Effect of randomness on surface critical phenomena

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We study surface critical phenomena of the S^4 model having Gaussian randomness. Using the renormalization-group $4-d$ expansion, we obtain surface critical exponents indicating the decay of the correlation function, $\eta_{\parallel}^{\rm random}$ and $\eta_{\perp}^{\rm random}$, at the random fixed point. These exponents are evaluated for both the ordinary and special transitions up to the second order, and the surface scaling relation $\eta^{\rm random}=2\eta_{\perp}^{\rm random}-\eta_{\parallel}^{\rm random}$ is confirmed in this random system. We also confirm the conformal invariance of the real-space correlation function at criticality.

I. INTRODUCTION

The effect of randomness on critical phenomena has been a problem of great interest. Critical phenomena of bulk magnetic systems with small Gaussian randomness in exchange couplings have especially been studied by using a renormalization-group field theory, i.e., the so-called $\epsilon=4-d$ expansion.[1-6] It has been pointed out that the effect of randomness does, in fact, affect critical phenomena for $\alpha>0$, while the pure fixed point is stable for $\alpha<0$. Here α is the specific-heat exponent of the corresponding system without randomness, i.e., of the pure system.[1,2]

Introducing a surface gives rise to another problem in critical phenomena.[7-14] It is now well known in the pure case that there are several universal classes according to the value of a surface exchange coupling; the ordinary transition which is intrinsically induced by the bulk ferromagnetic transition, the special transition at which both the bulk and the surface undergo the ferromagnetic transition simultaneously, and so forth. Renormalization-group studies[7-11,13,14] have clarified that critical phenomena at surfaces near the ordinary or special transition are determined by the bulk fixed point. From this point of view, it is natural to consider that critical exponents at surfaces will be modified from their pure counterparts when the random fixed point is stable.

In this paper, we study the effect of randomness on surface critical phenomena using the renormalization-group field theory ($\epsilon=4-d$ expansion), and derive such surface critical exponents. The Hamiltonian of the system reads

$$\mathcal{H}=\int d^{d-1}r_{\parallel}dz\{\tfrac{1}{2}[t(\mathbf{r})+c\delta(z-0)]\mathbf{S}^2(\mathbf{r})+\tfrac{1}{2}[\nabla\mathbf{S}(\mathbf{r})]^2+g_0[\mathbf{S}^2(\mathbf{r})]^2\}\ , \qquad (1.1)$$

where $\mathbf{r}=(\mathbf{r}_{\parallel},z)$, $\mathbf{S}(\mathbf{r})$ is the classical n-component spin with arbitrary length, and $t(\mathbf{r})$ denotes the local temperature which is a Gaussian random variable obeying

$$\langle t(\mathbf{r})\rangle=t_0 \qquad (1.2a)$$

and

$$\langle \delta t(\mathbf{r})\delta t(\mathbf{r}')\rangle=\zeta_0\delta(\mathbf{r}-\mathbf{r}') \qquad (1.2b)$$

with

$$\delta t(\mathbf{r})=t(\mathbf{r})-t_0\ . \qquad (1.2c)$$

We have assumed a semi-infinite geometry, and the type of transition is classified by the surface parameter c; the ordinary transition and the special transition occur, respectively, for $c>c^*$ and for $c=c^*$.

We organize the rest of this paper as follows. In Sec. II, we give a formulation of renormalizing multipoint vertex functions of the random semi-infinite system using minimal subtraction and dimensional regularization up to the two-loop order. We analyze the bulk fixed point, in Sec. III, and obtain critical exponents of the random bulk system in our formulation. In Sec IV, we calculate the two-point correlation function in a semi-infinite geometry at criticality and evaluate the surface critical exponents both for the ordinary and special transitions. Here we also refer to the surface scaling relation for random exponents. The real-space correlation function is explicitly obtained in Sec. V, and the conformal invariance of the correlation function is discussed. Section VI is devoted to the summary and discussions. The numerical estimate of the surface random exponents are given there.

II. FORMULATION OF RENORMALIZATION

It is convenient to introduce a momentum representation. We use a Fourier expansion[14] which diagonalizes the bilinear terms in the Hamiltonian (1.1),

$$\psi_{\mathbf{q}}(\mathbf{r})=\sqrt{2}e^{i\mathbf{q}_{\parallel}\cdot\mathbf{r}_{\parallel}}\sin(q_{\perp}z+\varphi)\ , \qquad (2.1a)$$

where φ is given by

$$\tan\varphi=q_{\perp}/c\ . \qquad (2.1b)$$

Expanding the spin $\mathbf{S}(\mathbf{r})$ as

$$\mathbf{S}(\mathbf{r})=\frac{1}{\sqrt{K}}\int_{\mathbf{q}}\sigma(\mathbf{q})\psi_{\mathbf{q}}(\mathbf{r})\ , \qquad (2.2)$$

we have the Hamiltonian in Fourier space

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2 \qquad (2.3)$$

with

$$\mathcal{H}_0=\tfrac{1}{2}\int_{\mathbf{q}}[t_0+q^2]\sigma(\mathbf{q})\cdot\sigma(-\nu\mathbf{q})\ , \qquad (2.4a)$$

$$\mathcal{H}_1=\tfrac{1}{2}\int_{\mathbf{q}_1\mathbf{q}_2}\delta t(\mathbf{q}_1,\mathbf{q}_2)\sigma(\mathbf{q}_1)\cdot\sigma(\mathbf{q}_2)\ , \qquad (2.4b)$$

$$\mathcal{H}_2=g_0\int_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\mathbf{q}_4}\sigma(\mathbf{q}_1)\cdot\sigma(\mathbf{q}_2)\sigma(\mathbf{q}_3)\cdot\sigma(\mathbf{q}_4)\times\Delta(q_{1\perp},q_{2\perp},q_{3\perp},q_{4\perp})(2\pi)^{d-1}\delta^{d-1}\left[\sum_{i=1}^{4}\mathbf{q}_{i\parallel}\right]\ , \qquad (2.4c)$$

where we put $\nu\mathbf{q}=(\mathbf{q}_{\parallel},-q_{\perp})$ and

$$\Delta(q_{1\perp},q_{2\perp},q_{3\perp},q_{4\perp})=\frac{\pi}{4}\sum_{\epsilon_i=\pm1,i=1\sim4}\epsilon_1\epsilon_2\epsilon_3\epsilon_4\left[\sum_{j=1}^{4}\epsilon_j q_{j\perp}\right]\cos\left[\sum_{j=1}^{4}\epsilon_j\varphi_j\right]-\frac{1}{8}\sum_{\epsilon_i=\pm1,i=1\sim4}\epsilon_1\epsilon_2\epsilon_3\epsilon_4\cot\left[\sum_{j=1}^{4}\epsilon_j q_{j\perp}\right]\sin\left[\sum_{j=1}^{4}\epsilon_j\varphi_j\right]\ . \qquad (2.5)$$

In this Fourier representation, Eqs. (1.2a) and (1.2b) are rewritten as

$$\langle\delta t(\mathbf{q}_1,\mathbf{q}_2)\rangle=0 \qquad (2.6a)$$

and

$$\langle\delta t(\mathbf{q}_1,\mathbf{q}_2)\delta t(\mathbf{q}_3,\mathbf{q}_4)\rangle=\zeta_0\Delta(q_{1\perp},q_{2\perp},q_{3\perp},q_{4\perp})(2\pi)^{d-1}\delta^{d-1}(\mathbf{q}_{1\parallel}+\mathbf{q}_{2\parallel}+\mathbf{q}_{3\parallel}+\mathbf{q}_{4\parallel}) \qquad (2.6b)$$

with the use of the same function Δ given by Eq. (2.5).

