci lattices. The off-diagonal model is employed, and the local electronic densities of states at several sites are numerically calculated for some generalized Fibonacci lattices.

I. INTRODUCTION

The one-dimensional (1D) quasiperiodic (aperiodic) Schrödinger equation has been extensively studied in recent years,[1-5] since it provides a mathematical model to describe the electronic states in such aperiodic systems intermediate between periodic crystals and amorphous materials. A typical example of 1D aperiodic systems is the Fibonacci lattice, of which the electronic properties were investigated by Kohmoto, Kadanoff, and Tang,[1] and independently by Ostlund *et al.*,[2] and for which a dynamical-map technique has been developed. In 1986, Niu and Nori[4] proposed a renormalization-group (RG) theory based on a decimation scheme to study the electronic properties of the Fibonacci lattice. Although this theory is only exact in a certain limit, it provides an intuitive and coherent picture about the hierarchical splitting in the energy spectrum and the scaling properties of the wave functions.[5]

The first experimental realization of the Fibonacci system is due to the work of Merlin *et al.*[3] by the molecular beam epitaxy. Because of the development in fabricating this system, it becomes possible to produce other 1D aperiodic systems in laboratory and provides further stimulus to the investigation of 1D aperiodic systems. Recently, the generalized Fibonacci lattices have received much attention.[7-11] Since these aperiodic systems are a straightforward generalization of the Fibonacci lattice, it is mathematical accessible to study their physical properties and easy to fabricate them in experiments. However, these simple aperiodic systems exhibit rich physical properties, much more than those of the Fibonacci lattice. For instance, three kinds of wave functions, i.e., extended, critical, and localized ones, are found in the generalized Fibonacci lattices and it is revealed that there may be mobility edges in these aperiodic systems.[10] In addition, transition phenomenon concerning the electronic states is also demonstrated by the adjustment of the system parameters.[11] In this paper we study in a systematic manner the RG of the generalized Fibonacci lattices. It will be shown that these RG can be conveniently used to determine the local Green's function (LGF) and then the local density of states (LDOS) of aperiodic systems of which the similar properties were calculated by Newman[12] with the analytic method. This RG method was previously employed to investigate the local electronic properties of the Fibonacci lattice,[13,14] and recently it was further used to deal with some of the generalized Fibonacci lattices.[15] As improvements, we deal with here all the generalized Fibonacci lattices and derive the unified RG transformations.

The present paper is organized as follows: in Sec. II, the renormalization group of generalized Fibonacci lattices are derived, which are composed of $2n(n+m-1)+1$ basic RG transformations $T_1,T_2,\ldots,T_{2n(n+m-1)-1}$, T_β, and T_γ. These RG transformations can be used to calculate the LGF and the LDOS at any site of a generalized Fibonacci lattice. In Sec. III, we apply the derived $2n(n+m-1)+1$ basic RG transformations to calculate the LDOS at several sites in some generalized Fibonacci lattices. Section IV is a brief summary.

II. RENORMALIZATION GROUP

Our theoretical work on electronic properties of quasiperiodic systems is concerned with 1D models described by a tight-binding Hamiltonian

$$H=\sum_i |i\rangle\epsilon_i\langle i|+\sum_{ij}|i\rangle V_{ij}\langle j| \;, \qquad (1)$$

where ϵ_i and $|i\rangle$ are the site energy and an atomiclike orbital centered at site i, respectively, and $\{V_{ij}\}$ are the nearest-neighbor hopping integrals. Here $\{V_{ij}\}$ takes two values V_A and V_B, which are arranged in the generalized

Fibonacci sequences S_∞ constructed by the inflation rule $\{A,B\}\rightarrow\{A^nB^m,A\}$ or alternatively by the recursion relation $S_{l+1}=\{S_l^n,S_{l-1}^m\}$ with $S_0=\{B\}$, $S_1=\{A\}$, $l\geq 1$, and m and n are positive integers. The site energy ϵ_i generally takes one of the following four values:

$$\epsilon_i=\begin{cases}\epsilon_\alpha & \text{if } V_{i-1,i}=V_{i,i+1}=V_A\ ,\\ \epsilon_\beta & \text{if } V_{i-1,i}=V_A \text{ and } V_{i,i+1}=V_B\ ,\\ \epsilon_\gamma & \text{if } V_{i-1,i}=V_B \text{ and } V_{i,i+1}=V_A\ ,\\ \epsilon_\delta & \text{if } V_{i-1,i}=V_{i,i+1}=V_B\ ,\end{cases} \tag{2}$$

according to the local environment of site i. The sites with the site energies ϵ_α, ϵ_β, ϵ_γ, and ϵ_δ are here referred to as the sites of types α, β, γ, and δ, respectively. Particularly, for the subfamily of the generalized Fibonacci lattices with $m=1$, the sites of type δ, which have the local environment $V_{i-1,i}=V_{i,i+1}=V_B$, do not exist, so the site energy ϵ_i only takes one of the values ϵ_α, ϵ_β, and ϵ_γ. Corresponding to Eq. (1), the Green's function is defined as

$$(Z-H)G(Z)=I\ , \tag{3}$$

where $Z=E+io^+$ and I denotes the unit matrix. From Eq. (3) it follows that the matrix element

