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Coupled nonlinear electrodynamics of type-II superconductors in the mixed state

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A theory of the nonlinear electrodynamics of isotropic high-κ type-II superconductors containing an array of vortices is presented. The theory generalizes a self-consistent approach to vortex dynamics wherein the effects of nonlocality, vortex inertia, pinning, flux flow, and flux creep are treated in a unified fashion. We derive and solve a single vector partial differential equation describing the nonlinear response of a type-II superconductor at frequencies well below the gap frequency. The generation of nth-order harmonics due to bilinear field nonlinearity is discussed.

This paper is concerned with the nonlinear electrodynamics of isotropic type-II superconductors in the mixed state. Our phenomenological theory self-consistently includes the coupling of the superconductor response and the vortex lattice dynamics. The theory which we present generalizes our theory of linear response[1–6] and leads to increased knowledge of electromagnetic screening and dissipation processes in type-II superconductors.

In this work we take into account nonlinearity present in the equation of motion of the vortex lattice and its conservation law for vortex density. The nonlinearity which we concentrate on is bilinear in the electrodynamic fields; such a combination of vortex density and velocity occurs in the vortex continuity equation. Other nonlinearities arise in the field-dependent coefficients such as the dc conductivity, penetration depth, pinning potential, and viscous drag. The dependences of the latter two quantities mean that in general the vortex dynamic mobility[2] is nonlinear in the fields. In this paper we briefly indicate how these nonlinearities in principle can be accommodated.

This paper is a condensation of a report[7] that provides a fuller exposition, including a more detailed discussion of the reduction and nature of our governing partial differential equations and results in the linear response regime. The work on linear response most closely related to the present theory is Ref. 6, while Ref. 8 is related concerning the treatment of nonlinearity. The present theory generalizes these and is primarily directed to the study of the dynamics of Abrikosov rather than Josephson vortices.

We assume that a London treatment of Abrikosov vortices is valid, using a continuum approximation of the London equation with vortex term.[9] Our theory employs a general vortex equation of motion with a complex-valued dynamic mobility.[2] This function is taken to be a scalar in this treatment. For a discussion of the related vortex diffusion coefficient and related complex diffusion constants, see Ref. 6. The dynamic mobility allows for the simultaneous inclusion of the effects of vortex inertia, pinning in a periodic potential, flux flow, and flux creep.[2,4,5,10] Restrictions on the form of the pinning potential, which uses an average potential height, are discussed elsewhere.[6]

Nonlinearity resulting from magnetic history or critical state effects is not treated here. In particular, the vortex displacements should not be large in comparison with the intervortex spacing. The frequencies of interest are much less than the superconducting gap frequency $\sim\Delta/\hbar$ so that pair-breaking effects are not present. Under these assumptions, we are able to derive a single vector partial differential equation (PDE) describing the nonlinear response. This approach leads to the formulation of an initial-boundary value problem for one of the total (or net) fields or densities. Once one of these quantities has been found, the others follow from various electrodynamic relations such as Maxwell's equations. This procedure is illustrated in a particular geometry.

We solve the nonlinear vector PDE in a planar geometry and discuss the generation of the nth-order harmonics due to bilinear field nonlinearity. Specific results for complex penetration depths and amplitudes are given. The fields and densities are presented explicitly for the second harmonic.

Our theory includes quasiparticle excitations through a normal current density contribution. Since the normal fluid is accounted for, our results hold through the transition temperature or upper critical field. In particular, when the normal state is reached, our partial differential equations reduce to the usual diffusion equations for the magnetic induction or current density. As discussed elsewhere,[5] the lack of a continuous superconductor to normal-state description is a fault of many other theories.

The theory given here is potentially applicable to a wide range of experiments involving vortex dynamics. It is usual in rf experiments to determine whether they are in the linear or nonlinear regime. However, the information on the amplitude dependence is often not analyzed. Specifically, if pinning is not too strong, so that a critical state is avoided, and the dynamics tend to be dominated by creep, flux flow, and/or the normal fluid, the present theory may provide an adequate description. The types of experiments that may be amenable to such an analysis of their nonlinear behavior include surface impedance, rf permeability, and vibrating reed.[11–15]

As we consider those situations where flux-flow losses dominate over hysteretic ones, we expect fairly strong frequency dependence of derived quantities such as the total power loss, which is calculated in Ref. 7. By contrast, in theories of bulk-pinning hysteretic losses, use of the Bean, Kim, and other models for the critical current density yield very little or no frequency dependence.[15]

