Resistivity of a one-dimensional interacting quantum fluid

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The frequency and temperature dependence of the conductivity of a one-dimensional fermion system with attractive interactions is studied by using a renormalization-group technique. At half filling the real part of the conductivity has it both a $\delta(\omega)$ part and a divergent frequency behavior $\omega^{-\nu}$ at finite frequencies, where ν is a nonuniversal exponent depending on the interactions. For the particular case of the attractive Hubbard model, logarithmic corrections appear and the conductivity behaves as $1/[\omega \ln^2(\omega)]$, plus a $\delta(\omega)$ part. Away from half filling the conductivity has a $\delta(\omega)$ part and a gap up to a critical frequency ω_c, where ω_c is proportional to the doping with a prefactor depending on the interactions. The results obtained for the fermion model can be straightforwardly extended to the conductivity of an interacting one-dimensional boson model.

I. INTRODUCTION

Strongly correlated fermionic or bosonic systems constitute nowadays one of the most interesting and challenging problems of solid-state physics. A possible approach to this very difficult problem is by understanding the physics of one-dimensional (1D) models. The one-dimensional models are usually much easier to handle than their counterparts in higher dimensions and can sometimes be exactly solvable, for example the 1D Hubbard model.[1] Even for more complicated models, very efficient techniques such as bosonization or renormalization calculations[2,3] are applicable and are expected to give the correct physics. Besides the physical insight that such one-dimensional interacting electron models can offer to understand higher dimensions, they have also proved to be of fundamental importance for purely one- or quasi-one-dimensional specific problems, e.g., quasi-one-dimensional organic conductors[4] or conducting polymers,[5] where interactions are known to play a major role.

As is well known in one dimension an interacting electron gas will be, for repulsive interactions, an insulator at half filling due to the existence of umklapp process,[2,3] whereas away from half filling the umklapp is expected to be irrelevant and the system is a conductor. There is, therefore, a metal-insulator transition as a function of doping, which has received much attention in the recent past.[6-11] For attractive interactions on the other hand the system has a superconducting ground state, at or away from half filling. A particular theoretical example of such a model would be the negative-U Hubbard model. More generally, in realistic one-dimensional conductors, such models with attractive interactions can also be viewed as reasonable effective models when electron-phonon interactions are present.[12] If the interactions change from attractive to repulsive, the system will undergo a superconducting-insulating or superconducting-metallic transition, depending on the filling. A similar metal-insulator transition driven by interactions occurs also for spinless fermions at half filling.[13]

In this paper I will study such a transition and more generally compute the full frequency and temperature dependence of the conductivity of an electron gas with attractive interactions at arbitrary filling, using the same technique as for the repulsive case.[11]

The interest of such a study is twofold. First it allows one to address the question of the superfluid-insulator transition in one dimension. It also provides an interesting example of what the physical properties of a strongly interacting superconductor can be. Secondly, in one dimension, an attractive fermion gas can be mapped straightforwardly to a repulsive boson gas. One can therefore extract the conductivity of a boson gas close to a superfluid-insulator transition as well as the various exponents of the transition. This allows us to check and complement the scaling hypothesis or numerical work that has been performed on such systems.[14,15]

The plan of the paper is as follows. In Sec. II I review the Hamiltonian of a general Luttinger liquid and the general formula for the conductivity. In Sec. III the conductivity is computed for arbitrary filling, temperature, and frequency, by using a memory function approximation.[16] Such an approximation is valid if the umklapp term is small. Section IV shows how the use of renormalization equations allows one to find the conductivity even in the case of a general umklapp process at and away from half filling. Section V discusses the boson problem.

II. HAMILTONIAN

The Hamiltonian describing, in the long-wavelength limit, the most general one-dimensional system with spin-isotropic short-range interactions, has been derived at length in Refs. 2, 3, and 11. I will therefore just quote the results here to fix the notations. The Hamiltonian is

$$H = H_\rho + H_\sigma + \frac{2g_3}{(2\pi\alpha)^2}\int dx \cos[\sqrt{8}\phi_\rho(x) + \delta x] + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx \cos[\sqrt{8}\phi_\sigma(x)], \quad (2.1)$$

where H_ρ and H_σ are defined by

$$H_\nu = \frac{1}{2\pi}\int dx \left[(u_\nu K_\nu)(\pi\Pi_\nu)^2 + \left(\frac{u_\nu}{K_\nu}\right)(\partial_x\phi_\nu)^2\right]. \tag{2.2}$$