Consider an evaluation of a two-point correlation function. In a high-temperature side, one may regard $\mathcal{H}_0$ as a nonperturbed Hamiltonian and set $\mathcal{H}_1=\mathcal{H}_2=0$ (equivalent to setting $g_0=\zeta_0=0$) as a starting approximation. Then we identify the two-point correlation function as

$$G^0(\mathbf{q}_1,\mathbf{q}_2)=\langle\phi_{\mathbf{q}_1}\phi_{\mathbf{q}_2}\rangle=(2\pi)^{d-1}\delta(\mathbf{q}_{1\parallel}+\mathbf{q}_{2\parallel})G^0_{q_{1\parallel}}(q_{1\perp},q_{2\perp}) \qquad (2.7)$$

with

$$G^0_{q_{1\parallel}}(q_{1\perp},q_{2\perp})=\frac{1}{t_0+q_1^2}\frac{2\pi}{2}[\delta(q_{1\perp}-q_{2\perp})-\delta(q_{1\perp}+q_{2\perp})]\ . \qquad (2.8)$$

Transforming this function into mixed space $(\mathbf{q}_{\parallel},z)$, we have

$$\frac{2}{(2\pi)^2}\int_{-\infty}^{\infty}dq_{1\perp}\sin(q_{1\perp}z_1+\varphi_1)\int_{-\infty}^{\infty}dq_{2\perp}\sin(q_{2\perp}z_1+\varphi_2)\frac{\delta(q_{1\perp}-q_{2\perp})-\delta(q_{1\perp}+q_{2\perp})}{t_0+q_{\parallel}^2+q_{1\perp}^2}=\frac{1}{2\sqrt{t_0+q_{\parallel}^2}}\left[e^{-\sqrt{t_0+q_{\parallel}^2}|z_1-z_2|}+\frac{\sqrt{t_0+q_{\parallel}^2}-c}{\sqrt{t_0+q_{\parallel}^2}+c}e^{-\sqrt{t_0+q_{\parallel}^2}(z_1+z_2)}\right]\ . \qquad (2.9)$$

Using this function as a free propagator which is diagrammatically drawn with a straight line, we may perform a perturbational expansion with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$. When we take the random average at each order in the expansion, in the present model the Gaussian distribution with zero mean enables us to neglect every odd power of δt and to rewrite every even power of δt by a summation of products of the pairwise average $\langle\delta t\delta t\rangle$. This procedure is diagrammatically represented by pairing all δt insertions with dotted lines in all possible ways as is shown in Fig. 1, which illustrates the summation of terms appearing at the fourth order with respect to δt.

In the case of the bulk system, Lubensky[2] originally derived a renormalizational recursion formula of t_0, g_0, and ζ_0 up to the two-loop order to obtain fixed points and

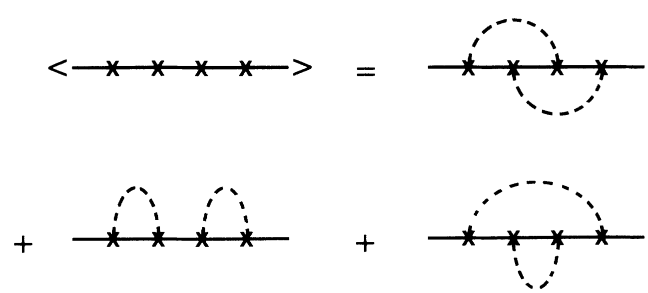

FIG. 1. A diagram of a correlation function at fourth order with the random iteration δt. This term corresponds to three second-order terms with respect to $\zeta_0\Delta$ after taking the random average.

bulk exponents. Because of the ambiguity in the choice of momentum cutoff in our spatially inhomogeneous system (see Refs. 8–10), however, here we alternatively use a formalism of minimal subtraction and dimensional regularization.[15] Below we mainly concern the system at criticality $t_0=0$. Let us introduce nondimensional coupling constants

$$u_0=\kappa^{-\epsilon}g_0, \quad w_0=\kappa^{-\epsilon}\zeta_0 , \tag{2.10}$$

where $\epsilon=4-d$, d being the spatial dimensionality, and write the renormalized bulk two-point vertex function and bulk four-point vertex function as

$$\Gamma_R^{(2)}(\mathbf{q}_i,u,w;\kappa)=Z_\phi(u,w;\kappa,\Lambda)\Gamma^{(2)}(\mathbf{q}_i,u_0,w_0;\Lambda) , \tag{2.11a}$$

$$\Gamma_{uR}^{(4)}(\mathbf{q}_i,u,w;\kappa)=Z_\phi^2(u,w;\kappa,\Lambda)\Gamma_u^{(4)}(\mathbf{q}_i,u_0,w_0;\Lambda) , \tag{2.11b}$$

$$\Gamma_{wR}^{(r)}(\mathbf{q}_i,u,w;\kappa)=Z_\phi^2(u,w;\kappa,\Lambda)\Gamma_w^{(4)}(\mathbf{q}_i,u_0,w_0;\Lambda) , \tag{2.11c}$$

where Z_ϕ indicates the field renormalization multiplier.

From now on, as a simplified diagrammatic representation, we will just use a single solid circle for the vertex g_0 and a single open circle for the vertex ζ_0, which corresponds to a pair of points connected by a dotted line in Fig. 1. Up to the two-loop order, Fig. 2 lists all the diagrams contributing to the bulk $\Gamma_u^{(4)}$, while those contributing to the bulk $\Gamma_w^{(4)}$ are given from Fig. 2 by interchanging solid and open circles. The weight for each diagram is the same as that given by Lubensky[2] for the bulk system, and also cited in Fig. 2; the above value is for $\Gamma_u^{(4)}$ and the below value is for $\Gamma_w^{(4)}$. Apart from these weights, all the integrals appearing in the diagrams are

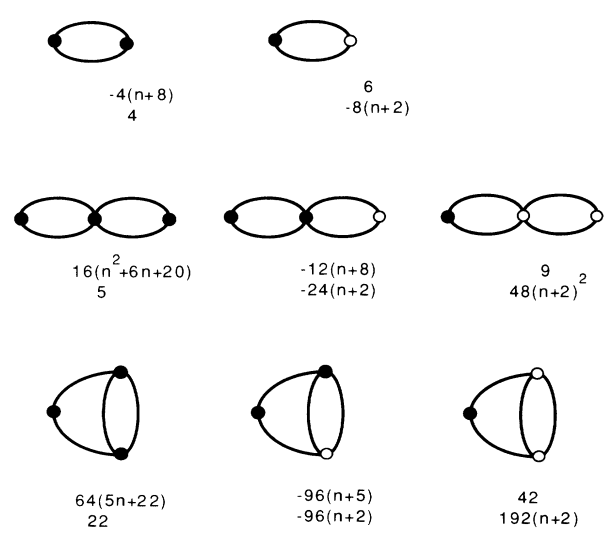

FIG. 2. The diagrams of the bulk four-point vertex contributing to the second loop. Solid and open circles correspond to the vertex g_0 and the vertex ζ_0, respectively.

the same as those appearing already in the pure system and given, respectively, by $I(\xi)$, $I^2(\xi)$, and $I_4(\xi)$ in Amit's book;[15] see Chaps. 7–9.