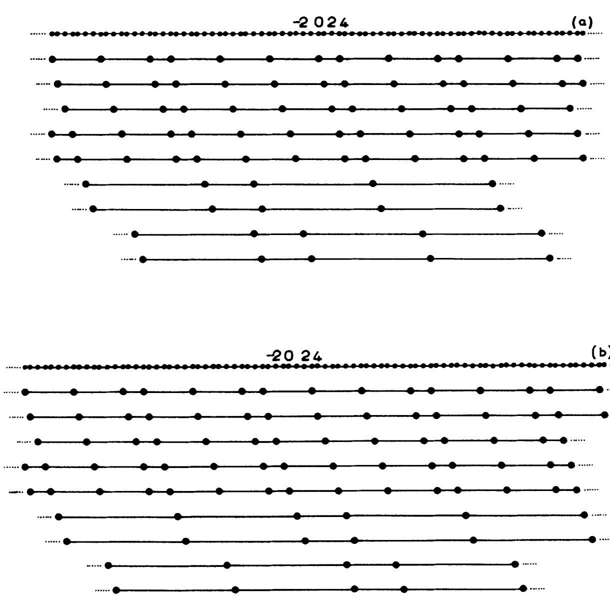

FIG. 1. A schematic representation of the decimation procedure for the generalized Fibonacci lattice with $(m,n)=(1,2)$. The top chain represents the original lattice and those from the second chain to the bottom one represent successively the new sublattices obtained by the basic RG transformations T_β, T_1, T_2, T_3, T_γ, T_4, T_5, T_6, and T_7, respectively. (a) β-type key site of the original lattice, which is assigned as site 0; (b) γ-type key site of the original lattice, which is assigned as site 0.

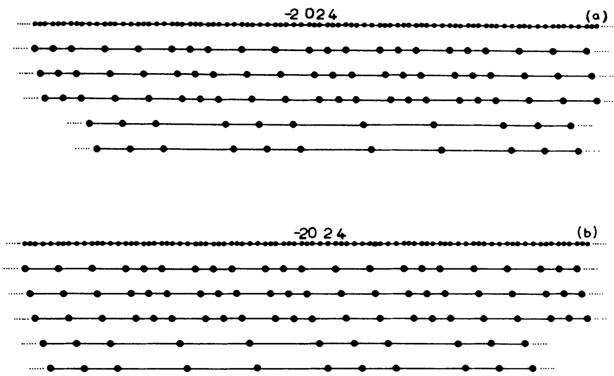

FIG. 2. Schematic representation of the decimation procedure for the generalized Fibonacci lattice with $(m,n)=(2,1)$. The top chain represents the original lattice and those from the second chain to the bottom one represent, successively, the new sublattices obtained by the basic RG transformations T_β, T_1, T_γ, T_2, and T_3, respectively. (a) β-type key site of the original lattice, which is assigned as site 0; (b) γ-type key site of the original lattice, which is assigned as site 0.

$G_{ij}=\langle i|G(Z)|j\rangle$ of Green's function $G(Z)$ is given by

$$(Z-\epsilon_i)G_{ij}=\delta_{ij}+\sum_k V_{ik}G_{kj}\ ,\quad i,j=0,\pm1,\pm2,\ldots\ , \tag{4}$$

where δ_{ij} is the Kronecker delta.

A. Basic RG transformations T_β and T_γ

In order to calculate the exact Green's function of generalized Fibonacci lattices, we first introduce two basic RG transformations T_β and T_γ, which correspond, respectively, to the decimation rules

$$\{(B^mA^n)^{n-1}B^mA^{n+m},B^mA^n\}\rightarrow\{A',B'\}\ , \tag{5}$$

and

$$\{(A^{n+m}B^m(A^nB^m)^{n-1},A^nB^m\}\rightarrow\{A',B'\}\ . \tag{6}$$

When $m=n=1$ in particular, T_β and T_γ are reduced to those corresponding to the decimation rules on the Fibonacci lattice: $\{BAA,BA\}\rightarrow\{A',B'\}$ and $\{AAB,AB\}\rightarrow\{A',B'\}$, respectively. For a generalized Fibonacci lattice, each of the RG transformations (5) and (6) yields a new inflated generalized Fibonacci lattice with only the parameters ϵ_i's and V_{ij}'s renormalized. Careful examination of transformations T_β and T_γ shows that there is a special site in a generalized Fibonacci lattice, which is called the key site of type β or γ with the same property as the key site in the Fibonacci lattice,[14] i.e., when T_β or T_γ is applied to a generalized Fibonacci lattice, the key site of type β or γ remains undecimated and its environment is not changed. In Figs. 1 and 2 the two types of key sites are shown and assigned as the sites 0, respectively. According to the geometric properties of the decimation rule (5), the RG equation of T_β is derived

$$\epsilon'_\alpha=\epsilon_\beta+\frac{V_B[S_{m-1,n+m-1}R_{m-2,n-1}U_{n-2}(X)-S_{m-2,n+m-1}Y]}{S_{m-1,n+m-1}Z-S_{m-2,n+m-1}S_{m-1,n-1}U_{n-2}(X)} + \frac{V_A[S_{m-1,n+m-2}Z-S_{m-2,n+m-2}S_{m-1,n-1}U_{n-2}(X)]}{S_{m-1,n+m-1}Z-S_{m-2,n+m-1}S_{m-1,n-1}U_{n-2}(X)} , \tag{7a}$$

$$\epsilon'_\beta=\epsilon_\beta+\frac{V_A[S_{m-1,n+m-2}Z-S_{m-2,n+m-2}S_{m-1,n-1}U_{n-2}(X)]}{S_{m-1,n+m-1}Z-S_{m-2,n+m-1}S_{m-1,n-1}U_{n-2}(X)}+\frac{V_BS_{m-2,n-1}}{S_{m-1,n-1}} , \tag{7b}$$