Π and ϕ are canonically conjugate boson fields, and $\pi\Pi = \partial_x\theta$. The ρ and σ parts of the Hamiltonian (2.1) describe, respectively, the charge and spin degrees of freedom of the system. The u, K, and g are constants depending on the microscopic model considered and are the only ones needed to describe the low-frequency, low-temperature, long-wavelength properties of the system. u is the velocity of the excitations, and K controls the decay of various correlation functions. The $g_{1\perp}$ term is the scattering between electrons of opposite spins with an exchange of momentum of $2k_F$. The umklapp process g_3 is the only process that does not conserve momentum. The term g_3 comes from the fact that the microscopic Hamiltonian, of which (2.1) is the continuum limit, is defined on a lattice, and momentum is therefore only conserved modulo 2π. The other interaction processes (hidden in the u and K parameters) conserve momentum. Also α is a short-distance cutoff that can be identified with the lattice constant, $\delta = 4k_F - 2\pi/\alpha$ measures the distance to half filling, and $\pi/(2\alpha)$ would be the Fermi wave vector for a half-filled band. Therefore, if d is the doping ($d = 0$ at half filling and $d = 1$ for a filled band), one has $d = (\alpha\delta)/(2\pi)$. Note that here we assume that we work at fixed number of particles, since k_F is directly related to the filling. For the Hubbard model the various coefficients in (2.1) and (2.2) are given by

$$\begin{aligned} u_\rho K_\rho = u_\sigma K_\sigma = v_F, \qquad & u_\rho/K_\rho = v_F + U/\pi, \\ u_\sigma/K_\sigma = v_F - U/\pi, \qquad & g_{1\perp} = g_3 = U, \end{aligned} \tag{2.3}$$

in a perturbation expansion in U. If U is not small compared to the bandwidth, the above expansion is no longer valid but one can still extract the parameters in (2.1) from the exact Bethe-ansatz solution[17–19,8] for various integrable models.

As visible from the Hamiltonian (2.1) there is for low-temperature and -frequency properties a decoupling between charge and spin degrees of freedom. For the charge transport properties, one need only consider the charge part of the Hamiltonian (2.1). Note that such a decoupling is only exact in the asymptotic limit $\omega \to 0$. The corrections to the Hamiltonian (2.1) are finite at finite frequencies and may affect the conductivity by adding regular corrections (going to zero when $\omega \to 0$). Since such corrections are regular and vanish when $\omega \to 0$, I will not consider them here and will restrict consideration in the following to the pure Luttinger liquid described by (2.1) for which spin charge separation is correct at all frequencies.[20]

For (2.1), one gets for the current[13,11]

$$j = \frac{\sqrt{2}}{\pi}\partial_t\phi_\rho(x) = \sqrt{2}(u_\rho K_\rho)\Pi_\rho. \tag{2.4}$$

It is also easy to show that the conductivity is given by[13]

$$\sigma(\omega) = \frac{i}{\omega}\left[\frac{2uK}{\pi} + \chi(\omega)\right], \tag{2.5}$$

where $\chi(\omega)$ is the retarded current-current correlation function. In (2.5) and in the following the ρ indices are dropped, since only charge variables will now be considered. Note that here what plays the role of the plasma frequency in the usual formulas for the conductivity[16] is uK. In the absence of umklapp (2.5) is easily evaluated to give

$$\sigma(\omega) = 2uK\left(\delta(\omega) + \frac{i}{\pi}\mathcal{P}\frac{1}{\omega}\right), \tag{2.6}$$

$\mathcal{P}$ being the principal part. Therefore, in the absence of an umklapp process the system is a perfect conductor, and the strength of the Drude peak is simply given by $2uK$.[8,13]

The umklapp term g_3 in the Hamiltonian (2.1) is a singular perturbation. The phase diagram and the renormalization equations of the various parameters in (2.1) are well known,[2,3,21–23,11]

$$\begin{aligned} \frac{dK}{dl} &= -\tfrac{1}{2}y_3^2K^2J_0(\delta(l)\alpha(l)), \\ \frac{dy_3}{dl} &= (2-2K)y_3, \\ \frac{du}{dl} &= -\frac{y_3^2}{2}uKJ_2(\delta(l)\alpha(l)), \\ \frac{d\delta}{dl} &= \frac{y_3^2}{2\pi\alpha(l)}J_1(\delta(l)\alpha(l)), \end{aligned} \tag{2.7}$$

where $y_3 = g_3/(\pi u)$ and l describes the renormalization of the cutoff α by $\alpha(l) = \alpha e^l$ and J are Bessel functions. The existence of the Bessel function is related to the use of a sharp cutoff in real space, whereas a smooth cutoff would have led to nonoscillatory functions.[22]

If one is at half filling $\delta = 0$ one then recovers the usual Kosterlitz-Thouless equations[24] with a separatrix at $K - 1 = |y_3|/2$ between a regime where g_3 is irrelevant and a regime where it is relevant, and leads to a gap in the charge spectrum. In the case $K - 1 > |y_3|/2$, which corresponds to the attractive case [see, e.g., (2.3)], g_3 renormalizes to zero and the fixed point Hamiltonian is the free one (2.2), with a renormalized K^* and u^*. Away from half filling, due to the oscillations from the Bessel functions the renormalization due to g_3 is stopped when $\delta(l) \sim 1/\alpha$. Note that if one is away from half filling, since $\delta \neq 0$, there is always one length scale at which δ stops the renormalization; thus g_3 is always an irrelevant operator. The fixed point Hamiltonian is again the free one with renormalized u^* and K^*.