The renormalization of the coupling constants is determined so as to subtract every pole of $\Gamma_{uR}^{(4)}$ and $\Gamma_{wR}^{(4)}$ at each order in u and w. We generally write the relation between the renormalized coupling constants and the bare coupling constants up to second order as

$$u_0=u+u(a_1u+a_2w)+u(a_3u^2+a_4uw+a_5w^2) , \tag{2.12a}$$

$$w_0=w+w(d_1w+d_2u)+w(d_3w^2+d_4wu+d_5u^2) , \tag{2.12b}$$

which can be rewritten inversely as $u=u(u_0,w_0)$, $w=w(u_0,w_0)$. Deriving the renormalization equation for $\Gamma_R^{(N)}$ in just the same way as the pure system, we introduce the β function

$$\beta_u(u,w)=\left[\kappa\frac{\partial u}{\partial\kappa}\right]_{g_0,\zeta_0} , \quad \beta_w(u,w)=\left[\kappa\frac{\partial w}{\partial\kappa}\right]_{g_0,\zeta_0} . \tag{2.13}$$

The fixed point (u^*,w^*) is determined from putting $\beta_u(u^*,w^*)=\beta_w(u^*,w^*)=0$. Then four fixed points are identified: the random fixed point $(u^*\neq0,\ w^*\neq0)$, the Gaussian fixed point $(u^*=w^*=0)$, the pure fixed point $(u^*\neq0,\ \omega^*=0)$, and the unphysical fixed point $(u^*=0,\ w^*\neq0)$.

III. CALCULATION OF FIXED POINTS AND BULK η

All the diagrams contributing to the bulk two-point vertex function $\Gamma^{(2)}$ up to two loops are given by Figs. 3(a)–3(c). The renormalization of coupling constants does not affect $\Gamma^{(2)}$ at this order and we may simply replace u_0 and w_0 by u and w. Therefore, we only need the renormalization of the wave function here. Let us write the renormalization factor as

$$Z_\phi=1+b_1u^2+b_2uw+b_3w^2 . \tag{3.1}$$

The relevant integrals appearing in the two-point vertex are all expressed by means of $E_e'|_{k^2=\kappa^2}$, which is given by Eq. (9-38) in Amit's book.[15] Note that the notation $u/4!$ in his book corresponds to our u since we are using the semi-infinite version of the Hamiltonian introduced by Lubensky.[2] Then we straightforwardly obtain

$$b_1=-\frac{4(n+2)}{\epsilon}, \quad b_2=\frac{n+2}{\epsilon}, \quad b_3=-\frac{1}{8\epsilon} . \tag{3.2}$$

Now that we have determined the renormalization factor (3.1) of the wave function let us turn our attention to the four-point vertex. Also, all the relevant integrals appearing in the four-point vertex are J_{sp} and J_{4sp} given by Eqs. (9-37) and (9-39) in Amit's book.[15] Thus, we obtain

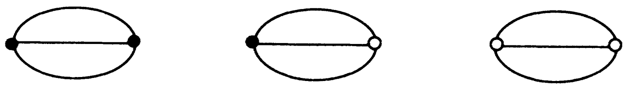

FIG. 3. The diagrams appearing in the bulk two-point function. Solid and open circles correspond to the vertex g_0 and the vertex ζ_0, respectively.

$$a_1=\frac{4(n+8)}{\epsilon},\quad a_2=-\frac{6}{\epsilon},\quad d_1=-\frac{4}{\epsilon},\quad d_2=\frac{8(n+2)}{\epsilon}, \tag{3.3}$$

$$a_3=\frac{6(n+8)^2}{\epsilon^2}-\frac{24(3n+14)}{\epsilon},\quad d_3=\frac{16}{\epsilon^2}-\frac{21}{4\epsilon}, \tag{3.4}$$

$$a_4=-\frac{12(5n+28)}{\epsilon^2}+\frac{2(11n+58)}{\epsilon},$$
$$d_4=-\frac{72(n+2)}{\epsilon^2}+\frac{22(n+2)}{\epsilon}, \tag{3.5}$$

$$a_5=\frac{30}{\epsilon^2}-\frac{41}{4\epsilon},\quad d_5=\frac{48(n+2)(n+4)}{\epsilon^2}-\frac{40(n+2)}{\epsilon}. \tag{3.6}$$

Then, solving $\beta_u(u^*,w^*)=\beta_w(u^*,w^*)=0$, we find the pure fixed point

$$u^*=\frac{\epsilon}{4(n+8)}+\frac{3(3n+14)}{4(n+8)^3}\epsilon^2+O(\epsilon^3),\quad w^*=0, \tag{3.7}$$

the random fixed point ($n\neq 1$)

$$u^*=\frac{\epsilon}{16(n-1)}\left[1+\frac{25n^2-248n+64}{128(n-1)^2}\epsilon\right]+O(\epsilon^3), \tag{3.8a}$$

$$w^*=\frac{\epsilon}{8(n-1)}\left[(4-n)+\frac{105n^3-364n^2+992n-256}{128(n-1)^2}\epsilon\right]+O(\epsilon^3), \tag{3.8b}$$

and the unphysical fixed point ($w^*\neq 0$ and $u^*=0$) besides the Gaussian fixed point ($\omega^*=u^*=0$). So far we have assumed $n>1$. In the case of the Ising system ($n=1$), the standard expansion with respect to ϵ breaks down and an alternative expansion in terms of $\sqrt{\epsilon}$ is required.[2,4] The corresponding random fixed point for the Ising system is given by

$$u^*=\sqrt{\epsilon/318}+O(\epsilon), \tag{3.9a}$$

$$w^*=\sqrt{6\epsilon/53}+O(\epsilon) \tag{3.9b}$$

up to the order $\sqrt{\epsilon}$.

Now that the fixed point is determined, next we evaluate the bulk critical exponent η from the relation [Eqs. (8-17) and (8-45) of Amit[15]]

$$\eta=\gamma_\phi(u^*,w^*)=\kappa\left[\frac{\partial \ln Z_\phi(u^*,w^*)}{\partial\kappa}\right]_{g_0,\zeta_0}. \tag{3.10}$$

From Eqs. (2.16) and (3.2), and taking Eq. (2.10) into account, we find

$$\eta=-2\epsilon(b_1u^{*2}+b_2u^*w^*+b_3w^{*2})$$
$$=8(n+2)u^{*2}-2(n+2)u^*w^*+\tfrac{1}{4}w^{*2}, \tag{3.11}$$

and, in turn,

$$\eta^{\text{random}}=\begin{cases}\dfrac{n(5n-8)}{256(n-1)^2}\epsilon^2+\dfrac{265n^4-1588n^3+2904n^2+1216n-512}{1024(n-1)^4}\epsilon^3+O(\epsilon^4), & n>1,\\ -\frac{1}{106}\epsilon+O(\epsilon^{3/2}), & n=1,\end{cases} \tag{3.12}$$

at the random fixed point and the usual

$$\eta=\frac{(n+2)}{2(n+8)^2}\left[1+\left[\frac{6(3n+14)}{(n+8)^2}-\frac{1}{4}\right]\epsilon\right]\epsilon^2+O(\epsilon^4) \tag{3.13}$$

at the pure fixed point. The ϵ^3 term in Eq. (3.12) for $n>1$ is our result for the bulk exponent, although the other results are the same as the previous ones.[5,15]

IV. RENORMALIZATION OF TWO-POINT FUNCTION IN THE SEMI-INFINITE GEOMETRY

According to Reeve and Guttmann,[9] we treat the two-point correlation function $G(\mathbf{q}_1,\mathbf{q}_2)$. Diagrammatically, this function is given by the sum of Figs. 4(a)–4(d). Here note that Fig. 4. collects only diagrams with different topologies and all the (four-point) vertices should be either