$$\epsilon'_\gamma=\epsilon_\beta+\frac{V_B[S_{m-1,n+m-1}R_{m-2,n-1}U_{n-2}(X)-S_{m-2,n+m-1}Y]}{S_{m-1,n+m-1}Z-S_{m-2,n+m-1}S_{m-1,n-1}U_{n-2}(X)}+\frac{V_AS_{m-1,n-2}}{S_{m-1,n-1}} , \tag{7c}$$

$$\epsilon'_\delta=\epsilon_\beta+\frac{V_BS_{m-2,n-1}+V_AS_{m-1,n-2}}{S_{m-1,n-1}} , \tag{7d}$$

$$V'_A=V_B/[S_{m-1,n+m-1}Z-S_{m-2,n+m-1}S_{m-1,n-1}U(x)] , \tag{7e}$$

$$V'_B=V_B/S_{m-1,n-1} , \tag{7f}$$

where

$$g=(E-\epsilon_\alpha)/2V_A , \quad h=(E-\epsilon_\delta)/2V_B , \tag{8a}$$

$$P_i=\frac{E-\epsilon_\beta}{V_A}U_i(g)-U_{i-1}(g) , \quad Q_i=\frac{E-\epsilon_\gamma}{V_B}U_i(h)-U_{i-1}(h) , \tag{8b}$$

$$R_{ij}=Q_iP_j-\frac{V_A}{V_B}U_i(h)P_{j-1} , \quad S_{ij}=\frac{V_B}{V_A}Q_iU_j(g)-U_i(h)U_{j-1}(g) , \tag{8c}$$

$$X=\tfrac{1}{2}(R_{m-1,n-1}-S_{m-2,n-1}) , \quad Y=S_{m-2,n-1}U_{n-2}(X)+U_{n-3}(X) , \tag{8d}$$

$$Z=R_{m-1,n-1}U_{n-2}(X)-U_{n-3}(X) , \tag{8e}$$

and $U_N(X)$ is the Nth Chebyshev polynomial of the second kind

$$U_N(X)=\frac{\sin[(N+1)\cos^{-1}(X)]}{\sin[\cos^{-1}(X)]} \tag{9}$$

which satisfies the recursion relation

$$U_N(X)=2XU_{n-1}(X)-U_{n-2}(X) . \tag{10}$$

Similarly, we can obtain the RG equation of the transformation T_γ

$$\epsilon'_\alpha=\epsilon_\gamma+\frac{V_A[S_{m-1,n+m-2}Z-R_{m-1,n+m-2}S_{m-1,n-1}U_{n-2}(X)]}{S_{m-1,n+m-1}Z-R_{m-1,n+m-1}S_{m-1,n-1}U_{n-2}(X)} + \frac{V_B[S_{m-1,n+m-1}S_{m-2,n-2}U_{n-2}(X)-R_{m-1,n+m-1}Y]}{S_{m-1,n+m-1}Z-R_{m-1,n+m-1}S_{m-1,n-1}U_{n-2}(X)} , \tag{11a}$$

$$\epsilon'_\beta=\epsilon_\gamma+\frac{V_B[S_{m-1,n+m-1}S_{m-2,n-2}U(X)-R_{m-1,n+m-1}Y]}{S_{m-1,n+m-1}Z-R_{m-1,n+m-1}S_{m-1,n-1}U_{n-2}(X)}+\frac{V_AS_{m-1,n-2}}{S_{m-1,n-1}} , \tag{11b}$$

$$\epsilon'_\gamma=\epsilon_\gamma+\frac{V_A[S_{m-1,n+m-2}Z-R_{m-1,n+m-2}S_{m-1,n-1}U_{n-2}(X)]}{S_{m-1,n+m-1}Z-R_{m-1,n+m-1}S_{m-1,n-1}U_{n-2}(X)}+\frac{V_BS_{m-2,n-1}}{S_{m-1,n-1}} , \tag{11c}$$

$$\epsilon'_\delta=\epsilon_\gamma+\frac{V_AS_{m-1,n-2}+V_BS_{m-2,n-1}}{S_{m-1,n-1}} , \tag{11d}$$

$$V'_A=V_B/[S_{m-1,n+m-1}Z-R_{m-1,n+m-1}S_{m-1,n-1}U_{n-2}(X)] , \tag{11e}$$

$$V'_B=V_B/S_{m-1,n-1} , \tag{11f}$$

where

$$g=(E-\epsilon_\alpha)/2V_A\ ,\quad h=(E-\epsilon_\delta)/2V_B\ , \tag{12a}$$

$$P_i=\frac{E-\epsilon_\beta}{V_A}U_i(g)-U_{i-1}(g)\ ,\quad Q_i=\frac{E-\epsilon_\gamma}{V_B}U_i(h)-U_{i-1}(h)\ , \tag{12b}$$

$$R_{ij}=Q_iP_j-\frac{V_B}{V_A}Q_{i-1}U_j(g)\ ,\quad S_{ij}=\frac{V_A}{V_B}P_jU_i(h)-U_{i-1}(h)U_j(g)\ , \tag{12c}$$

$$X=\tfrac{1}{2}(R_{m-1,n-1}-S_{m-1,n-2})\ ,\quad Y=R_{m-1,n-1}U_{n-2}(X)-U_{n-3}(X)\ , \tag{12d}$$

$$Z=S_{m-1,n-2}U_{n-2}(X)+U_{n-3}(X)\ . \tag{12e}$$

When $m=1$ in particular, since there is no site of type δ in this family of generalized Fibonacci lattices, Eqs. (7d) and (11d) should be removed. Using the properties of Chebyshev polynomial of second kind, $U_{-2}(X)=-1$ and $U_{-1}(X)=0$, it can be verified that the RG equations of the two basic RG transformations T_β and T_γ are still applicable to this particular family of generalized Fibonacci lattices as long as Eqs. (7d) and (11d) are removed.