For the attractive case $K - 1 \geq |y_3|/2$, i.e., the case where g_3 is an irrelevant operator and renormalizes to zero, even at half filling; the system has divergent superconducting fluctuations[2,3] and is a true superconductor at zero temperature although it does not possess long-range order.[25] The repulsive case $K - 1 < |y_3|/2$ has been examined in a previous paper.[11]

To get the correct frequency behavior of the conductivity for a given initial value of g_3, one has to take into account the fact that the g_3 term leads to a sin-

gular perturbative expansion[2,3] at low enough frequencies. One way to handle this difficulty is to vary the cutoff α in the Hamiltonian (2.1) to iterate the renormalization equations up to a point where a perturbative expansion in g_3 can be performed, but with the *renormalized* parameters.[26,11] In contrast to what happens for repulsive interactions where the renormalization equations flow toward strong coupling,[11] here one will always remains in the weak-coupling regime allowing us to extract results up to zero frequency or temperature from the renormalization-group equations.

III. PERTURBATIVE CALCULATION

Since such results will be needed for the full study, and, since it also exhibits the salient points of physics, let us look first at the pure perturbation expansion. As in Ref. 11, the simplest way to handle such a perturbative expansion is through the memory function formalism.[16] I will just recall here the main lines of the calculation and refer to Ref. 11 for more details. If one assumes that the system is a normal conductor (σ finite) at zero frequency, then from (2.5) one gets $\chi(0) = -2uK/\pi$, and one can express the conductivity in terms of the meromorphic memory function $M(\omega)$ by

$$\sigma(\omega) = \frac{i2uK}{\pi}\frac{1}{\omega + M(\omega)}, \tag{3.1}$$

where[16]

$$M(\omega) = \frac{\omega\chi(\omega)}{\chi(0) - \chi(\omega)}. \tag{3.2}$$

The calculation of the memory function can be carried out perturbatively to give at the lowest order

$$M(\omega) = \frac{[\langle F;F\rangle^0_\omega - \langle F;F\rangle^0_{\omega=0}]/\omega}{-\chi(0)}. \tag{3.3}$$

The F operators take into account that the current is not a conserved quantity $F = [j, H]$, and $\langle F;F\rangle^0_\omega$ stands for the retarded correlation function of the operator F at frequency ω computed in the *absence* of the scattering potential ($g_3 = 0$).

The memory function approximation is valid whenever an expansion in power of g_3 is possible. This is always the case at fixed frequency for a small enough initial g_3, but to get for a fixed g_3 the correct behavior for all frequencies one needs to couple the memory function formalism and a renormalization-group treatment as for the case of repulsive interactions.[11] This will be done in Sec. IV. As pointed out in Ref. 11, the memory function approximation alone corresponds to $|y_3| \ll |K-1|$.

The memory function for the Hamiltonian (2.1) is given by[11]

$$M(\omega) = \frac{g_3^2 K}{\pi^3\alpha^2}\left(\frac{2\pi\alpha T}{u}\right)^{4K-2}\frac{1}{\omega}[\,B(K-iS_+,1-2K)B(K-iS_-,1-2K) - B(K-iS^0_+,1-2K)B(K-iS^0_-,1-2K)] \tag{3.4}$$

with $S_\pm = (\omega \pm u\delta)/(4\pi T)$ and $S^0_\pm = S_\pm(\omega = 0)$. $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function.

A. Half filling

At half filling $\delta = 0$ and therefore $S_+ = S_-$. As shown in Ref. 11, from (3.4) the temperature dependence of the resistivity at $\omega = 0$ is simply given by

$$\rho(\omega) \simeq \frac{g_3^2}{\pi^2 u\alpha^2}B^2(K,1-2K) \times \cos^2(\pi K)\frac{1}{T}\left(\frac{2\pi\alpha T}{u}\right)^{4K-2}. \tag{3.5}$$

The physical implications of (3.5) have been examined in detail in.[11] Note that since $\rho(T) \sim T^{(4K-3)}$, and for attractive interactions one has $K > 1$, the resistivity vanishes when $T \to 0$. This is consistent with the fact that the system has divergent superconducting fluctuations.

The frequency dependence (at $T = 0$) shows some noteworthy features. One gets from (3.4) if $T \to 0$

$$M(\omega) \simeq \frac{g_3^2 K}{\pi^3\alpha^2}\sin(2\pi K)\Gamma^2(1-2K) \times e^{-i\pi(2K-1)}\frac{1}{\omega}\left(\frac{\alpha\omega}{2u}\right)^{4K-2}. \tag{3.6}$$

From (3.6), one can notice that $M(\omega)$ behaves as ω^{4K-3}, and is therefore negligible compared to ω when $\omega \to 0$ for attractive interactions. The real part of the conductivity is therefore given at finite frequency by

$$\mathrm{Re}\sigma(\omega) \simeq \frac{2g_3^2 uK^2}{\pi^4\alpha^2}\sin^2(2\pi K)\Gamma^2(1-2K)\frac{1}{\omega^3}\left(\frac{\alpha\omega}{2u}\right)^{4K-2}. \tag{3.7}$$

From (3.7), one sees that the frequency dependence of the conductivity is $\sigma(\omega) \sim \omega^{4K-5}$. Since for attractive interaction g_3 renormalizes to zero, the system is described in the long-wavelength regime by a fixed-point Hamiltonian, which is the free one (2.2), but with renormalized quantities u^* and K^*. Thus the conductivity is given in the $\omega \to 0$ limit by (2.6). The total real part of the conductivity is

$$\mathrm{Re}\sigma(\omega) \simeq 2u^*K^*\delta(\omega) + Ag_3^2\omega^{4K-5}, \tag{3.8}$$

where the coefficient A can be obtained from (3.7).