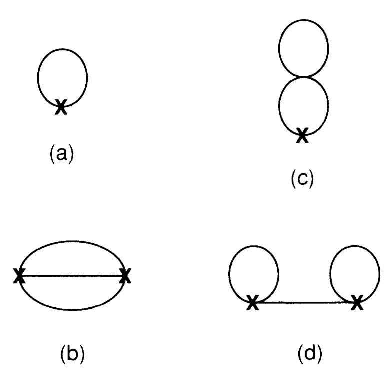

FIG. 4. The diagrams with different topologies, (a)–(d), appearing in the two-point correlation function in the semi-infinite geometry. All the diagrams are shown in Fig. 5.

solid or open circles; all the relevant diagrams are shown explicitly in Fig. 5. The diagrams consisting only of solid circles are the same as those for a pure system and the other diagrams are all new. The corresponding weights are written together in the same figure. It has been demonstrated[7,8] that, at the ordinary transition, the value of c moves toward $c=+\infty$ under the renormalization-group transformation, while the special transition moves from $c=0$ associated with mean-field theory to a nontrivial fixed point, $c=c^*$. Therefore, we start from two cases: $c=+\infty$ (the ordinary transition) and $c=0$ (the special transition).

From Eq. (2.6), we find that

$$\Delta(q_{1\perp},q_{2\perp},q_{3\perp},q_{4\perp})=\frac{\pi}{4}\sum_{\epsilon_i=\pm1}\epsilon_1\epsilon_2\epsilon_3\epsilon_4\delta\left[\sum_{j=1}^{4}\epsilon_j q_{j\perp}\right] \quad (4.1)$$

for the ordinary transition and

$$\Delta(q_{1\perp},q_{2\perp},q_{3\perp},q_{4\perp}) =\frac{\pi}{4}\mathrm{sgn}(q_{1\perp}q_{2\perp}q_{3\perp}q_{4\perp})\sum_{\epsilon_i=\pm1}\delta\left[\sum_{j=1}^{4}\epsilon_j q_{j\perp}\right] \quad (4.2)$$

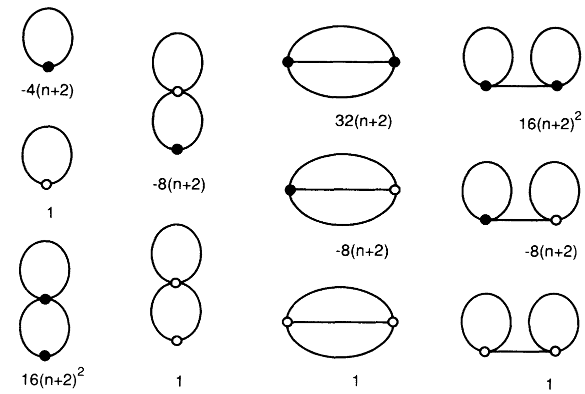

FIG. 5. All the diagrams relevant to the two-point correlation function in a semi-infinite geometry. Solid and open circles correspond to the vertex g_0 and the vertex ζ_0, respectively.

for the special transition. The integrals appearing in Figs. 4(a)–4(d) are given in Eqs. (4.1)–(4.4) and Eqs. (5.1)–(5.4) of Reeve and Guttmann,[9] which are written as $\mathcal{A}-\mathcal{D}$ here. Then we have

$$\begin{aligned}G(\mathbf{q}_1,\mathbf{q}_2)&=G^0(\mathbf{q}_1,\mathbf{q}_2)+[-4(n+2)u_0+w_0]\mathcal{A}+[32(n+2)u_0^2-8(n+2)u_0w_0+w_0^2]\mathcal{B}\\&\quad+[16(n+2)^2u_0^2-8(n+2)u_0w_0+w_0^2]\mathcal{C}+[16(n+2)^2u_0^2-8(n+2)u_0w_0+w_0^2]\mathcal{D}\\&=G^0(\mathbf{q}_1,\mathbf{q}_2)+[-4(n+2)u+w]+\frac{1}{\epsilon}[-16(n+2)(n+8)u^2+32(n+2)uw-4w^2]\mathcal{A}\\&\quad+[32(n+2)u^2-8(n+2)uw+w^2]\mathcal{B}+[16(n+2)^2u^2-8(n+2)uw+w^2]\mathcal{C}\\&\quad+[16(n+2)^2u^2-8(n+2)uw+w^2]\mathcal{D}\ ,\end{aligned} \quad (4.3)$$

where we used Eq. (2.12) for the renormalization of coupling constants.

To present explicitly the relevant integrals $\mathcal{A}-\mathcal{D}$ is relegated to Appendix A. Here one should note that the renormalization of the correlation function with respect to Z_ϕ of Eq. (2.16), $G^R(\mathbf{q}_1,\mathbf{q}_2)=Z_\phi^{-1}G(\mathbf{q}_1,\mathbf{q}_2)$, has an effect of just removing the pole of the local term proportional to δ in $\mathcal{B}$; we write such renormalized $\mathcal{B}$ as $\mathcal{B}^R$:

$$q_1^2q_2^2\mathcal{B}^R=q_1^2q_2^2\mathcal{B}+\frac{\delta}{8\epsilon}q_1^2\ . \quad (4.4)$$

The bulk η is determined by comparing the local term in $\mathcal{B}^R$ and the zeroth-order term, and we rederive Eq. (3.11). Next we show that, as a result of renormalization of coupling constants, all the poles in ϵ appearing in the nonlocal term proportional to δ' in $\mathcal{A}-\mathcal{D}$ exactly cancels out mutually. To see this we first notice that the pole in ϵ in these nonlocal terms appears just in the form

$$\frac{1}{\epsilon}\frac{1}{q_1^2q_2^2}(|\sigma_+|\mp|\sigma_-|)\ . \quad (4.5)$$

The coefficient of this function in $\mathcal{A}-\mathcal{D}$ is proportional to

$$\mathcal{A}:\ 4(n+2)(n+8)u^2-8(n+2)uw+w^2\ , \quad (4.6a)$$

$$\mathcal{B}^R:\ -24(n+2)u^2+6(n+2)uw-\tfrac{3}{4}w^2\ , \quad (4.6b)$$

$$\mathcal{C}:\ -4(n+2)^2u^2+2(n+2)uw-\tfrac{1}{4}w^2\ , \quad (4.6c)$$

$$\mathcal{D}:\ 0\ , \quad (4.6d)$$

and therefore the summation of all these terms gives zero. Therefore, we conclude that all the renormalization procedures done in the bulk random system are enough for the random semi-infinite system also.

In order to evaluate the surface critical exponents, we evaluate two functions

$$Q_{11}(q_\parallel)=G^R_{q_\parallel}(z_1,z_2)|_{z_1,z_2\sim0}\ , \quad (4.7a)$$

$$Q_1(q_\parallel)=\sum_{z_2}G^R_{q_\parallel}(z_1,z_2)|_{z_1\sim0}\ . \quad (4.7b)$$

One can determine the exponents $\eta_\parallel$ and $\eta_\perp$ from the asymptotic behavior $q_\parallel^{-1+\eta_\parallel}$ in $Q_{11}(q_\parallel)$ and $q_\parallel^{-2+\eta_\perp}$ in $Q_1(q_\parallel)$ in the limit $q_\parallel\to0$.