Because the two hopping integrals V_A' and V_B' follow to zero by infinitely iterating the basic RG transformation T_β or T_γ on a certain generalized Fibonacci lattice, the electronic LGF at the site of type β or γ is then given by

$$G_{00}=\frac{1}{E-\epsilon_\beta^*} \tag{13}$$

or

$$G_{00}=\frac{1}{E-\epsilon_\gamma^*}\ , \tag{14}$$

where ϵ_β^* or ϵ_γ^* is the value of the site energy ϵ_β or ϵ_γ after infinite iterations of the transformation T_β or T_γ.

B. Basic RG transformations $T_1, T_2, \ldots$ and $T_{2n(n+m-1)-1}$

In addition to T_β and T_γ, other basic RG transformations are needed to investigate the local physical properties at the sites different from the key ones. These additional basic RG transformations consist of $2n(n+m-1)-1$ transformations and are here classified into two groups: one group is composed of $n(n+m-1)-1$ basic RG transformations $T_1, T_2, \ldots, T_{n+m}, T_{n+m+1}, \ldots,$ and $T_{n(n+m-1)-1}$, which represent, respectively, the following decimation rules:

$$\begin{gathered}
\{(B^{m-1}A^nB)^{n-1}B^{m-1}A^{n+m}B, B^{m-1}A^nB\}\to\{A',B'\}\ ,\\
\{(B^{m-2}A^nB^2)^{n-1}B^{m-2}A^{n+m}B^2, B^{m-2}A^nB^2\}\to\{A',B'\}\ ,\\
\vdots\\
\{(B^mA^n)^{n-2}B^mA^{n+m}B^mA^n, B^nA^m\}\to\{A',B'\}\ ,\\
\{(B^{m-1}A^nB)^{n-2}B^{m-1}A^{n+m}B^mA^nB, B^{m-1}A^nB\}\to\{A',B'\}\ ,\\
\vdots\\
\{BA^{n+m}B^{m-1}(BA^nB^{m-1})^{n-1}, BA^nB^{m-1}\}\to\{A',B'\}\ ,
\end{gathered} \tag{15}$$

and the other group is composed of $T_{n(n+m-1)}, T_{n(n+m-1)-1}, \ldots, T_{(n+1)(n+m-1)-1}, T_{(n+1)(n+m-1)}, \ldots,$ and $T_{2n(n+m-1)-1}$, corresponding, respectively, to the following decimation rules:

$$\begin{gathered}
\{[A^{n+m-1}B^m(A^nB^m)^{n-1}A]^{n-1}A^{n+m-1}B^m(A^nB^m)^{n+m-1}A\ , A^{n+m-1}B^m(A^nB^m)^{n-1}A\}\to\{A',B'\}\ ,\\
\{[A^{n+m-2}B^m(A^nB^m)^{n-1}A^2]^{n-1}A^{n+m-2}B^m(A^nB^m)^{n+m-1}A^2\ , A^{n+m-2}B^m(A^nB^m)^{n-1}A^2\}\to\{A',B'\}\ ,\\
\vdots\\
\{[AB^m(A^nB^m)^{n-1}A^{n+m-1}]^{n-1}AB^m(A^nB^m)^{n+m-1}A^{n+m-1}\ , AB^m(A^nB^m)^{n-1}A^{n+m-1}\}\to\{A',B'\}\ ,\\
\{[A^{n+m-1}B^m(A^nB^m)^{n-1}A]^{n-2}A^{n+m-1}B^m(A^nB^m)^{n+m-1}A^{n+m}B^m(A^nB^m)^{n-1}A\ ,\\
A^{n+m-1}B^m(A^nB^m)^{n-1}A\}\to\{A',B'\}\ ,\\
\vdots\\
\{AB^m(A^nB^m)^{n+m-1}A^{n+m-1}[AB^m(A^nB^m)^{n-1}A^{n+m-1}]^{n-1}\ , AB^m(A^nB^m)^{n-1}A^{n+m-1}\}\to\{A',B'\}\ .
\end{gathered} \tag{16}$$

For example, when $m=n=1$ in particular, $2n(n+m-1)-1=1$. This indicates that for the well-known Fibonacci lattice, only one additional basic RG transformation T_1 is required, which is given by $\{ABABA, ABA\}\to\{A',B'\}$. However, for the 1D generalized Fibonacci lattice with $(m,n)=(2,1)$, three additional basic RG transformations T_1, T_2, and T_3 are needed, which are given by

$$\{BAAAB, BAB\}\to\{A',B'\}\ ,$$

$$\{AABBABBABBA, AABBA\}\to\{A',B'\}\ ,$$

and

$$\{ABBABBABBAA, ABBAA\}\to\{A',B'\}\ ,$$

respectively (see Fig. 2). When the $2n(n+m-1)+1$ basic RG transformations composed of $T_1,T_2,\dots,T_{2n(n+m-1)-1}$, T_β, and T_γ are applied to a generalized Fibonacci lattice, respectively, $2n(n+m-1)+1$ sublattices are obtained, of which each sublattice is similar to the original one. Moreover, any site in the original generalized Fibonacci lattice is also embodied in a certain sublattice.

