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Energy transfer between ripplons and phonons in liquid helium at low temperatures

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We discuss the energy transfer between ripplons and phonons in liquid helium at temperatures $T < 1$ K. The dominant process proves to be one wherein a phonon incident on the liquid surface is completely absorbed with the emission of two ripplons. The inelastic scattering of a phonon from the surface is found to be less effective. The thermal contact between ripplons and phonons is very poor at low temperatures; below 0.1 K it is worse than that between bulk helium and copper. This can have serious consequences for experiments on atomic hydrogen.

The structure and dynamics of the free surface of superfluid helium have been of interest for many years. The original idea of Atkins[1] in treating the long-wavelength excitations as quantized capillary waves, or ripplons, has been successfully used in further investigations (see, e.g., the review article by Edwards and Saam[2]). A very interesting problem associated with the free surface is the interaction of the ripplons with the bulk-helium phonons. This interaction determines the temperature difference between ripplon and phonon subsystems in the presence of heating. It also plays an essential role in determining the ripplon temperature distribution when the phonon temperature is nonuniform. Remarkably, no measurements of ripplon-phonon thermal contact have been published to our knowledge.

In this paper we report a calculation of the energy transfer between ripplons and phonons at low temperatures ($T < 1$ K). We find that the most important process is one wherein a phonon incident on the surface is completely absorbed with the emission of two ripplons, as was pointed out in Ref. 3. The inelastic scattering of phonons from the surface, discussed by Saam,[4] proves to be less effective. This follows from phase volume arguments since, at the low temperatures discussed here, the characteristic momentum of a thermal ripplon is much larger than that of a thermal phonon.

With ripplon temperature T_R and phonon temperature T_P, we calculate the energy flux $\mathcal{P}(T_R, T_P)$ from ripplons to phonons. For small differences in ripplon and phonon temperatures, this can be expressed in terms of a boundary conductance G, which should be readily accessible to experiment. Measurement of G would be a direct test of the quantum hydrodynamic theory of ripplon-phonon interaction. Additionally, the calculated conductance is low enough to have important consequences for proposed experiments on atomic hydrogen.[5]

The quantization of the hydrodynamics of a compressible liquid with a free surface was first done by Saam.[4] Beginning from the linearized equations of motion, he derived a complete and orthonormal set of eigenmodes (ripplons and surface-reflected phonons) and obtained a quadratic ripplon-phonon Hamiltonian in diagonal form. In order to discuss energy and momentum transfer between the elementary excitations, this Hamiltonian must be supplemented by higher order terms. The most important additional terms are those for three-particle interactions, that is, terms cubic in the ripplon and phonon creation and annihilation operators. The dominant cubic term involving the ripplons is

$$H = \frac{\rho}{2}\int\int dx\, dy\, (\mathbf{v}\cdot\zeta\mathbf{v})|_{z=0}, \tag{1}$$

where $\rho \simeq 145\,\mathrm{kg\,m^{-3}}$ is the equilibrium density and the liquid occupies the half-space $z \leq 0$. The fluid velocity can be written as a sum of phonon and ripplon velocities: $\mathbf{v} = \mathbf{v}_P + \mathbf{v}_R$ and the dominant contribution to the surface displacement ζ is due to ripplons. We use a simplified version of the quantum hydrodynamics of Ref. 4, appropriate for low-energy excitations. The velocity field due to phonons is

$$\mathbf{v}_P = \sum_{\mathbf{q},k}\left(\frac{\hbar c^2}{V\rho\omega_{qk}}\right)^{1/2}[i\mathbf{q}\sin(kz) + \hat{\mathbf{z}}k\cos(kz)] \times \exp(i\mathbf{q}\cdot\mathbf{r}_\parallel)(a_{\mathbf{q}k} + a^\dagger_{-\mathbf{q}k}), \tag{2}$$

where $c \simeq 238\,\mathrm{m\,s^{-1}}$ is the speed of sound, V is the volume of the liquid, $\omega_{qk} = c(q^2 + k^2)^{1/2}$ is the phonon frequency, and $a_{\mathbf{q}k}$ is the annihilation operator for a phonon of wave vector $\mathbf{q}$ parallel to the surface and $k > 0$ perpendicular to it. The velocity field due to ripplons is

$$\mathbf{v}_R = \sum_{\mathbf{q}}\left(\frac{\hbar\sigma q^2}{2A\rho^2\omega_q}\right)^{1/2}(i\mathbf{q} + \hat{\mathbf{z}}q) \times \exp(qz)\exp(i\mathbf{q}\cdot\mathbf{r}_\parallel)(b_{\mathbf{q}} + b^\dagger_{-\mathbf{q}}), \tag{3}$$