By using the relations

$$|\sigma_+|-|\sigma_-|=\mathrm{sgn}(q_{1\perp}q_{2\perp})\min(|q_{1\perp}|,|q_{2\perp}|)\ , \qquad (4.8a)$$

$$|\sigma_+|+|\sigma_-|=\mathrm{sgn}(q_{1\perp}q_{2\perp})\max(|q_{1\perp}|,|q_{2\perp}|)\ , \qquad (4.8b)$$

we evaluate the contribution of the relevant terms to $Q_{11}(q_\parallel)$ and $Q_1(q_\parallel)$, to give the values listed in Table I. Since we needed only the logarithmic correction in deriving the result given in this table, we first took the logarithmic derivative with respect to Λ and got the data. In order to derive $\eta_\perp$ from $Q_1(q_\parallel)$, we used the relation

$$\sum_z \sin q_\perp z \to \int_0^\infty dz\, \sin q_\perp z \to \frac{1}{q_\perp}\ . \qquad (4.9)$$

In the explicit integration in (4.9), one should introduce a cutoff function like $\exp[-z\Lambda^{-}1]$ in the integrand.

In the following, we separate the argument for the ordinary transition and the special transition to avoid the confusion.

(1) The ordinary transition. First we consider the case of the ordinary transition. From Table I (top) and the calculation given in Appendix B 1, we have

$$\eta_\parallel=2+[-4(n+2)u^*+w^*]\left[1+\frac{\epsilon}{2}\right]+8(n+2)(n+9)u^{*2}-18(n+2)u^*w^*+\tfrac{9}{4}w^{*2}+O(\epsilon^3)\ , \qquad (4.10a)$$

$$\eta_\perp=1+[-4(n+2)u^*+w^*]\left[\frac{1}{2}+\frac{\epsilon}{4}\right]+4(n+2)(n+10)u^{*2}-10(n+2)u^*w^*+\tfrac{5}{4}w^{*2}+O(\epsilon^3)\ . \qquad (4.10b)$$

For the pure fixed point given by (3.7), we readily obtain

$$\eta_\parallel^{\mathrm{pure}}=2-\frac{n+2}{n+8}\epsilon-\frac{(n+2)(17n+76)}{2(n+8)^3}\epsilon^2+O(\epsilon^3)\ , \qquad (4.11a)$$

TABLE I. The short-distance behaviors of terms appearing in the correlation function in mixed space. All the relevant terms are listed in this table both for the ordinary transition (top) and for the special transition (bottom). The first column corresponds to the terms in Fourier space.

	Ordinary transition	
Function $\times q_1^2q_2^2$	Contribution to $Q_{11}(q_\parallel)/q_\parallel$	Contribution to $Q_1(q_\parallel)q_\parallel$
$q_1^2\delta$	-1	1
$q_1^2\ln q_1^2\Lambda^{-2}\delta$	$-2\ln q_\parallel\Lambda^{-1}$	$2\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert-\lvert\sigma_-\rvert)\delta'$	$\frac{2}{\pi}[-1+\ln2\pm\ln q_\parallel\Lambda^{-1}]$	$\frac{1}{\pi}[1-2\ln2-\ln q_\parallel\Lambda^{-1}]$
$(\lvert\sigma_+\rvert\ln\lvert\sigma_+\rvert-\lvert\sigma_-\rvert\ln\lvert\sigma_-\rvert)\delta'$	$\frac{1}{\pi}(\ln q_\parallel\Lambda^{-1})^2$	$-\frac{1}{2\pi}[2\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert-\lvert\sigma_-\rvert)\ln[q_\parallel^2+(\lvert\sigma_+\rvert+\lvert\sigma_-\rvert)^2]\Lambda^{-2}\delta'$	$\frac{2}{\pi}[-2+4\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$	$\frac{1}{\pi}[2-4\ln2-\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert-\lvert\sigma_-\rvert)\ln[q_\parallel^2+(\lvert\sigma_+\rvert-\lvert\sigma_-\rvert)^2]\Lambda^{-2}\delta'$	$\frac{4}{\pi}[-2+2\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$	$\frac{2}{\pi}[1-3\ln2-\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$
$q_\parallel\tan^{-1}\frac{\lvert\sigma_+\rvert-\lvert\sigma_-\rvert}{q_\parallel}\delta'$	$\frac{1}{\pi}[2\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$	$-\frac{1}{\pi}\ln2\ln q_\parallel\Lambda^{-1}$
$\frac{\sigma_+^2-\sigma_-^2}{q_\parallel}\tan^{-1}\frac{q_\parallel}{\lvert\sigma_+\rvert+\lvert\sigma_-\rvert}\delta'$	$\frac{2}{\pi}\ln q_\parallel\Lambda^{-1}$	$-\frac{1}{\pi}\ln q_\parallel\Lambda^{-1}$

	Special transition	
Function $\times q_1^2q_2^2$	Contribution to $Q_{11}(a_\parallel)/q_\parallel$	Contribution to $Q_1(q_\parallel)q_\parallel$
$q_1^2\delta$	1	1
$q_1^2\ln q_1^2\Lambda^{-2}\delta$	$2\ln q_\parallel\Lambda^{-1}$	$2\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert+\lvert\sigma_-\rvert)\delta'$	$-\frac{2}{\pi}[\ln2+\ln q_\parallel\Lambda^{-1}]$	$-\frac{1}{\pi}\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert\ln\lvert\sigma_+\rvert-\lvert\sigma_-\rvert\ln\lvert\sigma_-\rvert)\delta'$	$-\frac{1}{\pi}(\ln q_\parallel\Lambda^{-1})^2$	$\frac{1}{2\pi}[2\ln2-\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$
$(\lvert\sigma_+\rvert+\lvert\sigma_-\rvert)\ln[q_\parallel^2+(\lvert\sigma_+\rvert+\lvert\sigma_-\rvert)^2]\Lambda^{-2}\delta'$	$-\frac{2}{\pi}[\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$	$-\frac{1}{\pi}(\ln q_\parallel\Lambda^{-1})^2$
$(\lvert\sigma_+\rvert+\lvert\sigma_-\rvert)\ln[q_\parallel^2+(\lvert\sigma_+\rvert-\lvert\sigma_-\rvert)^2]\Lambda^{-2}\delta'$	$-\frac{4}{\pi}[2\ln2+\ln q_\parallel\Lambda^{-1}]\ln q_\parallel\Lambda^{-1}$	$-\frac{2}{\pi}(\ln q_\parallel\Lambda^{-1})^2$
$q_\parallel\tan^{-1}\frac{q_\parallel}{\lvert\sigma_+\rvert+\lvert\sigma_-\rvert}\delta'$	0	0
$\frac{\sigma_+^2-\sigma_-^2}{q_\parallel}\tan^{-1}\frac{q_\parallel}{\lvert\sigma_+\rvert-\lvert\sigma_-\rvert}\delta'$	$-\frac{2}{\pi}\ln2\ln q_\parallel\Lambda^{-1}$	0

$$\eta_\perp^{\rm pure}=1-\frac{n+2}{2(n+8)}\epsilon-\frac{(n+2)(4n+167)}{(n+8)^3}\epsilon^2+O(\epsilon^3)\ , \tag{4.11b}$$

which is the same as the one obtained earlier by Diehl and Dietrich,[8] Reeve and Guttmann,[9] and Ohno and Okabe.[16] Next, inserting the random fixed point (3.8) into (4.10), we have

$$\eta_\parallel^{\rm random}=\begin{cases}2-\dfrac{3n}{8(n-1)}\epsilon-\dfrac{n(135n^2-708n+96)}{1024(n-1)^3}\epsilon^2+O(\epsilon^3), & n>1\ ,\\ 2-\sqrt{6\epsilon/53}+O(\epsilon), & n=1\ ,\end{cases} \tag{4.12a}$$

$$\eta_\perp^{\rm random}=\begin{cases}1-\dfrac{3n}{16(n-1)}\epsilon-\dfrac{n(115n^2-656n+64)}{2048(n-1)^3}\epsilon^2+O(\epsilon^3), & n>1\ ,\\ 1-\sqrt{3\epsilon/106}+O(\epsilon), & n=1\ .\end{cases} \tag{4.12b}$$

Now it is easy to see that the scaling relation $\eta_\parallel=1/\nu$ proposed by Bray and Moore[16] [in the present case this relation should read as $\eta_\parallel^{\rm random}=1/\nu^{\rm random}$, and $\nu^{\rm random}$ is given in Eq. (6.2) in Sec. VI] is also violated at the random fixed point, while the surface scaling relation[6–11,19] $\eta^{\rm random}=2\eta_\perp^{\rm random}-\eta_\parallel^{\rm random}$ holds up to this order.