For the sake of convenience, we successively number the sites in a 1D generalized Fibonacci lattice with integers from the left to the right and assign the key site as site 0 (see Figs. 1 and 2). If the site 0 is the key site of type β, it remains undecimated and its environment does not change after the iteration of the transformation T_β. However, transformations $T_1,T_2,\dots,T_{n(n+m-1)-1}$, and T_γ make the sites $1,2,\dots,n(n+m-1)-1$ and $n(n+m-1)$ in the original generalized Fibonacci lattice become the β-type key sites of the sublattices, respectively, while other $n(n+m-1)$ basic RG transformations $T_{n(n+m-1)},T_{n(n+m-1)+1},\dots$, $T_{(n+1)(n+m-1)-1},T_{(n+1)(n+m-1)},T_{(n+1)(n+m-1)+1},\dots$, $T_{2n(n+m-1)-2}$, and $T_{2n(n+m-1)-1}$ make the sites $-[n^2(n+m-1)+n-1],-[n^2(n+m-1)+n-2],\dots$, $-[(n^2-1)(n+m-1)+n]$, $-[(n^2-n-1)(n+m-1)+n-2],-[(n^2-n-1)(n+m-1)+n-3],\dots$, -2, and -1 in the original lattice become the γ-type key sites of the sublattices, respectively. When the site 0 in the original generalized Fibonacci lattice is the key of type γ, transformations T_β, $T_1,T_2,\dots$, and $T_{n(n+m-1)-1}$ transfer the sites $-n(n+m-1),-[n(n+m-1)-1]$, $-[n(n+m-1)-2],\dots$ and -1 to γ-type key sites of the sublattices, respectively, while transformations $T_{n(n+m-1)}$, $T_{n(n+m-1)+1},\dots$, $T_{(n+1)(n+m-1)-1}$, $T_{(n+1)(n+m-1)}$, $T_{(n+1)(n+m-1)+1},\dots$, and $T_{2n(n+m-1)-1}$ transfer the sites $1,2,\dots,(n+m-1)$, $[(n+1)(n+m-1)+2]$, $[(n+1)(n+m-1)+3],\dots$, and $[n^2(n+m-1)+n-1]$ to the β-type key sites of the sublattices, respectively. For each sublattice, when the above RG transformation procedure is repeated, $2n(n+m-1)+1$ new subsublattices are produced, of which each new subsublattice has a key site of type β or γ. For instance, the sites $-[(n+1)(n+m-1)+1]$, $-[(n+1)(n+m-1)],\dots$ and $-(n+m)$ in the generalized Fibonacci lattice with β-type key site becomes sites -1 of the new sublattices by the applications of T_β, $T_1,T_2,\dots$, $T_{n(n+m-1)-1}$, and T_γ to the original lattice, respectively, and then the transformation $T_{2n(n+m-1)-1}$ further transfers each of the above sites to the γ-type key site of a new subsublattice. In this way, any site in a original generalized Fibonacci lattice can be converted into the key site of type β or γ. When $(m,n)=(1,2)$ in particular, $2n(n+m-1)+1=9$. All nine basic RG transformations are shown in Figs. 1(a) and 1(b), corresponding to the cases in which the original generalized Fibonacci lattices with $(m,n)=(1,2)$ have the key sites of types β and γ, respectively. In Fig. 1(a), site 0 in the original lattice is the key site of type β. Sites 1, 2, 3, and 4 are converted into the β-type key sites of the new sublattices as transformations T_1, T_2, T_3, and T_γ are applied, respectively, while transformations T_4, T_5, T_6, and T_7 transfer the sites -9, -8, -2, and -1 to the γ-type key sites of the new sublattices, respectively. As for the sites -7, -6, -5, -4, and -3, they can be first transfered to the sites -1 in the sublattices by the transformations T_β, T_1, T_2, T_3, and T_γ, respectively, and then to the key sites of the subsublattices by applications of T_7 to each sublattice. In Fig. 1(b), the site 0 in the original generalized Fibonacci lattice is the key site of type γ. Similar to Fig. 1(a), transformations T_β and $T_1,T_2,\dots,T_7$ transfer the -4, -3, -2, -1, 1, 2, 8, and 9 to the key sites of the new sublattices, respectively. For other sites, they can also be converted into the key sites of certain new sublattices by choosing suitable combinations of the basic RG transformations, and applying them to the original Fibonacci lattice with $(m,n)=(1,2)$. As to the five basic RG transformations for the generalized Fibonacci lattice with $(m,n)=(2,1)$ as shown in Fig. 2, their properties are similar to those of the nine basic RG transformations shown in Fig. 1.

Analogous to T_β and T_γ, according to the geometric properties of the transformations $T_1,T_2,\dots$, and $T_{2n(n+m-1)-1}$, we can obtain $2n(n+m-1)-1$ sets of RG equations corresponding to $T_1,T_2,\dots$, and $T_{2n(n+m-1)-1}$. For instance, it follows from Eq. (5) that the set of RG equations corresponding to transformation $T_{2n(n+m-1)-1}$ can be written as

$$\epsilon'_\alpha=\epsilon_\alpha+\frac{V_A[P_{n+m-2}Z_3-Q_{n+m-2}Z_4]}{P_{n+m-2}Z_1-Q_{n+m-2}Z_2}+\frac{V_A[P_{n+m-3}Z_1-Q_{n+m-3}Z_2]}{P_{n+m-2}Z_1-Q_{n+m-2}Z_2}\ , \tag{17a}$$

