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Coefficients of the second viscosity in bulk liquid helium

Chung-In Um
Department of Physics, College of Science, Korea University, Seoul 136-701, Korea

Chul-Won Jun
Department of Physics, College of Science, Kyung-Nam University, Ma-san 630-701 Kyung-Nam, Korea

Thomas F. George*
Departments of Chemistry and Physics, Washington State University, Pullman, Washington 99164-1046

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The coefficients of the second viscosity in bulk liquid helium are evaluated explicitly as a function of temperature via interactions between the anomalous excitation spectrums. The four coefficients of the second viscosity exhibit a T^{-1} dependence at low temperatures (well below 0.6 K), which is due to three-phonon processes originating from the anomalous dispersion at very low momenta and temperatures.

Since Peshkov[1] has confirmed experimentally the existence of a temperature wave called the second sound in bulk liquid helium, Landau[2] evaluated at first the ordinary second-sound velocity in terms of the macroscopic thermal quantities and later derived the second sound as a collective density wave in the elementary excitation spectrum. Transport coefficients, i.e., thermal conductivity, viscosity, and other kinetic coefficients, were also calculated above 0.6 K by means of kinetic equations by Landau[3] and Khalatnikov.[4]

It is well known that at low temperatures and low pressures the thermal properties of superfluid helium are dominated by low-momentum phonons, which do not possess the normal dispersion, but rather an anomalous dispersion. In fact, these two cases are quite different in microscopic processes. The former is governed by four-phonon processes (4PP) while the latter is by three-phonon processes (3PP). Recently, several authors[5] have evaluated the temperature variations of the first and second sound by using the wrong normal dispersion. However, in more recent articles,[6] we have derived a Landau-type elementary excitation spectrums, which has anomalous phononlike behavior and rotonlike behavior at low and high momenta, respectively, through the ring-diagram approximation in two- and three-dimensional liquid helium. In the bulk case, the obtained excitation spectrum is given by

$$E(p)=c_0p[1+\delta_1p^2-\delta_2p^3+\delta_3p^4+\cdots], \tag{1}$$

$$E(P)=\Delta+\frac{(P-P_0)^2}{2\mu}, \tag{2}$$

where Δ, P_0, and μ are the roton parameters, c_0 is the sound velocity at absolute zero temperature, and δ_1 and δ_2 are positive constants which can be determined by the potential parameters. Here, we have adopted a soft potential with a Lennard-Jones-type tail, which helps make a smooth connection between the attractive part and a soft repulsive core. Using these excitation spectra, we have successfully derived and explained the various sounds,[7] sound attenuations,[8] and transport coefficients[9] in two- and three-dimensional liquid ^{4}He.

Concerning the coefficients of the second viscosity in bulk liquid helium, Kirkpatrick and Dorfman[10] evaluated these coefficients at very low temperature ($na\lambda^1 \gg 1$) on the basis of their kinetic equations[11] for a dilute superfluid, where n, a, and λ represent the number density, s-wave scattering length, and thermal wavelength, respectively. In a previous paper[12] we have successfully evaluated the coefficients ζ_1, ζ_2, ζ_3, and ζ_4 of the second viscosity in two-dimensional liquid ^{4}He as a function of temperature by solving the superfluid hydrodynamics through the theory of the kinetic phenomena developed by Khalatnikov.

In this paper, adopting the above theory, we carry out a three-dimensional calculation to evaluate the coefficients of the second viscosity of bulk liquid helium via 3PP in the region of very low temperatures, which was not done by Landau and Khalatnikov. Near zero temperature, the important mechanism is 3PP: the direct process of emission of a phonon $\mathbf{p}$ by $\mathbf{p}_1\rightarrow\mathbf{p}_2+\mathbf{p}$ and the reverse process of absorption of a phonon $\mathbf{p}$ by $\mathbf{p}_2+\mathbf{p}\rightarrow\mathbf{p}_1$. The differential decay rate for 3PP in three dimensions is defined as

$$d\omega=\frac{2\pi}{\hbar}|\langle F|H_3|I\rangle|^2\delta(E_F-E_I)\frac{V^2d\mathbf{p}_2d\mathbf{p}}{(2\pi\hbar)^6}, \tag{3}$$

where H_3 is the Hamiltonian for 3PP.[9] The transition amplitude between the temperature-dependent initial state $|I\rangle$ and final states $|F\rangle$ for the direct process is given by[9]

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$$\langle F|H_3|I\rangle=\frac{3!}{2}\frac{(2\pi\hbar)^3}{(2V)^{3/2}}\delta(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p})\left\{\left[\frac{c_0p}{\rho p_1p_2}\right]^{1/2}(\mathbf{p}_1\cdot\mathbf{p}_2)+\frac{\rho^2}{3c_0^2}\frac{\partial}{\partial\rho}\left[\frac{c_0^2}{\rho}\right]\left[\frac{c_0p_1p_2p}{\rho}\right]^{1/2}\right\}[n_{p_1}(n_{p_2}+1)(n_p+1)]^{1/2}\ . \tag{4}$$