For the imaginary part of the conductivity one has also a ω^{4K-5} correction from (3.6), negligible compared to the $1/\omega$ part of the fixed point Hamiltonian. One notices two interesting features, first, in contrast to what happens for repulsive interactions, the finite frequency conductivity is proportional to g_3^2 because the expansion in g_3 is well defined. Secondly, although the system is a genuine superconductor at $T = 0$, the conductivity behaves quite differently from the usual higher-dimensional case, where the real part of the conductivity is zero up to frequencies equal to the superconducting gap. Even if there is no gap in one dimension, one would naively expect a regular conductivity at finite frequency (going to zero for $\omega \to 0$) reminiscent of the gap. This is only the case for sufficiently attractive interactions ($K > 5/4$); otherwise the conductivity has, in addition to the $\delta(\omega)$ part, a *divergent* finite frequency part.

The $K = 1$ case (more precisely the separatrix $K-1 = |y_3|/2$ in the Kosterlitz-Thouless equations (2.7), if one considers a finite y_3) needs special care. In particular, it is easily seen that it is impossible for the conductivity to have a real part behaving at finite frequency in $1/\omega$, since this would violate the sum rule[27–29]

$$\int_0^{+\infty} d\omega\sigma(\omega) = \langle K\rangle, \tag{3.9}$$

where K is the kinetic energy, by making the conductivity nonintegrable. There are in fact logarithmic corrections to take into account, as will be seen in Sec. IV. This manifests itself in the expression (3.6) as a divergence of the prefactor when $K \to 1$.

B. Away from half filling

Away from half-filling the formulas for the conductivity are identical to those for repulsive interactions and have been derived in Ref. 11. I will merely recall the results and discuss the physical properties.

Two interesting regimes occur depending on whether $T \ll (\omega, u\delta)$ or $T \gg (\omega, u\delta)$. I will not consider here the cases $(\omega, T) \gg u\delta$, since in that case we are led back to expressions similar to those obtained for the half-filled case: at sufficiently large temperatures or frequencies the system is unable to distinguish whether or not it is at half filling.

The temperature dependence is formally identical whether or not one has attractive or repulsive interactions (putting of course the correct K), and one gets an exponential increase of the conductivity due to the freezing of the umklapp process in a one-dimensional system. More details can be found in Ref. 11.

For the frequency behavior at $T = 0$, the memory function will depend on whether $\omega > u\delta$ or not. For $\omega < u\delta$

$$M(\omega) = \frac{g_3^2K}{\pi^3\alpha^2}\sin(2\pi K)\Gamma^2(1-2K)\left(\frac{\alpha}{2u}\right)^{4K-2}\frac{1}{\omega}\{[(u\delta)^2-\omega^2]^{2K-1} - (u\delta)^{4K-2}\}. \tag{3.10}$$

By expanding in ω, one sees that the first term is proportional to ω

$$M(\omega) \simeq \frac{g_3^2K}{\pi^3\alpha^2}\sin(2\pi K)\Gamma^2(1-2K)\left(\frac{\delta\alpha}{2}\right)^{4K-2}(1-2K)\frac{\omega}{(u\delta)^2}. \tag{3.11}$$

The memory function has no imaginary part, which means that the resistivity strictly vanishes when $\omega < u\delta$ [except for the $\delta(\omega)$ part]. The fact that the real part of the memory function is proportional to ω traduces at the lowest order the renormalization of K by the g_3 term, and the fact that the conductivity will be given by the fixed point Hamiltonian (2.2) as (2.6),

$$\sigma(\omega) = 2u^*K^*\left(\delta(\omega) + \frac{i}{\pi}\mathcal{P}\frac{1}{\omega}\right), \tag{3.12}$$

which is the result suggested by (3.10). Again irrelevant couplings between charge and spin or other irrelevant operators could, in principle, modify these results and give a finite conductivity even if $\omega < u\delta$. Such a contribution should, however, vanish when $\omega \to 0$.

If $\omega > u\delta$, it is easy to see from (3.4) that $M(\omega)$ acquires an imaginary part,

$$\mathrm{Im}M(\omega) = \frac{g_3^2K}{\pi^3\alpha^2}\sin^2(2\pi K)\Gamma^2(1-2K)\left(\frac{\alpha}{2u}\right)^{4K-2}\frac{1}{\omega}[\omega^2-(u\delta)^2]^{2K-1}. \tag{3.13}$$

If $K > 1$ the correction due to the real part of $M(\omega)$ remains always small even if one goes closer to half filling $\delta \to 0$, since it behaves in δ^{4K-4}. This is to be contrasted with what happens for repulsive interactions, where this correction diverges, traducing the fact that the umklapp term g_3 becomes pertinent at half filling. For $\omega > u\delta$ the conductivity start to increase as

$$\sigma(\omega) \sim g_3^2\frac{[\omega^2-(u\delta)^2]^{2K-1}}{(u\delta)^3}. \tag{3.14}$$

IV. FINITE g_3

We are now in a position to combine the results of Sec. III with the renormalization equations (2.7) to get

the full temperature or frequency dependence of the conductivity.