(2) The special transition. The argument for the special transition is the same as the ordinary transition. From Table I (bottom) and Appendix B 2, we have

$$\eta_\parallel=[-4(n+2)u^*+w^*]\left[1+\frac{\epsilon}{2}\right]+8(n+2)(n+21)u^{*2}-42(n+2)u^*w^*+\tfrac{21}{4}w^{*2}+O(\epsilon^3)\ , \tag{4.13a}$$

$$\eta_\perp=[-4(n+2)u^*+w^*]\left[\frac{1}{2}+\frac{\epsilon}{4}\right]+4(n+2)(n+22)u^2-22(n+2)u^*w^*+\tfrac{11}{4}w^{*2}+O(\epsilon^3)\ . \tag{4.13b}$$

For the pure fixed point given by (3.7), we readily obtain

$$\eta_\parallel^{\rm pure}=-\frac{n+2}{n+8}\epsilon-\frac{5(n+2)(n-4)}{2(n+8)^3}\epsilon^2+O(\epsilon^3)\ , \tag{4.14a}$$

$$\eta_\perp^{\rm pure}=-\frac{n+2}{2(n+8)}\epsilon-\frac{(n+2)(n-7)}{(n+8)^3}\epsilon^2+O(\epsilon^3)\ , \tag{4.14b}$$

which were already obtained earlier.[8,9,16] For the random fixed point (3.8), we have

$$\eta_\parallel^{\rm random}=\begin{cases}-\dfrac{3n}{8(n-1)}\epsilon+\dfrac{3n(35n^2+28n+96)}{1024(n-1)^3}\epsilon^2+O(\epsilon^3), & n>1\ ,\\ -\sqrt{6\epsilon/53}+O(\epsilon), & n=1\ ,\end{cases} \tag{4.15a}$$

$$\eta_\perp^{\rm random}=\begin{cases}1-\dfrac{3n}{16(n-1)}\epsilon-\dfrac{n(125n^2+32n+32)}{2048(n-1)^3}\epsilon^2+O(\epsilon^3), & n>1\ ,\\ -\sqrt{3\epsilon/106}+O(\epsilon), & n=1\ .\end{cases} \tag{4.15b}$$

We also confirmed the surface scaling relation[6–11,16] $\eta^{\rm random}=2\eta_\perp^{\rm random}-\eta_\parallel^{\rm random}$ for the special transition.

V. THE REAL-SPACE CORRELATION FUNCTION: CONFORMAL INVARIANCE

It has been demonstrated that the correlation function at criticality generally has the conformal invariance[17–21,11] in real space. Figure 6 illustrates the geometry specifying the two-point vertex or correlation function between two points $\mathbf{r}_1=(\mathbf{r}_{1\parallel},z_1)$ and $\mathbf{r}_2=(\mathbf{r}_{2\parallel},z_2)$. As a consequence of the conformal invariance, the two-point correlation (or vertex) function depends only on the real distance $r=|\mathbf{r}_1-\mathbf{r}_2|$ and the image distance $\bar{r}=[\rho^2+(z_1+z_2)^2]^{1/2}$ with $\rho=|\mathbf{r}_{1\parallel}-\mathbf{r}_{2\parallel}|$. Now we show that this statement actually holds also for the random system.

The mean-field propagator in mixed space is given by

$$G^0_{q_\parallel}(z,z')=\sqrt{zz'}\int_0^\infty \frac{kdk}{q_\parallel^2+k^2}J_{\mu_0}(kz)J_{\mu_0}(kz')$$
$$=\begin{cases}\sqrt{zz'}I_{\mu_0}(q_\parallel z)K_{\mu_0}(q_\parallel z'), & \text{for } z<z' ,\\ \sqrt{zz'}K_{\mu_0}(q_\parallel z)I_{\mu_0}(q_\parallel z'), & \text{for } z>z' ,\end{cases} \tag{5.1}$$

where $\mu_0=\frac{1}{2}$ for the ordinary transition ($c=\infty$) and $\mu_0=-\frac{1}{2}$ for the special transition ($c=0$); J, I, and K denote the (modified) Bessel functions.[21] Transforming this function into real space by using an integral formula given in p. 686 (6.578.11) of Ref. 21, we get

$$G^0_\rho(z,z')=\frac{2}{(2\pi)^{(d-1)/2}}\frac{\sqrt{zz'}}{\rho^{(d-3)/2}}\int dq_\parallel q^{(d-1)/2}J_{(d-3)/2}(q_\parallel\rho)I_{\mu_0}(q_\parallel z)K_{\mu_0}(q_\parallel z')$$
$$=\frac{1}{(2\pi)^{d/2}}(zz')^{1-d/2}e^{-[(d-2)/2]\pi i}\frac{Q^{(d-2)/2}_{\mu_0-1/2}(v)}{(v^2-1)^{(d-2)/4}} , \tag{5.2}$$

where we put $\rho=|r_\parallel-r'_\parallel|$ and

$$v=\frac{\rho^2+z^2+z'^2}{2zz'} . \tag{5.3}$$

By performing the second-order calculation along the same line as in the pure case,[19] we finally obtain a complete form of the correlation function in real space as

$$G_\rho(z,z')=(zz')^{(2-d-\eta)/2}[f^{(1)}(v)+f^{(2)}(v)] , \tag{5.4}$$

$$f^{(1)}(v)=\frac{2^{(d-3)/2}\Gamma[(1/2)(d-1)]K_{d-1}}{\sqrt{2\pi}}e^{-(d-2)\pi i/2}\frac{Q^{(d-2)/2}_{\mu-1/2}(v)}{(v^2-1)^{(d-2)/4}} , \tag{5.5}$$

$$f^{(2)}(v)=\eta g^{(0)}(v)\int^v\frac{dv}{y}\int_\infty^v dv\, y(v^2-1)^{-1}\int_\infty^v\frac{dv}{y}\int_\infty^v dv\, y^3(v^2-1)^{-3} \tag{5.6}$$

with

$$\begin{aligned} &y=1,\ g^{(0)}(v)=1/(2\pi)^2(v^2-1)\ \text{ for the ordinary transition} ,\\ &y=v,\ g^{(0)}(v)=v/(2\pi)^2(v^2-1)\ \text{ for the special transition} .\end{aligned} \tag{5.7}$$

This correlation function has the same form as in the pure system and satisfies the conformal invariance: It is only the function of real distance r and image distance $\bar{r}$, in other words, it is only the function of zz' and v. One can very easily obtain the real-space correlation function at the random fixed point if one inserts the random values for the exponents η and μ (strictly speaking, we should use instead the notation η^{random} and μ^{random}, but here we omit all these superscripts); the exponent μ is related to the usual surface critical exponent $\eta_\parallel$ as $\mu=(\eta_\parallel-1)/2$ both for the ordinary transition and for the special transition. Exactly the same argument also holds in the case of $n=1$ where we need the expansion with respect to $\sqrt{\epsilon}$; since the exponent $\eta_\parallel$ has been obtained only to the first order in $\sqrt{\epsilon}$ in this paper, the explicit form of the real-space correlation function, Eq. (5.5), might be thought of as $O(\sqrt{\epsilon})$. However, if this exponent $\eta_\parallel$ could be obtained up to the second order in $\sqrt{\epsilon}$ as well as the exponent η [see Eq. (3.14)], the exact same form of the correlation function, Eq. (5.5), should also be obtained in this case.