$$\epsilon'_\beta=\epsilon_\alpha+\frac{V_A[P_{n+m-3}Z_1-Q_{n+m-3}Z_2]}{P_{n+m-2}Z_1-Q_{n+m-2}Z_2}+\frac{V_A W_{n-2,n+m-2}}{F_{n-2,n+m-2}}\ , \tag{17b}$$

$$\epsilon'_\gamma=\epsilon_\alpha+\frac{V_A[P_{n+m-2}Z_3-Q_{n+m-2}Z_4]}{P_{n+m-2}Z_1-Q_{n+m-2}Z_2}+\frac{V_A F_{n-2,n+m-3}}{F_{n-2,n+m-2}}\ , \tag{17c}$$

$$\epsilon'_\delta=\epsilon_\alpha+\frac{V_A[F_{n-2,n+m-3}+W_{n-2,n+m-2}]}{F_{n-2,n+m-2}} , \tag{17d}$$

$$V'_A=\frac{V_A}{P_{n+m-2}Z_1-Q_{n+m-2}Z_2} , \quad V'_B=\frac{V_A}{F_{n-2,n+m-2}} , \tag{17e}$$

where

$$g=(E-\epsilon_\alpha)/2V_A , \quad h=(E-\epsilon_\delta)/2V_B , \tag{18a}$$

$$G_i=\frac{E-\epsilon_\gamma}{V_A}U_i(g)-U_{i-1}(g) , \quad H_i=\frac{E-\epsilon_\beta}{V_B}U_i(h)-U_{i-1}(h) , \tag{18b}$$

$$P_i=G_iH_{m-1}-\frac{V_B}{V_A}H_{m-2}U_i(g) , \quad Q_i=\frac{V_A}{V_B}U_{m-1}(h)G_i-U_{m-2}U_i(g) , \tag{18c}$$

$$X=(P_{n-1}-Q_{n-2})/2 , \quad R_i=P_{n-1}U_i(X)-U_{i-1}(X) , \tag{18d}$$

$$S_i=Q_{n-2}U_i(X)+U_{i-1}(X) , \quad F_{ij}=R_iP_j-P_{n-2}Q_jU_i(X) , \tag{18e}$$

$$W_{ij}=Q_{n-1}P_jU_i(X)-S_iQ_j , \quad Y=\tfrac{1}{2}(F_{n-2,n+m-1}-W_{n-2,n+m-2}) , \tag{18f}$$

$$Z_1=F_{n+m-2,n+m-1}[F_{n-2,n+m-1}U_{n-3}(Y)-U_{n-4}(Y)]-F_{n+m-2,n+m-2}W_{n-2,n+m-1}U_{n-3}(Y) , \tag{18g}$$

$$Z_2=F_{n+m-2,n+m-1}F_{n-2,n+m-2}U_{n-3}(Y)-F_{n+m-2,n+m-2}[W_{n-2,n+m-2}U_{n-3}(Y)+U_{n-4}(Y)] , \tag{18h}$$

$$Z_3=W_{n+m-2,n+m-1}[F_{n-2,n+m-1}U_{n-3}(Y)-U_{n-4}(Y)]-W_{n+m-2,n+m-2}W_{n-2,n+m-1}U_{n-3}(Y) , \tag{18i}$$

$$Z_4=W_{n+m-2,n+m-1}F_{n-2,n+m-2}U_{n-3}(Y)-W_{n+m-2,n+m-2}[W_{n-2,n+m-2}U_{n-3}(Y)+U_{n-4}(Y)] . \tag{18j}$$

When $m=1$ in particular, although the forms of $T_1,T_2,\ldots,T_{2n(n+m-1)-1}$ are applicable to the family of 1D generalized Fibonacci lattices with $m=1$, eq. (17d) should be removed. The reason is that the sites of type δ do not exist in these generalized Fibonacci lattices. As a matter of fact, it can be easily verified that, similar to the case concerning the key sites (see Sec. II A). Equations (17a)–(17c), (17e), and (18a)–(18j) are still applicable to this family of the generalized Fibonacci lattices. As discussed above, any site can be transfered to the key site of a certain new sublattice, therefore we can study the LGF at this site in terms of the technique developed in Sec. II A for dealing with the key sites.

III. NUMERICAL RESULTS FOR LDOS

Many physical properties of a system are associated with the Green's function of the system. For instance, the electronic LDOS at site i is given by

$$\rho_i(E)=-\frac{1}{\pi}\mathrm{Im}G_{ii}(E+i0^+) , \tag{19}$$

where Im denotes the imaginary part of a complex quantity. As discussed above, the LGF of generalized Fibonacci lattices can be calculated in terms of the $2n(n+m-1)+1$ basic RG transformations introduced in Sec. II. Here the off-diagonal tight-binding model is studied, which has the parameters $\epsilon_\alpha=\epsilon_\beta=\epsilon_\gamma=\epsilon_\delta=0$ and $V_B/V_A=1.5$ and, as typical examples, we present the electronic LDOS at several sites of the generalized Fibonacci lattices with $(m,n)=(1,1)$, (1,2), (1,3), (2,1), (2,2), and (3,2).