Here n_p is the phonon distribution function with momentum $\mathbf{p}$. In the case of the reverse process, the last bracket in Eq. (4) should be replaced by $[(n_{p_1}+1)n_{p_2}n_p]^{1/2}$. Then the total decay rate for the direct process becomes

$$\omega_D=\frac{\pi c_0}{2\hbar\rho}(u+1)^2\int p_1p_2pn_{p_1}(n_{p_2}+1)(n_{p_1}+1)\times\delta(E_F-E_I)\frac{d\mathbf{p}_2}{(2\pi\hbar)^3}\ , \tag{5}$$

where $u=(\rho/c)(\partial c/\partial\rho)$ is the Grüneisen constant. For the reverse process, $n_{p_1}(n_{p_2}+1)(n_p+1)$ is replaced by $(n_{p_1}+1)n_{p_2}n_p$.

From the total decay rate for the direct and reverse processes, the collision integral for the 3PP becomes

$$J_{3\mathrm{PP}}=-\frac{\pi c_0}{2\hbar\rho}(u+1)^2\int p_1p_2p\delta n(n_{p_{10}}-n_{p_{20}})\times\delta(E_F-E_i)\frac{d\mathbf{p}_2}{(2\pi\hbar)^3}\ , \tag{6}$$

where δn is equal to $n_p-n_{p_0}$, and $n_{p_{10}}$ and $n_{p_{20}}$ represent the equilibrium distribution functions for the phonons with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Since 3PP occur largely in the small angle events, the phonon distribution, which depends on the chemical potential α',[13] can be written as

$$n=\{\exp[(\alpha'+pc)/k_BT]-1\}^{-1}\ , \tag{7}$$

and expanding δn in a power series in terms of α', we obtain $\delta n=n-n_0=-n_0(n_0+1)\alpha'$. The relation[4] between the collision integral and the kinetic coefficient can be expressed by

$$\frac{1}{k_BT}\int J_{3\mathrm{PP}}(n)\frac{d\mathbf{p}}{(2\pi\hbar)^3}=\alpha'\Gamma_{3\mathrm{PP}}\ , \tag{8}$$

and solving Eq. (8) for the kinetic coefficient, we get

$$\Gamma_{3\mathrm{PP}}=\frac{3!\pi\zeta(3)(u+1)^2}{60\hbar^7\rho c_0^8}(k_BT)^7\ . \tag{9}$$

To obtain the coefficients of the second viscosity, we follow the calculation in the third way given by Khalatnikov and show below only the results near absolute-zero temperature:

$$\zeta_1(T)=\frac{9N_{\mathrm{ph}}^2(T)}{\rho\Gamma_{3\mathrm{PP}}}\left[\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right]^2\ , \tag{10}$$

$$\zeta_2(T)=\frac{9N_{\mathrm{ph}}^2(T)}{\Gamma_{3\mathrm{PP}}}\left[\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right]^2\ , \tag{11}$$

$$\zeta_3(T)=\frac{9N_{\mathrm{ph}}^2(T)}{\rho^2\Gamma_{3\mathrm{PP}}}\left[\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right]^2\ , \tag{12}$$

$$\zeta_4(T)=\zeta_1(T)\ . \tag{13}$$

Here, the Grüneisen constant comes from differentiation of N_{ph} with respect to ρ and can be taken as constant near absolute-zero temperature. $\zeta_1(T)$ and $\zeta_4(T)$ are equal and thus satisfy Onsager's reciprocity symmetric principle. $N_{\mathrm{ph}}(T)$ is the number of phonons per unit volume at temperature T, and taking up to order p^4 in Eq. (1), we can express this as

$$N_{\mathrm{ph}}(T)=\alpha_0T^3[1+\alpha_1T^2-\alpha_2T^3+\alpha_3T^4]\ , \tag{14}$$

where

$$\begin{aligned}\alpha_0&=\frac{36\times4!\zeta(4)}{\pi^3}\left[\frac{k_B}{hc_0}\right]^3,\\ \alpha_1&=\frac{3\times6!\zeta(6)}{4!\zeta(4)}\delta_1\left[\frac{k_B}{c_0}\right]^2,\\ \alpha_2&=\frac{4\times8!\zeta(8)}{4!\zeta(4)}\delta_2\left[\frac{k_B}{c_0}\right]^3,\\ \alpha_3&=\frac{5\times8!\zeta(8)}{4!\zeta(4)}\delta_3\left[\frac{k_B}{c_0}\right]^4.\end{aligned} \tag{15}$$