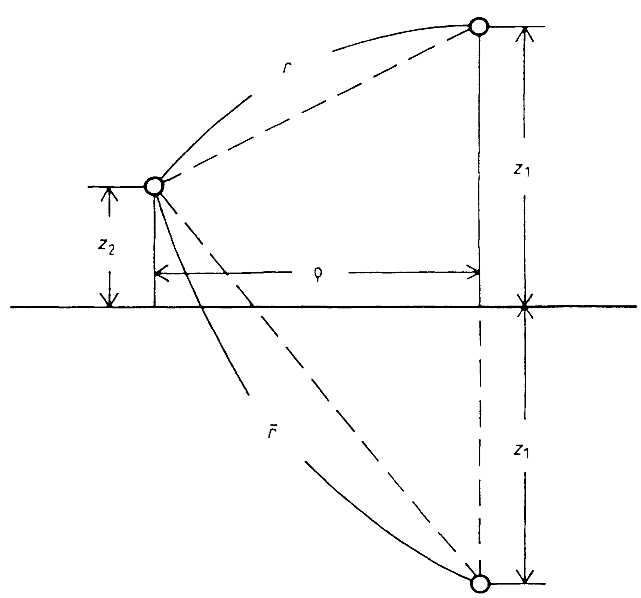

FIG. 6. The illustration of the conformal invariance. The two-point functions depend only on the real distance r and the image distance $\bar{r}$, hence, is the function of only zz' and v.

VI. SUMMARY AND DISCUSSIONS

In this paper we have studied the critical phenomena of the S^4 model having a Gaussian randomness in the semi-infinite geometry. The renormalization-group equation using minimal subtraction and dimensional regularization has led us to the surface decay exponents in the ϵ expansion. The random surface exponents are given in (4.12) and (4.15) for the ordinary and special transitions, respectively. We have confirmed the surface scaling relation $\eta^{\text{random}}=2\eta_{\perp}^{\text{random}}-\eta_{\parallel}^{\text{random}}$ and the conformal invariance of the two-point correlation function for both transitions.

Other surface critical exponents such as the susceptibility exponents γ_1^{random} and $\gamma_{11}^{\text{random}}$ or the magnetization exponent β_1^{random}, which are rather relevant in real experiments, are obtained from the surface scaling relations[6,7]

$$\gamma_1=\nu(2-\eta_{\perp}) ,$$
$$\gamma_{11}=\nu(1-\eta_{\parallel}) , \qquad (6.1)$$
$$\beta_1=\frac{\nu}{2}(d-2+\eta_{\parallel}) ,$$

where we omitted the suffix *random* because these relations also hold for the pure system. For the random system, the bulk correlation exponent should read as

$$\nu^{\text{random}}=\begin{cases}\frac{1}{2}+\frac{3n}{32(n-1)}\epsilon+\frac{n(127n^2-572n-32)}{4096(n-1)^3}\epsilon^2+O(\epsilon^3), & n>1 ,\\ \frac{1}{2}+\frac{1}{4}\sqrt{6\epsilon/53}+O(\epsilon), & n=1 .\end{cases} \qquad (6.2)$$

TABLE II. Several surface and bulk critical exponents for the pure Ising system and for the random Ising system in three dimensions, which are obtained from the direct use of the $\epsilon=4-d$ expansion up to the first order (for the surface) and second order (for the bulk).

Exponent surface	Pure Ordinary	Pure Special	Random Ordinary	Random Special
$\eta_{\parallel}$	1.60	−0.30	1.66	−0.34
$\eta_{\perp}$	0.81	−0.14	0.83	−0.17
γ_1	0.70	1.35	0.68	1.26
γ_{11}	−0.36	0.82	−0.38	0.78
β_1	0.80	0.23	0.77	0.17

Bulk	Pure	Random
η	0.032	−0.009
γ	1.24	1.17
β	0.33	0.29

The comparison of pure and random exponents are listed in Table II for the Ising system in three dimensions, where the random fixed point is expected to be stable because $\alpha>0$. The bulk exponents are also given in Table II for comparison. The effect of randomness is rather small, but can be observable in experiments.

A short comment will be given on the formulation of treating randomness. We have obtained all the results without a replica trick in the present study. The same result should, in principle, be derived from the replica theory.

Finally, we mention the effect of surface randomness in the present result. As far as the bulk upper critical dimension is $d_c>3$, which is expected in the usual case, this effect is irrelevant as Diehl and Nüsser[22] analyzed recently. However, they showed that, at the bulk tricritical point where $d_c=3$, the effect of surface randomness indeed affects the pure surface critical behavior. In this respect, the extension of the present work to the case at the bulk tricritical point may be the next problem.

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APPENDIX A

The integrals $\mathcal{A}$–$\mathcal{D}$ are evaluated as (the upper sign corresponds to the ordinary transition and the lower sign corresponds to the special transition)

$$q_1^2q_2^2\mathcal{A}=-\frac{2\pi}{4}\left[1+\frac{\epsilon}{2}\right](1-\epsilon\ln 2)[|\sigma_+|^{1-\epsilon}\mp|\sigma_-|^{1-\epsilon}]\delta'+O(\epsilon^2) ,$$

$$q_1^2q_2^2\mathcal{B}=\frac{\delta}{K_d^2}\int_{\mathbf{k}_1\mathbf{k}_2}\frac{1}{k_1^2k_2^2(\mathbf{q}_1-\mathbf{k}_1+\mathbf{k}_2)^2}+B ,$$

$$B=-\frac{3\pi}{2}\left[1+\frac{\epsilon}{2}\right](1-\epsilon\ln 2)\left[\frac{1}{\epsilon}+\frac{1}{2}\right][|\sigma_+|^{1-\epsilon}\mp|\sigma_-|^{1-\epsilon}]\delta'$$
$$+\frac{3\pi}{2}(|\sigma_+|\mp|\sigma_-|)\left\{\frac{1}{2}(\pm1-1)\pm\left[\frac{|\sigma_+|+|\sigma_-|}{q_1}\right]^{\pm1}\tan^{-1}\frac{q_1}{|\sigma_+|+|\sigma_-|}+\frac{1}{2}\ln[(|\sigma_+|+|\sigma_-|)^2+q_1^2]\Lambda^{-2}\right\}\delta' ,$$

$$q_1^2 q_2^2 \mathcal{C} = -\frac{2\pi}{4}\left[1+\frac{\epsilon}{2}\right]^2 (1-\epsilon\ln 2)^2 \frac{1}{\epsilon}[|\sigma_+|^{1-\epsilon} \mp |\sigma_-|^{1-\epsilon}]\delta' \mp \frac{2\pi}{8}[|\sigma_+|-|\sigma_-|]\delta' ,$$

$$q_1^2 q_2^2 \mathcal{D} = \frac{2\pi}{16}\left\{ (|\sigma_+| \mp |\sigma_-|)\left[\pm 2 + \ln[q_1^2 + (|\sigma_+|+|\sigma_-|)^2] - \ln[q_1^2+(|\sigma_+|-|\sigma_-|)^2] \pm 2\frac{|\sigma_+| \pm |\sigma_-|}{q_1}\tan^{-1}\frac{q_1}{|\sigma_+|\pm|\sigma_-|}\right] - 2q_1 \tan^{-1}\left[\frac{|\sigma_+|\mp|\sigma_-|}{q_1}\right]^{\pm 1}\right\}\delta' . \quad \text{(A1)}$$