Figure 3(a)–3(f) are the LDOS at the β-type key sites 0 of the above generalized Fibonacci lattices, respectively. It can be seen from Figs. 3(a)–3(c) that the LDOS are multifractal, which are self similar and exhibit hierarchical structures. We have also numerically calculated the LDOS at β-type key sites of several generalized Fibonacci lattices with $m=1$ and $n\geq 4$, and found that the LDOS are multifractal as well. These results are consistent with the numerical results obtained by means of the dynamical-map technique.[7–10] When $m\neq 1$, the situation becomes complicated. For the generalized Fibonacci lattices with $(m,n)=(2,1)$ and (2,2), the LDOS shown at the β-type key sites in Figs. 3(d) and 3(e) are multifractal, except for those in the vicinity of $E=0$. One sees that each LDOS has a smooth part in the regions near $E=0$, which means that the electronic states are extended here. However, the LDOS at the β-type key site of the generalized Fibonacci lattice with $(m,n)=(3,2)$ has a smooth part in each of the regions in the vicinity of $E=\pm 1.5$ [see Fig. 3(f)].

In order to compare the LDOS at different sites, we employ the generalized Fibonacci lattice with $(m,n)=(2,2)$ and present its LDOS at sites -1, 1, 2, and 3 in Figs. 4(a)–4(d). It can be seen that the LDOS at these sites and that at the β-type key site 0 are different from one another in regard to their amplitudes. This implies that at different sites the local electronic properties sensitive to their amplitudes of the LDOS are much different from each other. Besides the above difference, the LDOS at different sites have yet common features. For instance, the LDOS at sites -1, 1, 2, and 3 have the same structure as that at the key site 0. From Figs.

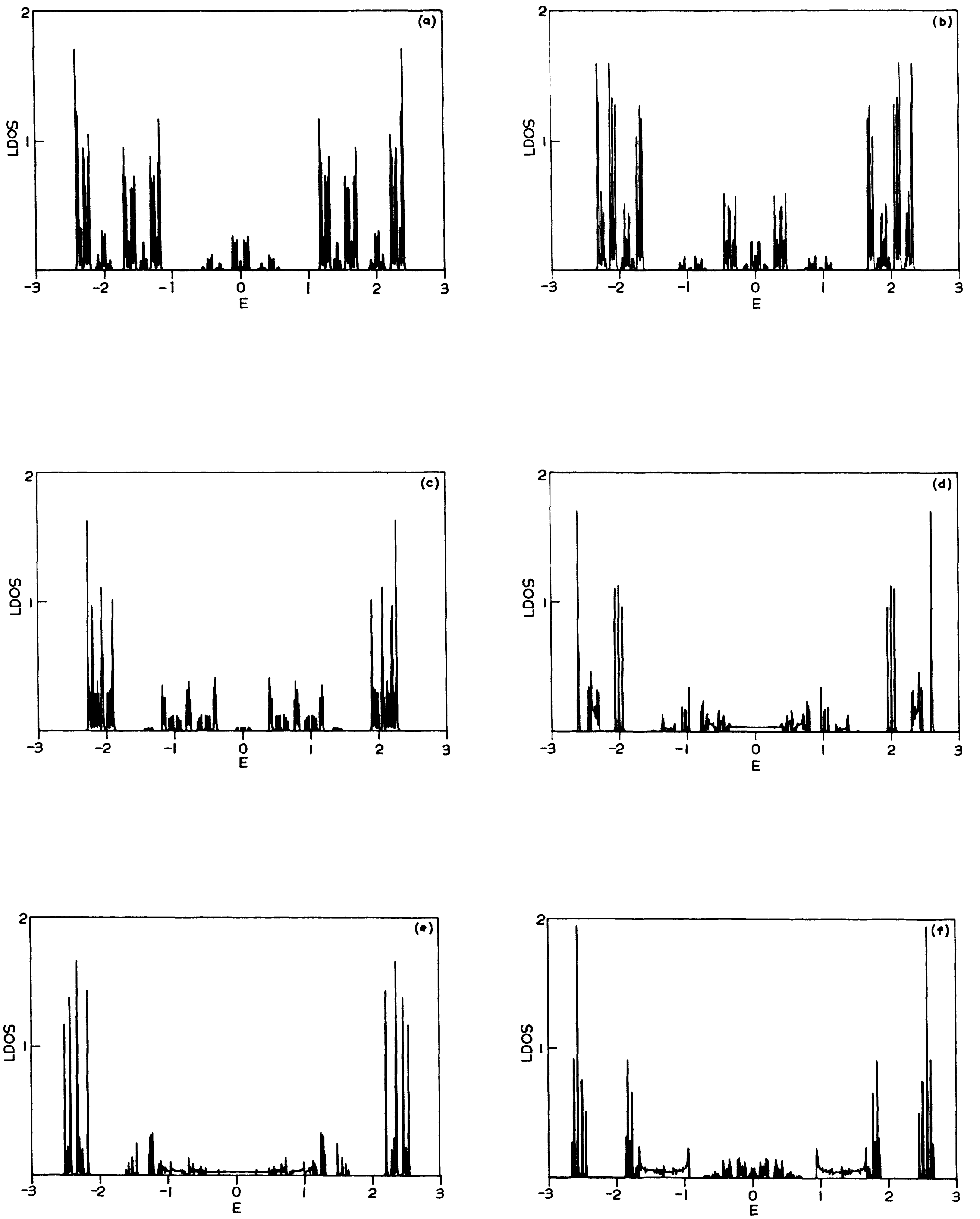

FIG. 3. the LDOS (arbitrary units) at β-type key site of some generalized Fibonacci lattices. $V_A=1$, $V_B=1.5$, and $\epsilon_\alpha=\epsilon_\beta=\epsilon_\gamma=\epsilon_\delta=0$ (in units of V_A). (a) $(m,n)=(1,1)$; (b) $(m,n)=(1,2)$; (c) $(m,n)=(1,3)$; (d) $(m,n)=(2,1)$; (e) $(m,n)=(2,2)$; (f) $(m,n)=(3,2)$.