We present the governing nonlinear equations in our theory and, under the above assumptions, combine them to yield a single nonlinear vector partial-differential equation for the magnetic induction. To model the electrodynamics of the superconductor we use Maxwell's curl equations,

$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}},\ \nabla\times\mathbf{H}=\mathbf{J}\,, \tag{1}$$

and the supercurrent source equation[2-6]

$$\nabla\times\mathbf{J}_s=-\frac{1}{\mu_0\lambda^2}(\mathbf{B}-\mathbf{B}_v)\,, \tag{2}$$

where $\mathbf{B}_v\equiv n(\mathbf{x},t)\phi_0\hat{\mathbf{B}}_0$ is the local vortex magnetic field, $n(\mathbf{x},t)$ is the local areal density of vortices, $\hat{\mathbf{B}}_0=\mathbf{B}_0/B_0$ is their local direction, and ϕ_0 is the flux quantum. We further employ the two-fluid equation

$$\mathbf{J}=\mathbf{J}_n+\mathbf{J}_s\,, \tag{3}$$

where the normal current density is given by the constitutive relation $\mathbf{J}_n=\sigma_{\rm nf}\mathbf{E}$, $\sigma_{\rm nf}(B,T)$ being the local electrical conductivity of the normal fluid. The coupling of vortex motion and the total current density $\mathbf{J}$ is completed by the inclusion of a vortex equation of motion, which we take to be given by the general relation

$$\mathbf{v}(\mathbf{x},t)=\tilde{\mu}_v(\omega,B,T)\mathbf{f}(\mathbf{x},t) \tag{4}$$

where $\mathbf{v}$ is the vortex velocity, $\tilde{\mu}_v$ is the (complex) dynamic mobility,[2,4-6] and the Lorentz force $\mathbf{f}(\mathbf{x},t)=\mathbf{J}(\mathbf{x},t)\times\phi_0\hat{\mathbf{B}}_0$ is the driving force per unit length.

It is well known[4-6] that Eqs. (1)–(3) apart from the vortex equation of motion can be combined to give a generalized diffusion–London equation for the magnetic induction $\mathbf{B}(\mathbf{x},t)$:

$$\nabla^2\mathbf{B}=D_{\rm nf}^{-1}\dot{\mathbf{B}}+(1/\lambda^2)(\mathbf{B}-\mathbf{B}_v)\,. \tag{5}$$

In writing Eq. (5) we have assumed the normal-fluid conductivity to be a constant and have set the normal-fluid diffusion coefficient $D_{\rm nf}(B,T)=\rho_{\rm nf}/\mu_0$. Rewriting Eq. (5) as

$$\mathbf{B}_v=\mathbf{B}-\lambda^2\nabla^2\mathbf{B}+D_{\rm nf}^{-1}\lambda^2\dot{\mathbf{B}} \tag{6}$$

gives the local variation of the vortex density. In particular, Eq. (6) shows how the vortex and total magnetic fields differ due to the incomplete Meissner effect ($\lambda\neq0$) and the presence of the normal fluid. We ignore the B dependence of the London penetration depth λ. This point is discussed further on.

In order to combine Eqs. (4) and (5) a relation between the vortex velocity and magnetic induction is required. Such a relation is provided by the vortex continuity equation (conservation of flux lines),

$$\frac{\partial\mathbf{B}_v}{\partial t}=-\nabla\times(\mathbf{B}_v\times\mathbf{v})\,, \tag{7}$$

which is equivalent to Faraday's law with $\mathbf{E}_v=\mathbf{B}_v\times\mathbf{v}$ being the induced electric field. Upon taking the time derivative of Eq. (5) and using Eq. (7) we have

$$\nabla^2\dot{\mathbf{B}}=D_{\rm nf}^{-1}\ddot{\mathbf{B}}+(1/\lambda^2)[\dot{\mathbf{B}}+\nabla+(\mathbf{B}_v\times\mathbf{v})]\,. \tag{8}$$