A. Half filling

In this case only the temperature or frequency stops the renormalization. The simple limit of Sec. III is equivalent to neglecting the renormalization of K due to g_3 and to considering only the renormalization of g_3 itself.[11]

Since for a finite g_3 the renormalization of K has to be taken into account, the simple power law (3.7) is no longer valid. By numerically integrating (2.7) and using (3.5) one can obtain the full temperature or frequency dependence of the conductivity for various values of the interactions. The results are shown in Fig. 1, for various values of y_3 and K. One notices that the power-law behavior ω^{4K-5} (T^{3-4K}) that is valid at high frequency (temperature) is modified by the renormalization of K, and the exponent is weakened to become at low frequency (temperature)

$$\sigma(\omega) \simeq 2u^* K^* \delta(\omega) + A g_3^2 \omega^{4K^*-5},$$
$$\sigma(\omega) \sim \frac{1}{g_3^2} T^{3-4K^*}, \tag{4.1}$$

where K^* is the renormalized value of the K. The exact value of K^* can be obtained from renormalization in weak coupling and is $1+\frac{1}{2}\sqrt{4(K-1)^2-y_3^2}$ or, for some models, directly from the Bethe-ansatz solution.[17–19,8] The renormalization of the exponent from $4K-5$ to $4K^*-5$ can change qualitatively the behavior of the conductivity, as can be seen from Fig. 1. Since $K^* < K$ and there is a nonzero conductivity at finite frequency, this traduces the fact that the umklapp term, although irrelevant, will steal some weight from the $\delta(\omega)$ part to push it at finite frequency.

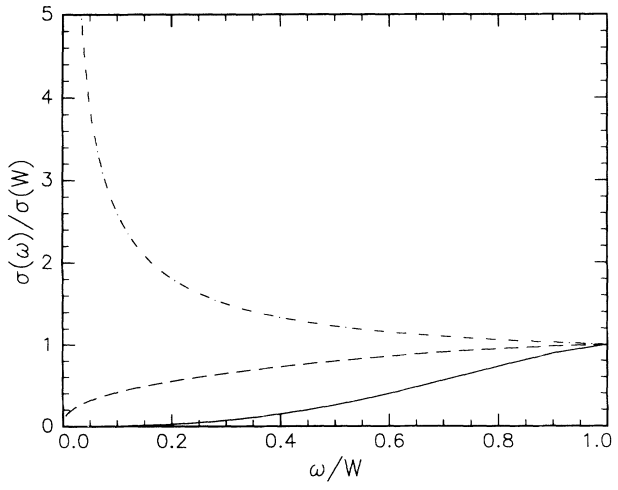

FIG. 1. Frequency dependence of the conductivity at half filling. $W \simeq u/\alpha$ is half the bandwidth, and all curves have been normalized with $\sigma(W)$. The full line, dashed line, and dash-dotted line are for (K, y_3) of $(2, 0.5)$, $(1.5, 0.5)$, and $(1.3, 0.5)$, respectively. The renormalized K^* is, respectively, of 1.87, 1.35, and 1.027.

The asymptotic expression (4.1) is valid as long as one has $K(l) - K^* \ll y_3(l)$. It is easily seen from the equations (2.7) that this is the case at sufficiently low frequencies (or large l) unless one is exactly on the separatrix $K - 1 = |y_3|/2$. In that case one has always $K(l) - 1 = y_3(l)/2$ and (4.1) never correctly gives the low-frequency behavior (here $K^* = 1$). Such a case corresponds in particular to the attractive Hubbard model as can be seen from the weak-coupling expressions (2.3). For the attractive Hubbard model one remains on the separatrix even at strong coupling, since at half filling a particle hole transformation on one spin species maps the attractive Hubbard model into the repulsive one. The charge part of the former becoming the spin part of the latter. The fact that one sticks to the separatrix is then just a consequence of spin isotropy in the transformed Hamiltonian.

On the separatrix $K - 1 = |y_3|/2$ one has from (2.7)

$$y(l) = \frac{y_0}{1 + y_0 l}, \tag{4.2}$$

where $l = \ln[\alpha(l)/\alpha(0)]$ and $K = 1 + y(l)/2$. If one chooses to stop the renormalization when

$$\frac{\alpha(l)\omega}{2u} = 1, \tag{4.3}$$

then the memory function reads

$$M(\omega) = (\pi u)^2 \frac{y^2(l) K(l)}{\pi \alpha^2(l)} \left(i - \frac{1}{\pi y(l)} \right) \frac{1}{\omega \Gamma^2[2K(l)]}. \tag{4.4}$$

Replacing l by its value, the conductivity reads

$$\sigma(\omega) = \frac{u}{2\omega} y^2(l) \frac{K(l)}{\Gamma^2[2K(l)]}, \tag{4.5}$$

which at low frequency $y_3 \ln[(2u)/(\alpha\omega)] \gg 1$ gives

$$\sigma(\omega) \simeq \frac{u}{2\omega} \frac{1}{\ln^2(2u/\alpha\omega)}. \tag{4.6}$$