In the case of the ordinary transition (upper sign), these formulas correspond to Eqs. (4.1)–(4.4) of Reeve and Guttmann[9] with several corrections in $\mathcal{D}$: the third term is missing in Eq. (4.4) of Reeve and Guttmann[9] and the fourth term has a different sign. Furthermore, in the case of the special transition (lower sign), our Eqs. (4.1)–(4.4) correspond to Eqs. (5.1)–(5.4) of Reeve and Guttmann,[9] in which several corrections are also necessary: the last term both in $\mathcal{B}$ and in $\mathcal{C}$ has a different sign. The other formulas are all the same.

APPENDIX B

We give the detailed calculation of $\eta_\parallel$ and $\eta_\perp$ in this appendix.

1. The ordinary transition

From Table I (top) we derive

$$\begin{aligned} Q_{11}(q_\parallel)/q_\parallel = & -1-[-4(n+2)u+w](-1+\ln 2+\ln q_\parallel \Lambda^{-1}) \\ & +\ln q_\parallel \Lambda^{-1}\{-\epsilon[-4(n+2)u+w](\tfrac{1}{2}-\ln 2) - [-16(n+2)(n+8)u^2+32(n+2)uw-4w^2](\tfrac{1}{2}-\ln 2) \\ & +[32(n+2)u^2-8(n+2)uw+w^2](-\tfrac{13}{4}+6\ln 2) \\ & +[16(n+2)^2u^2-8(n+2)uw+w^2](\ln 2)+\text{irrelevant terms}+O(\epsilon^3)\} . \end{aligned} \quad \text{(B1a)}$$

$$\begin{aligned} Q_1(q_\parallel)q_\parallel = & 1+[-4(n+2)u+w](-\tfrac{1}{2}+\ln 2+\tfrac{1}{2}\ln q_\parallel \Lambda^{-1}) \\ & +\ln q_\parallel \Lambda^{-1}\{\epsilon[-4(n+2)u+w](\tfrac{1}{4}-\ln 2)+[-16(n+2)(n+8)u^2+32(n+2)uw-4w^2](\tfrac{1}{4}-\ln 2) \\ & +[32(n+2)u^2-8(n+2)uw+w^2](\tfrac{7}{4}-6\ln 2) \\ & +[16(n+2)^2u^2-8(n+2)uw+w^2](\tfrac{1}{4}-\tfrac{3}{2}\ln 2)+\text{irrelevant terms}+O(\epsilon^3)\} . \end{aligned} \quad \text{(B1b)}$$

In this result at second order the square of the first-order term divided by 2! is also included. In order to estimate $\eta_\parallel$ and $\eta_\perp$, one should subtract that squared term; doing this explicitly, one gets

$$\begin{aligned} \eta_\parallel = & 2+[-4(n+2)u^*+w^*]\left[1+\frac{\epsilon}{2}-\epsilon\ln 2\right] \\ & +u^2[8(n+2)(n+9)-16(n+2)(n+8)\ln 2]+[8(n+2)u^*w^*-w^{*2}](-\tfrac{9}{4}+4\ln 2)+O(\epsilon^3) , \end{aligned} \quad \text{(B2a)}$$

$$\begin{aligned} \eta_\perp = & 1+[-4(n+2)u^*+w^*]\left[\frac{1}{2}+\frac{\epsilon}{4}-\epsilon\ln 2\right] \\ & +u^2[4(n+2)(n+10)-16(n+2)(n+8)\ln 2]+[8(n+2)u^*w^*-w^{*2}](-\tfrac{5}{4}+4\ln 2)+O(\epsilon^3) . \end{aligned} \quad \text{(B2b)}$$

Now we write the coefficient of ln2 as L: The coefficients for Eqs. (B2a) and (B2b) are the same and read

$$L = -\epsilon[-4(n+2)u^*+w^*]-16(n+2)(n+8)u^{*2}+32(n+2)u^*w^*-4w^{*2}+O(\epsilon^3) . \quad \text{(B3)}$$

It is easy to see at the pure and random fixed points given, respectively, by (3.7) and (3.8) that this value L vanishes exactly up to order ϵ^2. Therefore, we derive Eqs. (4.10a) and (4.10b) in the text.

2. The special transition

The argument for the special transition is the same as the ordinary transition. From Table I (bottom) in this case we derive

$$
\begin{aligned}
Q_{11}(q_\parallel)/q_\parallel = 1 + [-4(n+2)u+w](\ln 2 + \ln q_\parallel \Lambda^{-1}) \\
+ \ln q_\parallel \Lambda^{-1}\{\epsilon[-4(n+2)u+w](\tfrac{1}{2} - \ln 2) + [-16(n+2)(n+8)u^2 + 32(n+2)uw - 4w^2](\tfrac{1}{2} - \ln 2) \\
+ [32(n+2)u^2 - 8(n+2)uw + w^2](\tfrac{25}{4} - 6\ln 2) \\
+ [16(n+2)^2u^2 - 8(n+2)uw + w^2](1 - n2) + \text{irrelevant terms} + O(\epsilon^3)\} \ .
\end{aligned} \tag{B4a}
$$

$$
\begin{aligned}
Q_1(q_\parallel)q_\parallel = 1 + [-4(n+2)u+w]\tfrac{1}{2} \ln q_\parallel \Lambda^{-1} \\
+ \ln q_\parallel \Lambda^{-1}\{\epsilon[-4(n+2)u+w]\tfrac{1}{4} + [-16(n+2)(n+8)u^2 + 32(n+2)uw - 4w^2]\tfrac{1}{4} \\
+ [32(n+2)u^2 - 8(n+2)uw + w^2]\tfrac{13}{4} \\
+ [16(n+2)^2u^2 - 8(n+2)uw + w^2]\tfrac{1}{2} + \text{irrelevant terms} + O(\epsilon^3)\} \ .
\end{aligned} \tag{B4b}
$$

Subtracting the (squared) first-order term as in the ordinary transition, one gets

$$
\begin{aligned}
\eta_\parallel = [-4(n+2)u^* + w^*]\left[1 + \frac{\epsilon}{2} - \epsilon \ln 2\right] \\
+ u^{*2}[8(n+2)(n+21) - 16(n+2)(n+8)\ln 2] + [8(n+2)u^*w^* - w^{*2}](-\tfrac{21}{4} + 4\ln 2) + O(\epsilon^3)
\end{aligned} \tag{B5}
$$

and Eq. (4.13b) for $\eta_\perp$. Again, the coefficient of ln2 in Eq. (B5) is the same as L defined in (B3) and, at pure and random fixed points, this L vanishes exactly up to order ϵ^2. Consequently, Eq. (B5) gives Eq. (4.13a) in the text.

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