The vortex velocity $\mathbf{v}$ can be eliminated in favor of $\mathbf{B}$ from Eq. (8) by using the equation of motion (4) and Faraday's law (1). Then using Eq. (6) for the vortex induction $\mathbf{B}_v$ gives a single nonlinear vector PDE for the total magnetic induction $\mathbf{B}(\mathbf{x},t)$:

$$\lambda^2\nabla^2\dot{\mathbf{B}}-\lambda^2D_{\rm nf}^{-1}\ddot{\mathbf{B}}-\dot{\mathbf{B}}=(\phi_0\tilde{\mu}_v/\mu_0)\nabla\times(\{\mathbf{B}-\lambda^2\nabla^2\mathbf{B}+D_{\rm nf}^{-1}\lambda^2\dot{\mathbf{B}}\}\times[(\nabla\times\mathbf{B})\times\hat{\mathbf{B}}_0])\,. \tag{9}$$

We have written the linear terms in Eq. (9) on the left-hand side. In linear response theory, the $\mathbf{B}_v$ terms (in curly brackets) on the right-hand side (RHS) of Eq. (9) are replaced by $\mathbf{B}_0$, the constant applied magnetic induction. Then Eq. (9) can be immediately integrated once with respect to time and the result agrees with the governing vector PDE derived in Ref. 6.

It is possible to include the displacement current term in Ampère's law (1) in the derivation of the generalized nonlinear flux diffusion-wave equation (9).[7] However, in the following the corresponding term in Eq. (9) is not retained.

The nonlinear terms on the RHS of Eq. (9), which are due to the motion of the vortices, have a special structure (when λ is assumed to be independent of B). Each term on the RHS is bilinear in the field and its derivatives. This fact has several important implications. In particular, the bilinearity is key in deriving an analytic solution in a specialized geometry, to which we now turn.

Here we present an application of Eq. (9) to planar geometry. The superconductor is chosen to occupy the half space $x\geq0$ and the applied magnetic field, a combination of static and time-varying fields, is taken to lie along the z direction. (We assume for convenience that any static field producing vortices satisfies $B_0/\mu_0\gtrsim 2H_{c1}$ where H_{c1} is the lower critical field.)

For this geometry, where $\mathbf{B}(\mathbf{x},t)=B(x,t)\hat{\mathbf{z}}$, $\hat{\mathbf{B}}_0=\hat{\mathbf{z}}$, and $\mathbf{J}(x,t)=-(\hat{\mathbf{y}}/\mu_0)\partial B/\partial x$, Eq. (9) becomes

$$\lambda^2\partial_{xx}\dot{B}-\lambda^2D_{\rm nf}^{-1}\ddot{B}-\dot{B}=-(\phi_0\tilde{\mu}_v/\mu_0)\partial_x\{[B-\lambda^2\partial_{xx}B+D_{\rm nf}^{-1}\lambda^2\dot{B}]\partial_xB\}\,, \tag{10}$$

where the notation $\partial_x\equiv\partial/\partial x$ is used. Equation (10) is a basic one-dimensional (in space) nonlinear PDE investigated in this paper. The bilinearity present on the RHS of Eq. (10) is reminiscent of equations which arise from certain nonlinear evolution and wave equations when Hirota's method of solution[16,17] is applied. This is an extensive subject—we note that the direct algebraic method of solving nonlinear PDEs (Refs. 18 and 19) has many similar features.

Before considering the nth-order harmonics, it is useful to check the linear dispersion relation that results from Eq. (10). Taking $B=B_0+B_1$ with $B_1 \ll B_0$, we have the linear equation

$$\lambda^2\partial_{xx}\dot{B}_1-\lambda^2 D_{\rm nf}^{-1}\ddot{B}_1-\dot{B}_1=-\tfrac{1}{2}\,\omega\tilde{\delta}_{vc}^2\partial_{xx}B_1\,, \qquad (11)$$

where we have introduced the complex effective skin depth $\tilde{\delta}_{vc}\equiv(2\tilde{\rho}_v/\mu_0\omega)^{1/2}$ and the complex effective resistivity associated with vortex motion and creep,[2,4-6] $\tilde{\rho}_v(\omega,B_0,T)=B_0\phi_0\tilde{\mu}_v(\omega,B_0,T)$. By taking for the rf or microwave induction $B_1=b_0e^{-i\omega t}e^{-x/\tilde{\lambda}}$ we obtain

$$\tilde{\lambda}^2=\frac{\lambda^2+\frac{1}{2}i\tilde{\delta}_{vc}^2}{1-2i\lambda^2\delta_{\rm nf}^{-2}} \qquad (12)$$

for the complex self-consistently determined penetration depth $\tilde{\lambda}=\tilde{\lambda}(\omega,B_0,T)$. (Here the square of the normal-fluid skin depth is given by $\delta_{\rm nf}^2=2D_{\rm nf}/\omega$.) Due to the semi-infinite geometry, Eq. (11), or Eq. (14) below, e.g., could be solved by Laplace transformation with respect to x.[6] The complex phenomenological penetration depth given in Eq. (12) generalizes that of several other theories of dynamic vortex response (e.g., Refs. 8, 12, 20, and 21).