In addition to the expected $1/\omega$ behavior corresponding to a renormalized value of $K^* = 1$, logarithmic corrections to the frequency dependence with an exponent -2 appear at the superconducting-insulator transition. Such corrections also exist on the critical line for other correlation functions but here seem to arise only from the logarithmic variation of the coupling constant g_3 and not from the need to renormalize the correlation function itself.[30] Note that the $\ln^{-2}(\omega)$ correction is sufficient to make the singularity of $\sigma(\omega)$ integrable and therefore keep it compatible with the sum rule (3.9). On the separatrix ($K^* = 1$) the stiffness constant is $D = 2u$. Therefore, using (4.6), one can see that $\sigma(\omega)/u$ should be a universal quantity up to the order $1/[\omega \ln^2(1/\omega)]$. It is unclear whether this result remains true when one goes beyond the Luttinger-liquid approximation. A similar treatment for the temperature dependence would give

$$\sigma(T) \simeq \frac{2u}{\pi} \frac{\ln^2[u/(2\pi\alpha T)]}{T}. \tag{4.7}$$

B. Away from half filling

In this case there are two possible cutoffs for the renormalization equations, T or ω and δ. If $T \gg u\delta$, the renormalization equations are stopped at a length $l \simeq \ln(W/T)$ and δ introduces few changes in the renormalization flow (2.7) except for a small renormalization of the velocity u. One therefore basically recovers the half-filled behavior described in Sec. IV A. On the other hand, if $\omega \ll u\delta$, the renormalization will be cut off by the Bessel functions.

An interesting quantity to compute is the frequency at which the real part of the conductivity vanishes, which was just $u\delta$ if $g_3 \to 0$. For attractive interactions, one can obtain its dependence on g_3 analytically. For small δ one can use $J_\nu(x) \simeq (x/2)^\nu$. The renormalization of δ becomes

$$\frac{d\delta}{dl} = \frac{y_3^2 \delta(l)}{4\pi}, \tag{4.8}$$

which can be integrated to give

$$\ln[\delta(l)/\delta(0)] = \int_0^l dl \frac{y^2(l)}{2\pi}, \tag{4.9}$$

one stops the renormalization when the condition (4.3) is reached. Then the critical value of ω is given by $\omega = u\delta(l)$, where we have neglected the renormalization of u due to g_3, since it would be of higher order. From (2.7) one gets

$$y_3^2(l) = \frac{\Delta^2}{\sinh^2[\Delta l + \operatorname{arctanh}(\Delta/\sqrt{\Delta^2 + y_3^2})]}, \tag{4.10}$$

where $\Delta = 2(K^* - 1) = \sqrt{4(K-1)^2 - y_3^3}$ and $\Delta = 0$ on the separatrix. From (4.8) one has

$$\ln\left(\frac{\delta(l)}{\delta}\right) = \frac{1}{4\pi}\left(\sqrt{\Delta^2 + y_3^2} - \frac{\Delta}{\tanh[\Delta l + \operatorname{arctanh}(\Delta/\sqrt{\Delta^2 + y_3^2})]}\right). \tag{4.11}$$

The conductivity starts to be nonzero when $\omega_c = \delta(l)$. Since one stops the renormalization when $l^* = \ln[u/(\alpha\omega)]$, one has for small δ

$$\omega_c = \delta(l^*) \simeq \delta(\infty) = \delta e^{(1/4)\pi[\sqrt{\Delta^2+y_3^2}-\Delta]}. \tag{4.12}$$

So in the case where the doping is small the critical value of ω at which the real part of the conductivity of a Luttinger liquid starts to be nonzero is still proportional to the doping but with a prefactor depending on the interactions and greater than one.

V. ONE-DIMENSIONAL BOSONS

The results obtained in the preceding sections and in Ref. 11 can easily be applied to a boson gas with repulsive interactions in one dimension. To see that fact, we use a representation of boson operators in terms of phase fields introduced by Haldane.[19] The single-boson creation operator is written

$$\Psi_b^\dagger(x) = [\rho(x)]^{\frac{1}{2}} e^{i\theta(x)}, \tag{5.1}$$

where $\rho(x)$ is the particle density operator and $\phi(x)$ the phase of the boson field. Taking the discrete nature of the particle density into account, the density operator is

$$\rho(x) = \frac{1}{\pi}\frac{\partial\phi(x)}{\partial x} \sum_{m=-\infty}^{+\infty} \exp[2im\phi(x)], \tag{5.2}$$

where $\partial\phi(x)/\partial x = \pi[\rho_0 + \Phi(x)]$, ρ_0 is the average density, and $\Phi(x)$ obeys the canonical commutation relations:

$$[\theta(x), \Phi(x')] = i\delta(x - x'). \tag{5.3}$$

If one sets $\Phi(x) = \partial_x\tilde{\phi}(x)$, the long-wavelength–low-energy properties of the interacting boson gas are described by the Hamiltonian[19]

$$H = \frac{1}{2\pi}\int dx \left[(v_b K_b)(\partial_x\tilde{\phi})^2 + \left(\frac{v_b}{K_b}\right)(\partial_x\theta)^2\right], \tag{5.4}$$