where $\sigma \simeq 3.78\times10^{-4}\,\mathrm{J\,m^{-2}}$ is the (zero-temperature) surface tension, A is the surface area, $\omega_q = (\sigma q^3/\rho)^{1/2}$ is the ripplon frequency, and $b_{\mathbf{q}}$ is the annihilation operator for a ripplon of wave vector $\mathbf{q}$. The surface displacement due to ripplons is given by

$$\zeta = \sum_{\mathbf{q}} \left(\frac{\hbar\sigma q^2}{2A\rho^2\omega_q}\right)^{1/2} \frac{q}{\omega_q} \exp(i\mathbf{q}\cdot\mathbf{r}_\parallel)(b_{\mathbf{q}} + b^\dagger_{-\mathbf{q}}). \tag{4}$$

The ripplon-phonon coupling due to H involves four distinct processes. Denoting a ripplon by R and a phonon by P, these processes are R⇌P+P, R⇌P+R, R+P⇌P, and R+R⇌P. The first two processes are forbidden on kinematic grounds. The third process was considered by Saam, in the context of a calculation of the energy absorbed from a phonon beam incident on the free surface at zero temperature.[4] This process is inefficient for energy transfer, as we will discuss later. The most efficient channel for transfer of energy between ripplons and phonons is the process R+R⇌P.

We have calculated the energy flux from the surface of liquid helium to the bulk due to the process R+R⇌P in the case where the ripplons and phonons have finite (but different) temperatures T_{R} and T_{P}. The flux (power per unit area) is

$$\mathcal{P}(T_{\mathrm{R}}, T_{\mathrm{P}}) = \int V \frac{d^2q'\, d^2q\, dk}{(2\pi)^4\pi} \frac{2\pi}{\hbar} \delta(\omega_{q'} + \omega_{q''} - \omega_{qk})\,\omega_{qk}\,|\langle \mathbf{q}\,k|H|\mathbf{q}', \mathbf{q}''\rangle|^2 \times[N_{q'}N_{q''}(1+n_{qk}) - n_{qk}(1+N_{q'})(1+N_{q''})], \tag{5}$$

where $\mathbf{q}'' = \mathbf{q} - \mathbf{q}'$ (conservation of momentum parallel to the surface), and $N_{q'}(T_{\mathrm{R}})$, $N_{q''}(T_{\mathrm{R}})$, and $n_{qk}(T_{\mathrm{P}})$ are the equilibrium occupation numbers for ripplons and phonons, respectively. The transition amplitude $\langle \mathbf{q}\,k|H|\mathbf{q}', \mathbf{q}''\rangle$ refers to the process R+R→P with initial state $N_{\mathbf{q}'} = N_{\mathbf{q}''} = 1$, $n_{\mathbf{q}\,k} = 0$. In order not to have an overcomplete set of ripplon states, the integration over d^2q' in Eq. (5) should be performed only over half of the momentum space.

The transition amplitude $\langle \mathbf{q}\,k|H|\mathbf{q}', \mathbf{q}''\rangle$ is found in first-order perturbation theory. The perturbing Hamiltonian H given by Eq. (1) is expressed in terms of creation and annihilation operators using Eqs. (2), (3), and (4). Since the momenta of the ripplons are larger than the momentum of the phonon, we approximate $\mathbf{q}'' \simeq -\mathbf{q}'$. The transition amplitude is then

$$\langle \mathbf{q}\,k|H|\mathbf{q}', \mathbf{q}''\rangle \simeq \left(\frac{\hbar^3c^2}{V\rho\omega_{qk}}\right)^{1/2} q'\,k\,\delta_{\mathbf{q},\mathbf{q}'+\mathbf{q}''}. \tag{6}$$