For numerical calculations, we have adopted the parameters which were deduced from the analysis of the elementary excitation of the bulk liquid-helium data,[6,14] as listed in Table I. We have also taken the numerical value of u to be 1.8.[15] Using these parameters and substituting Eqs. (9) and (14) into Eqs. (10)–(12), we obtain

$$\zeta_1(T)=\frac{6.313\times10^{-4}}{T}A(T)\ , \tag{16}$$

$$\zeta_2(T)=\frac{9.156\times10^{-5}}{T}A(T)\ , \tag{17}$$

TABLE I. Parameters for bulk liquid helium.

n (Å^{-3})	Δ/k_B (K)	q_0 (Å^{-1})	μ	c_0 (m/s)	δ_1 (Å^2)	δ_2 (Å^3)	δ_3 (Å^4)
2.18×10^{-2}	8.616	1.930	$0.153m_{\mathrm{He}}$	238.21	1.51 ± 0.13	3.25 ± 0.20	~16

$$\zeta_3(T)=\frac{4.355\times10^{-3}}{T}A(T)\ , \tag{18}$$

where

$$A(T)=1+2\alpha_1T^2-2\alpha_2T^3+(2\alpha_3+\alpha_1^2)T^4 -2\alpha_1\alpha_2T^5+(\alpha_2^2+2\alpha_1\alpha_3)T^6+\cdots\ . \tag{19}$$

In Eqs. (16)–(18), all four coefficients include various T-dependent terms. The T^{-1} dependence also appears as the main contribution in the coefficient of the first viscosity,[9]

$$\eta(T)=\frac{3!\zeta(3)\zeta(2)\rho c_0^2}{\pi^5k_B(u+1)^2}\frac{1}{T}=1.90\times10^{-1}\frac{1}{T}\ . \tag{20}$$

This means that as the temperature approaches absolute zero, the roton contribution disappears, and thus the main contribution comes from 3PP. Therefore the temperature dependence of the first viscosity coefficient is changed dramatically from T^{-5} to T^{-1}.

Kirkpatrick and Dorfman[10] have introduced the classical kinetic theory technique to derive two-fluid hydrodynamics in an expansion of the unknown functions in terms of orthogonal polynomials, keeping only the first term, which represents the first Enskog approximation in the classical theory.[16] They have also introduced an isotropic linear collision operator L that can be divided into two types of collision processes: an excited particle scatters with a condensed particle to produce two excited particle, or vice versa (L_{12}), and two excited particles scatter with each other and produce two new excited particles (L_{22}). These two processes correspond exactly to the processes 3PP ($\mathbf{p}_1\rightleftarrows\mathbf{p}_2+\mathbf{p}_3$) and 4PP ($\mathbf{p}_1+\mathbf{p}_2\rightleftarrows\mathbf{p}_3+\mathbf{p}_4$). The collision operator L_{12} holds only for very small momenta. When the number of particles decreases with decreasing temperature, there are not enough collisions to keep a local equilibrium, and thus their hydrodynamic solution breaks down, which may also occur in very dilute classical gases. This means that Kirkpatrick and Dorfman have not taken into account 3PP, but only 4PP.

A finite lifetime of an elementary excitation in a quantum liquid originates from two processes: one is the collision between elementary excitations, and the other is the spontaneous decay of a phonon into two, three, or more phonons. Since the collision probability tends to zero at very low temperatures, the collision process becomes less important and negligible. At very low temperatures, the main excitations in liquid ^{4}He are phonons. On the other hand, we note that 3PP are the most important process in the anomalous elementary excitation spectrum of liquid ^{4}He at very low temperatures and very small momenta.

Before Kirkpatrick and Dorfman, Ma[17] and Popov[18] obtained the same results, i.e., a damping factor on the order of q^2T^{-9}, where q is the wave number of the disturbance in the calculation of the hydrodynamic eigenvalue describing second-sound propagation. Comparing this with that of Kirkpatrick and Dorfman, we find the relation $q^2\eta/\rho_n\sim q^2T^{-9}$, and thus have $\eta\sim T^{-5}$, where ρ_n is proportional to T^4. Popov has also derived a T^{-5} dependence of η through the fundamental integral analysis, which turns out to be mainly the same contribution from 4PP in Landau-Khalatnikov theory.[19]