where from Galilean invariance one has $v_b/(\pi K_b) = \rho_0/m$, and $\pi v_b K_b = \kappa/(\pi^2\rho_0^2)$, where κ is the compressibility. Clearly, the excited states of H are sound waves with phase velocity v_b, which from (5.1) are the phonon modes typical of a Bose superfluid. The existence of such modes is sufficient for true superfluidity to exist.[25] The coefficient K_b determines the asymptotic behavior of the correlation functions:[19]

$$\langle\Psi_B^\dagger(r)\Psi_B(0)\rangle = B\rho_0(\rho_0 r)^{-K_b/2}, \tag{5.5}$$

$$\langle\rho(r)\rho(0)\rangle = \frac{2}{K_b}(2\pi\rho_0 r)^{-2} + A\rho_0^2(\rho_0 r)^{-2/K_b}\cos(2\pi\rho_0 r)$$

with some numerical constants A and B.

A commensurate potential with the periodicity of the lattice will give rise to terms similar to the umklapp term[19]

$$H_{\text{comm}} = \int dx \cos[2\phi(x) + \delta x], \tag{5.6}$$

where $\delta = 2\pi(1-\nu)/\alpha$ is again the distance to commensuration. Here α is the lattice constant and ν the filling of the lattice ($\nu = 1$ for one boson per site). If one rescales the ϕ field by $\sqrt{2}$ then the boson Hamiltonian becomes exactly the charge part of (2.1) with the identification

$$u = v_b, \qquad K = 1/(2K_b). \tag{5.7}$$

This Hamiltonian is also identical (for $\nu = 1$) to the one studied for the spinless fermions metal insulator transition.[13]

One can therefore straightforwardly use the above results and the results derived in Ref. 11 to study the conductivity of a one-dimensional boson system. Note that here the coefficient of the $\delta(\omega)$ part in the conductivity can be identified with the superfluid stiffness of the boson gas. All the formulas for the conductivity obtained for fermions have to be divided by a factor of 2 because of the absence of spin degrees of freedom for the bosons.

A. $\nu = 1$

The system undergoes a superfluid–Mott insulator transition for $K_b = 1/2$, of the Kosterlitz-Thouless type.[19] The superfluid stiffness is simply given by $v_b/(2K_b)$ and therefore has a jump at the transition, since it is zero in the insulating phase. In the superfluid phase the conductivity is a power law of the temperature or frequency as given by formulas (3.5) and (3.7) by using the substitution (5.7) or in general, similar to the one given in Sec. IV A. Exactly at the transition the conductivity exhibits logarithmic corrections as given by (4.6) and (4.7).

B. $\nu \neq 1$

Then the transition exponents are just those of the commensurate-incommensurate phase transition.[21] One has $\rho - \rho_c = \rho_c(\nu - 1) = (\mu - \mu_c)^{(1/2)}$, where μ is the chemical potential and μ_c, ρ_c are the values of the chemical potential and the density at the transition. If $K_b > 1/2$ the stiffness is given by[11]

$$D \propto \frac{\delta}{\Delta} = \frac{\rho - \rho_c}{\Delta} = \frac{\rho_c(\nu - 1)}{\Delta}, \tag{5.8}$$

where Δ is the Mott gap for $\nu = 1$. Close to $\nu = 1$ one has $K_b \to 1$,[21,11] which fixes the decay of the various correlation functions. Note that these exponents agree with the one obtained by a different method in.[14,15] The frequency or temperature dependence of the conductivity is the same than obtained in Sec. IV B for $K_b < 1/2$ or in Ref. 11 for $K_b > 1/2$.

VI. CONCLUSION

I have studied in this paper the influence of umklapp scattering on the conductivity of a one-dimensional Luttinger liquid with attractive interactions. By using a renormalization group method and a memory function approximation, it is possible to obtain the temperature and frequency dependence of the conductivity for arbitrary fillings. For a very small umklapp term the conductivity behaves at half filling as $\sigma(T) \sim T^{3-4K}$ or $\sigma(\omega) \sim \omega^{4K-5}$, where K is an exponent characteristic of the Luttinger liquid and enters into the decay of various correlation functions. K depends on the interactions and $K > 1$ for attractive ones. For a finite umklapp, the renormalization of K by the umklapp has to be taken into account. The temperature dependence becomes $\sigma \sim T^{3-4K^*}$, and the frequency one $\sigma(\omega) = 2u^*K^*\delta(\omega) + A\omega^{4K^*-5}$, where u^* and K^* are the renormalized values of the velocities of excitations and the parameter K. The conductivity has therefore both a $\delta(\omega)$ part *and* a divergent finite frequency part at least for not too attractive ($K^* < 5/4$) interactions. This is to be contrasted with what one would expect for a superconductor in higher dimensions.

For models that are exactly on the transition line ($K^* = 1$), as is the negative U Hubbard model, logarithmic correction appears and the finite frequency part becomes

$$\sigma(\omega) \simeq \frac{u}{2\omega} \frac{1}{\ln^2(2u/\alpha\omega)}. \tag{6.1}$$

Away from half filling there is a crossover temperature above which the temperature behavior is similar to the one at half filling and below, which the conductivity starts to increase exponentially with the temperature. At $T = 0$, for the pure Luttinger liquid the conductivity has a gap for frequencies lower that a characteristic frequency ω_c and a $\delta(\omega)$ part with $\omega_c = Au\delta$, where δ is the doping and A a prefactor depending on the interactions and $A \to 1$ if the umklapp becomes infinitesimally small.