From Eq. (5) we obtain

$$\mathcal{P}(T_{\mathrm{R}}, T_{\mathrm{P}}) = \left(\frac{\rho}{\sigma}\right)^{4/3} \frac{\hbar^2}{\rho c^3} \left(\frac{k_{\mathrm{B}}T_{\mathrm{R}}}{\hbar}\right)^{20/3} \frac{8}{9\pi^2} \int_0^\infty dx\, x^{17/3} \left[\frac{1}{(e^x-1)^2} - \frac{1}{e^{2\beta x}-1}\left(\frac{e^x+1}{e^x-1}\right)\right], \tag{7}$$

where $\beta = T_{\mathrm{R}}/T_{\mathrm{P}}$. This expression for $\mathcal{P}$ may be used to determine the power available to cool the ripplons or a system coupled to the liquid helium through the ripplons. The maximum cooling power at a given ripplon temperature is obtained with $T_{\mathrm{P}} \ll T_{\mathrm{R}}$, and is

$$\mathcal{P}(T_{\mathrm{R}}, 0) = (8.4\times10^3\,\mathrm{W\,m^{-2}\,K^{-20/3}})\,T_{\mathrm{R}}^{20/3}. \tag{8}$$

For small differences in ripplon and phonon temperatures, we can define a Kapitza conductance

$$G \equiv \left(\frac{\partial\mathcal{P}(T_{\mathrm{R}}, T)}{\partial T_{\mathrm{R}}}\right)_{T_{\mathrm{R}}=T} = (5.3\times10^4\,\mathrm{W\,m^{-2}\,K^{-20/3}})\,T^{17/3}. \tag{9}$$

At temperatures below 0.1 K this conductance is smaller than that between liquid helium and copper, and it becomes rapidly worse with decreasing temperature. Consequently, situations may be realized in which the ripplons are heated so that T_{R} is appreciably higher than T_{P}. This heating may be due, for example, to ripplon thermal conduction from warmer regions, or to accommodation of atomic hydrogen gas.

We have also calculated the energy transfer for the process R+P⇌P of inelastic scattering of a phonon from the surface. We find a flux

$$\mathcal{P}_{\mathrm{R+P\rightleftharpoons P}} \simeq (6\times10^2\,\mathrm{W\,m^{-2}\,K^{-8}})\,T_{\mathrm{P}}^7\,(T_{\mathrm{R}} - T_{\mathrm{P}}) \tag{10}$$

corresponding to a conductance which is much smaller than that for the two-ripplon process. For energy transfer from ripplons, inelastic scattering of phonons is not as effective as phonon emission and absorption.

The finite ripplon-phonon coupling plays an essential role in determining the temperature distribution on the surface (in the ripplon subsystem) resulting from a nonuniform phonon temperature. A temperature inhomogeneity in the phonons will manifest itself in the ripplons, but smeared out over a characteristic length $l = (\kappa/G)^{1/2}$, where κ is the ripplon thermal conductivity and G is the conductance to phonons given by Eq. (9). For ripplons on a 100-Å-thick helium film, $\kappa \simeq 40\,\mathrm{nW\,K^{-1}}$ at $T = 0.1\,\mathrm{K}$.[6] Using Eq. (9) we find the characteristic length $l \simeq 0.6\,\mathrm{mm}$. For this film thickness the ripplon thermal conductivity is determined by the interaction of the ripplons with the substrate. For thicker films or bulk helium κ (and hence l) will be even larger.

The long characteristic length l has grave consequences for proposed "cold spot" experiments on atomic

hydrogen.[5] It is desired to attain the Kosterlitz-Thouless transition in atomic hydrogen adsorbed on the surface of liquid helium. In order to have a reasonable recombination lifetime and heat load, only a small spot on the liquid helium film coating the walls can be cooled to a (ripplon) temperature where surface adsorption is appreciable. This spot should be bulk liquid helium or capillary-filled sinter, and should have a radius smaller than 1 mm. The ripplon temperature should be ~ 0.1 K (the bulk gas temperature being ~ 0.3 K). Additionally, in order to make a cold spot with a well-defined ripplon temperature, the spot size should be much larger than the characteristic length l. The results of this paper indicate that both requirements may not be satisfied simultaneously.

One may hope that this problem can be circumvented by operating at higher temperatures, thereby reducing l. But then, in order to achieve the Kosterlitz-Thouless transition, the bulk density of atomic hydrogen should be very high. In this case, the thermal contact between the hydrogen gas and ripplons is much better than that between ripplons and phonons, and the ripplon temperature will be very close to the temperature of the hot hydrogen gas, and hence homogeneous. In this sense there will be no "cold spot."

In conclusion, we have calculated the energy transfer between ripplons and phonons in superfluid liquid helium. For small temperature differences we can define a ripplon-phonon thermal boundary conductance G which, for temperatures below 0.1 K, is worse than that between bulk liquid helium and copper. The weak thermal contact between ripplons and phonons can be very important in proposed experiments on atomic hydrogen and must be taken into account.

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