We proceed to examine the complex penetration depths which arise in our theory for nonlinear vortex response. We first obtain the equation governing the jth-order harmonic ($j\geq 2$). We employ an infinite formal expansion

$$B(x,t)=\sum_{n=0}^{\infty}B_n(x,t)\epsilon^n \qquad (13)$$

in powers of an arbitrary parameter ϵ where $B_0\equiv$const. Upon substituting Eq. (13) into Eq. (10), equating separately the coefficients of like powers of ϵ to zero, and noting the simplifications arising for terms with $m=0$ and $m=n$, we find the recursion relation

$$\lambda^2\partial_{xx}\dot{B}_n-\lambda^2D_{\rm nf}^{-1}\ddot{B}_n-\dot{B}_n+\tfrac{1}{2}\,\omega\tilde{\delta}_{vc}^2\partial_{xx}B_n=-\frac{\phi_0\tilde{\mu}_v}{\mu_0}\partial_x\left\{\sum_{m=1}^{n-1}[B_m-\lambda^2\partial_{xx}B_m+D_{\rm nf}^{-1}\lambda^2\dot{B}_m]\partial_xB_{n-m}\right\}. \qquad (14)$$

Due to the special bilinear form of the RHS of Eq. (14), this equation can possess solutions of the form $B_j\sim B_{j0}e^{-ij\omega t}e^{-jpx}$ where the complex wave number p and amplitudes B_{j0} are to be determined by solving a nonlinear recursion relation. We recall that in the direct algebraic method[18,19] the solution of a nonlinear PDE with constant coefficients can be built up from the exponentials solving the linear part. The solution of the PDE (10) is similar.

An approximate field dependence of the London penetration depth is[22] $\lambda^2(B,T)\simeq\lambda^2(0,T)/[1-B/B_{c2}(T)]$. As can be seen from Eq. (9) or (10), the inclusion of such dependence results in factors $\partial_x\lambda^2=2\lambda(\partial\lambda/\partial B)\partial_xB$ which generally remove the bilinearity of the RHS of these equations. The inclusion of the London penetration depth's field dependence thus complicates the analytic solution for the field although in principle it is still possible. It is expected that the results presented here will nonetheless be applicable over a wide range of field, at least for the known high-temperature superconductors, due to their large values of upper critical field.

For the case of the second harmonic, bilinear terms in only B_1 appear on the RHS of Eq. (14). The solution for the second harmonic, subject to the boundary condition $B_2(x=0,t)=0$, can be found in the form

$$B_2(x,t)=B_{20}e^{-2i\omega t}(e^{-2x/\tilde{\lambda}_1}-e^{-x/\tilde{\lambda}_2})\,. \qquad (15)$$

Using the same form of B_1 as above gives a solution[7] provided that $\tilde{\lambda}_1=\tilde{\lambda}$,

$$\tilde{\lambda}_2^2=\frac{\lambda^2+\frac{1}{4}i\tilde{\delta}_{vc}^2}{1-4i\lambda^2\delta_{\rm nf}^{-2}}\,, \qquad (16)$$

and

$$B_{20}=\frac{b_0^2}{4B_0}\frac{\tilde{\lambda}^{-2}\tilde{\delta}_{vc}^4(1-2i\lambda^2\delta_{\rm nf}^{-2})}{(3\lambda^2+\frac{1}{2}i\tilde{\delta}_{vc}^2-4i\lambda^4\delta_{\rm nf}^{-2})}\,. \qquad (17)$$

The similarity of Eq. (16) to Eq. (12) can be noted. In fact, in certain special cases, typically at high temperature, we have $\tilde{\lambda}_2(\omega,B_0,T)\simeq\tilde{\lambda}(2\omega,B_0,T)$. This approximate relation will hold in the normal state, due to the form of the normal-fluid skin depth, or when viscous flux flow dominates the vortex dynamics. For then the complex penetration depths are dominated by the flux-flow skin depth $\delta_f=(2B_0\phi_0/\mu_0\eta\omega)^{1/2}$,[1-5,14] where η is the viscous drag coefficient (e.g., Ref. 23). At $T=T_{c2}$, the field-dependent transition temperature, or $B_0=B_{c2}$, the upper critical field, $\tilde{\delta}_{vc}$ vanishes and so does B_{20}.