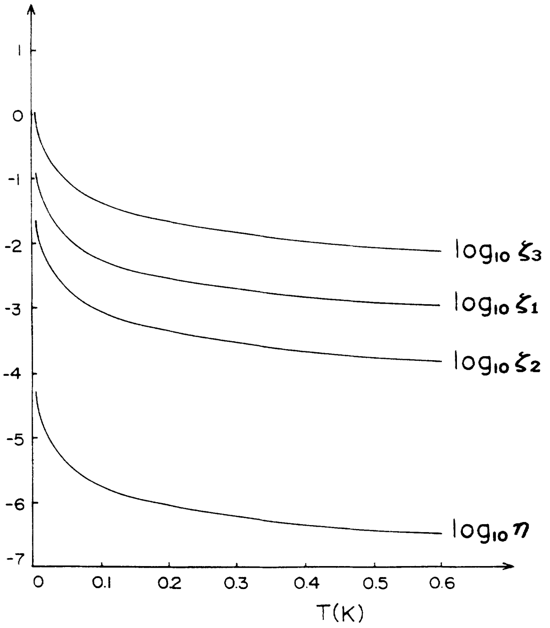

FIG. 1. Temperature variation of the coefficient of the first viscosity (η) and the four coefficients of the second viscosity, ζ_1 ($=\zeta_4$), ζ_2, and ζ_3. The curves represent only the magnitudes of these coefficients, where the units are g/cm s for η and ζ_2, cm^2/s for ζ_1 ($=\zeta_4$), and cm^5/g s for ζ_3.

In the calculation for $\eta(T)$ and $\zeta(T)$ given by Kirkpatrick and Dorfman, $\eta(T)$ is proportional to T^{-5}, while the thermal conductivity $\kappa(T)$ is proportional to T^2, and all four coefficients of the second viscosity have only a T^3 dependence for very low temperature. However, Eqs.

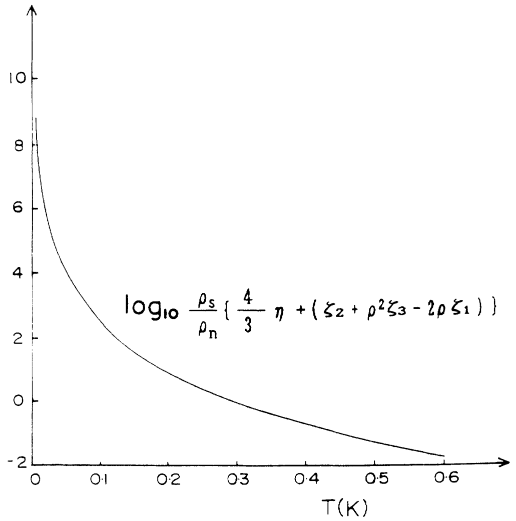

FIG. 2. Coefficients of the first and sound viscosity as a function of temperature in the functional form by which they appear in the absorption coefficient of second sound given by Khalatnikov. The quantities ρ_s and ρ_n are the superfluid and normal densities, respectively.

(16)–(18) include not only T^3 terms but also an important T^{-1} dependence. As $T \rightarrow 0$, their four coefficients tend directly to zero. However, Eqs. (16)–(18) show a divergence, which is quite contrary to their results. We emphasize again the difference between contributions from 3PP and 4PP. The discrepancy in the results obtained by us and by Kirkpatrick and Dorfman is due to the fact that their hydrodynamic solution accommodates only 4PP, which does not appear at very low temperatures in the Landau-Khalatnikov theory, and breaks down when 3PP are included, while our theory includes only 3PP, which are the main contribution to the transport coefficients near zero temperature and at very small momenta. We note that there are generally thirteen independent dissipation coefficients.[20] Here we only consider five coefficients and do not discuss the pressure and frequency variation[21] of the coefficients of the second viscosity.

Figure 1 illustrates the temperature variations and magnitudes of the four coefficients of the second viscosity and the coefficient of the first viscosity. The above five coefficients are all positive. For very low temperatures (well below $T < 0.6$ K), the condition $\zeta_1^2 < \zeta_2 \zeta_3$ holds. Comparing the order of magnitudes for the four coefficients, we can write $\zeta_3 > \zeta_1 > \zeta_2$, and all four coefficients have larger values than that of the coefficient of the first viscosity. We note that the thermal conductivity (κ) and five coefficients, as mentioned above, for thin liquid-helium films have the same properties as those for bulk liquid helium.[12]

Figure 2 represents the values of the coefficients[22] of the first and second viscosity as a function of temperature in the functional form by which they appear in the absorption coefficient of the second sound given by Khalatnikov. The behavior of this curve increases moderately as temperature decreases in the range $T > 0.2$ K and rapidly diverges near zero temperature. It shows a similar behavior to that in the high-temperature region ($T > 0.6$ K).

In conclusion, the coefficients of the sound viscosity in bulk liquid helium exhibit a T^{-1} dependence for temperatures below 0.6 K, like that of the first viscosity, which is due to 3PP originating from the anomalous excitation spectrum of bulk liquid helium at low momenta and low temperatures. However, there are no experimental results. A precise measurement of the temperature variation of the second viscosities would be desirable.

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*Author to whom correspondence should be addressed.

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