The results for the fermion system as well as those obtained for the repulsive case in Ref. 11 are straightforwardly extended to the metal-insulator transition of repulsive boson system. A superfluid-localized transition occurs now for increasingly repulsive interactions.[19] The conductivity is similar to the one obtained for the fermion system and depends in particular whether or not the boson system has a density commensurate with the lattice. For a commensurate density the same logarithmic corrections appear at the transition.

As for the repulsive case, a question of interest would be the influence of disorder on both the $\delta(\omega)$ part and on the divergence at finite frequency. Although the purely disordered system[26] and the purely commensurate system are reasonably well understood, the situation when both are present is yet unclear. If one neglects the forward scattering due to disorder, keeping only the backward scattering part, the disorder becomes pertinent only if $K > 2/3$ for the boson gas[26] and therefore should not affect the behavior in the commensurate case, since the interaction-driven transition occurs for $K = 1/2$. Away from half filling or for the fermionic system where the disorder is strongly pertinent, the situation is much less clear. If forward scattering is taken into account the situation changes and disorder kills the commensurate potential, at least if the commensurate potential is weak enough.[14] Again the situation for a finite commensurate potential and weak disorder is unclear. The renormalization equation for the disorder in the boson case are also Kosterlitz-Thouless like but in the variables $(K, D^{1/2})$,[26] where D is the disorder. Nevertheless, since the conduc-

tivity at finite frequency has an expansion in D (instead of y_3^2 for the umklapp term), one would expect, from a similar analysis that the one in Sec. IV A, the *same* kind of $1/[\omega \ln^2(\omega)]$ behavior to hold even if the transition is disorder driven. This is consistent with previous results[26] and the scaling analysis of Ref. 31.

Another interesting question is how irrelevant operators, coming from the curvature of the band or higher-order umklapp, not taken into account in the Luttinger-liquid Hamiltonian (2.1), would affect the conductivity. Although they give a vanishing contribution to any correlation function in the limit $\omega \to 0$, they may change the finite frequency, or temperature, conductivity and, in particular, suppress the gap that exists in the optical conductivity of a Luttinger liquid away from half filling. Such a question is still open.

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[1] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).
[2] J. Sólyom, Adv. Phys. **28**, 209 (1979).
[3] V. J. Emery. *Highly Conducting One-Dimensional Solids* (Plenum, New York, 1979), p. 247.
[4] D. Jérome and H.J. Schulz, Adv. Phys. **31**, 299 (1982).
[5] A.J. Heeger, S. Kivelson, J.R. Schrieffer, and W.P. Su, Rev. Mod. Phys. **60**, 781 (1988).
[6] B.S. Shastry and B. Sutherland, Phys. Rev. Lett. **65**, 243 (1990).
[7] C.J. Hamer, G.R.W. Quispel, and M.T. Batchelor, J. Phys. A **20**, 5677 (1987).
[8] H.J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).
[9] N. Kawakami and S.K. Yang, Phys. Rev. Lett. **63**, 3063 (1990).
[10] N. Kawakami and S.K. Yang, Phys. Rev. B **44**, 7844 (1991).
[11] T. Giamarchi, Phys. Rev. B **44**, 2905 (1991).
[12] J. Voit and H.J. Schulz, Phys. Rev. B **34**, 7429 (1986).
[13] R. Shankar, Int. J. Mod. Phys. B **4**, 2371 (1990).
[14] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).
[15] G.G. Batrouni, R.T. Scalettar, and G.T. Zimanyi, Phys. Rev. Lett. **65**, 1765 (1990).
[16] W. Götze and P. Wölfle, Phys. Rev. B **6**, 1226 (1972).
[17] A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).
[18] F. D. M. Haldane, Phys. Rev. Lett. **45**, 1358 (1980).
[19] F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
[20] F.D.M. Haldane, J. Phys. C **14**, 2585 (1981).
[21] H.J. Schulz, Phys. Rev. B **22**, 5274 (1980).
[22] B. Horowitz, T. Bohr, J.M. Kosterlitz, and H.J. Schulz, Phys. Rev. B **28**, 6596 (1983).
[23] T. Giamarchi and H.J. Schulz, J. Phys. (Paris) **49**, 819 (1988).
[24] J.M. Kosterlitz, J. Phys. C **7**, 1046 (1974).
[25] H.J. Mikeska and H. Schmidt, J. Low Temp. Phys. **2**, 371 (1970).
[26] T. Giamarchi and H.J. Schulz, Phys. Rev. B **37**, 325 (1988).
[27] D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966).
[28] P. Maldague, Phys. Rev. B **16**, 2437 (1977).
[29] D. Baeriswyl, J. Carmelo, and A. Luther, Phys. Rev. B **33**, 7247 (1986).
[30] T. Giamarchi and H.J. Schulz, Phys. Rev. B **39**, 4620 (1989).
[31] M.P.A. Fisher, G. Grinstein, and S.M. Girvin, Phys. Rev. Lett. **64**, 587 (1990).