Once the magnetic induction has been found, the other fields and densities follow from various electrodynamic relations. We have found that $B(x,t)=B_0+B_1(x,t)+B_2(x,t)$ is given by

$$B(x,t)=B_0+b_0e^{-i\omega t}e^{-x/\tilde{\lambda}}+B_{20}e^{-2i\omega t}(e^{-2x/\tilde{\lambda}}-e^{-x/\tilde{\lambda}_2})\,. \qquad (18)$$

The resulting total rf current density from Ampère's law is

$$J_y(x,t)=\frac{b_0}{\mu_0\tilde{\lambda}}e^{-i\omega t}e^{-x/\tilde{\lambda}}+\frac{B_{20}}{\mu_0}e^{-2i\omega t}\left[\frac{2}{\tilde{\lambda}}e^{-2x/\tilde{\lambda}}-\frac{1}{\tilde{\lambda}_2}e^{-x/\tilde{\lambda}_2}\right] \qquad (19)$$

and the resulting total electric field from Faraday's law is

$$E_y(x,t)=-\tilde{\lambda}i\omega b_0e^{-i\omega t}e^{-x/\tilde{\lambda}}+B_{20}2i\omega e^{-2i\omega t}\times(-\tfrac{1}{2}\tilde{\lambda}e^{-2x/\tilde{\lambda}}+\tilde{\lambda}_2e^{-x/\tilde{\lambda}_2})\,. \qquad (20)$$

The vortex velocity field is given by Eq. (4), so that integrating with respect to time yields the displacement field.[7] As shown in Ref. 7, Eqs. (19) and (20) can be used to compute $\mathbf{E}\cdot\mathbf{J}$ losses in the superconductor. Such a calculation generalizes the usual surface resistance (R_s) calculation to a nonlinear regime. By using Eq. (6), the vortex density $n(x,t)$ can be computed[7] and we find, e.g., that the ratio $|\tilde{\delta}_{vc}/\tilde{\lambda}|$ gives a measure of the vortex density variation in linear response. The vortex density variation and displacement vanish at $T=T_{c2}$ or $B_0=B_{c2}$, as they should.

There are several difficulties associated with the analytic solution of Eq. (9) in general geometries even for linear response, due to the vector nature of the equation. For a discussion of these topics in the context of linear response, Ref. 6 may be consulted. A brief discussion of the application of the nonlinear equation (9) to cylindrical geometry is given elsewhere.[7]

In summary, our self-consistent approach to vortex dynamics, including the effect of nonlocal vortex interaction, has been generalized to a nonlinear response regime. This nonlinear theory does not include critical state effects so that pinning should be weak for it to apply. We discussed the possible application of the nonlinear theory to rf experiments involving vortex dynamics. We derived a single vector partial-differential equation, Eq. (9), describing the nonlinear response in the mixed state and discussed the appearance of nth-order harmonics due to bilinear field nonlinearity. The solution of the nonlinear PDE, including complex penetration depths and amplitudes, was presented for a special planar geometry.

All aspects of our previous linear response theory are recovered in the limit of small driving forces. In addition, our results for nonlinear response hold through the transition temperature or upper critical field; when the normal state is reached, our governing partial differential equations reduce to the usual diffusion equations for the magnetic induction or current density.

In this paper we concentrated on problems where attenuation of the applied radiation dominated. We mentioned how the displacement current term in Ampère's law could be included if desired, which might be important at higher frequencies (e.g., in the infrared range). Furthermore, since we know that our linear response theory for attenuation-dominated problems can be extended to finite thickness samples,[4,5] we can expect that it can similarly be extended to propagation-dominated problems. The latter theory could be useful in the description of, e.g., microwave transmission and reflection measurements.

It is then possible to expect that our nonlinear response theory of radiation flow through a type-II superconductor can be extended to propagation-dominated problems. This theory would provide a description of nth-harmonic generation for transmission-type experiments, possibly in analogy to that encountered in nonlinear optics.

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