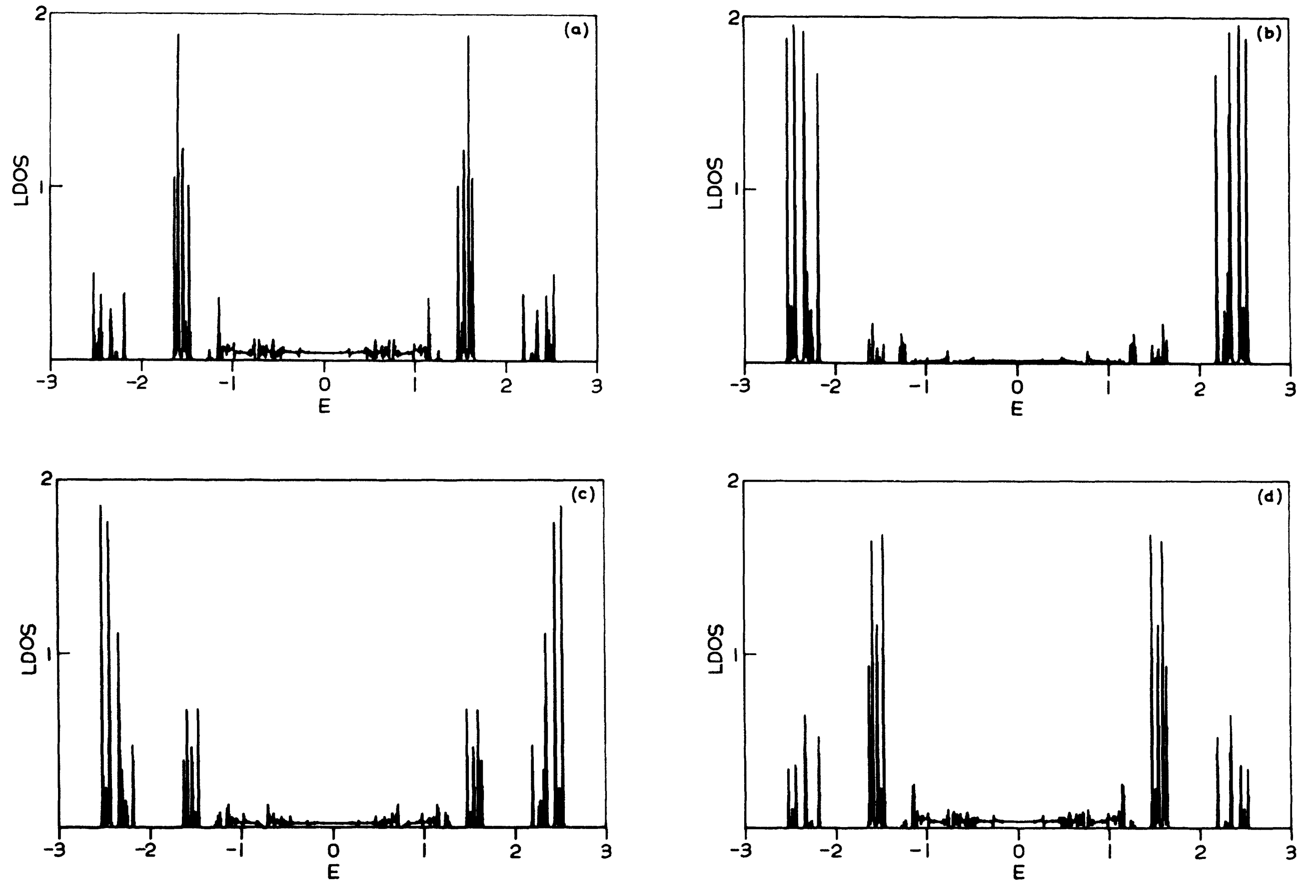

FIG. 4. The LDOS (arbitrary units) at several nonkey sites of the generalized Fibonacci lattice with $(m,n)=(2,2)$. $V_A=1$, $V_B=1.5$, and $\epsilon_\alpha=\epsilon_\beta=\epsilon_\gamma=\epsilon_\delta=0$ (in units of V_A). (a) nearest-neighbor site left to the β-type key site; (b) nearest-neighbor site right to the β-type key site; (c) second-neighbor site right to the β-type key site; (d) third-neighbor site right to the β-type key site.

4(a)–4(d), it can be seen that each LDOS has a smooth part in the vicinity of $E=0$, analogous to that at the key site 0 as shown in Fig. 3(e). In addition, it can also be seen that the LDOS shown in Figs. 4(a)–4(d) have the same multifractal structure as that at the key site 0.

IV. SUMMARY

We present an exact renormalization-group approach for the study of the local electronic properties at any site of generalized Fibonacci lattices associated with the aperiodic sequences as produced by the inflation rule: $\{A,B\}\to\{A^nB^m,A\}$ in which m and n are positive integers. The renormalization group consists of $2n(n+m-1)+1$ basic RG transformations, which can be used to determine the LGF and the LDOS of the generalized Fibonacci lattices. The LDOS of some generalized Fibonacci lattices are numerically calculated for the off-diagonal tightbinding model. It is found that the LDOS of the generalized Fibonacci lattices with $m=1$ are self similar and exhibit heirarchical structures. As for the generalized Fibonacci lattices with $m\neq1$, the LDOS have smooth parts in certain energy regions, which implies that there are extended electronic states in these energy regions.

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