RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 46, NUMBER 5 1 AUGUST 1992-I

Flux-periodic persistent current in mesoscopic superconducting rings close to T_c

Felix von Oppen and Eberhard K. Riedel
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 18 December 1991)

We study the fluctuation-induced persistent current close to T_c in effectively one-dimensional superconducting rings threaded by a magnetic flux ϕ. Using a transfer-operator approach to the Ginzburg-Landau theory we calculate the dependences of the persistent current on temperature, flux, and T_c. In agreement with a simple physical argument we find that the temperature dependence of the current amplitude is similar to that of the order-parameter fluctuations $\langle|\Delta|^2\rangle$. We discuss the observability of a flux-periodic response below T_c in view of the existence of metastable states and phase slips. Our predictions apply to experiments on superconductors whose T_c is larger than the correlation energy E_c.

During the past few years there has been considerable interest in mesoscopic superconducting structures.[1-5] Several mesoscopic effects in nonsuperconducting systems have direct analogs in superconductors, such as conductance fluctuations[3] and conductance quantization.[4] Recently Ambegaokar and Eckern have extended their work on the collective contribution[6,7] to the mesoscopic persistent current to superconducting materials[1] above T_c.

The flux response of superconducting rings has long been studied in mean-field theory. We consider a mesoscopic superconducting ring of circumference L, whose transverse dimension $L_\perp$ is smaller than both the superconducting coherence length $\xi(T)$ and the magnetic penetration depth λ. Such effectively one-dimensional superconductors do not exhibit a sharp transition. However, we denote by T_c the (zero-flux) mean-field transition temperature, which enters the Ginzburg-Landau (GL) functional as a parameter. Furthermore, we assume that the self-inductance of the ring may be neglected so that the flux threading the ring is equal to the applied flux $\varphi=\phi/\phi_0$ (here $\phi_0=hc/e$ denotes the *normal-metal* flux quantum).[1] In particular there is *no* flux quantization. For such rings fluxoid quantization predicts a mean-field current, which vanishes for temperatures above the flux-dependent transition temperature[8] $T_c(\varphi)$. However, for sufficiently small systems there is a sizable "precursor" current above $T_c(\varphi)$ induced by superconducting fluctuations.[1,9]

In this paper we study this fluctuation-induced persistent current in superconducting rings both below and above but close to T_c within the one-dimensional GL theory. A GL description limits our results to samples, whose[10] T_c is larger than the correlation energy E_c.[1] Using the transfer operator method, we reduce the functional integral for the GL partition function to a one-particle quantum-mechanical problem, which may be solved by straightforward numerical techniques. This method has been applied previously to one-dimensional superconductors in the absence of flux.[11] We find that the magnetic flux enters the equivalent quantum Hamiltonian as an imaginary magnetic field. We also consider the problem in the Gaussian, Hartree,[12] and Hartree-Fock[13] approximations.

Physically one expects that the temperature dependence of the fluctuation-induced current is similar to that of the density of superconducting electrons $\langle|\Delta|^2\rangle$, where $\Delta=|\Delta|\exp(i\chi)$ is the complex order (or gap) parameter. Indeed, we find that sufficiently far above T_c the current amplitude decreases exponentially with temperature, while below T_c it approaches smoothly the linear temperature dependence obtained from mean-field theory. In particular the current does not diverge at $T_c(\varphi)$, in contrast to an inference from a recent diagrammatic calculation.[1] The scale for the exponential decrease above T_c is given by the correlation energy E_c.[1] This may be understood by noting that the fluctuation contribution to the free energy becomes flux sensitive, when the superconducting fluctuations extend around the ring, i.e., when $L\lesssim\xi(T)$ or equivalently $T-T_c\lesssim E_c/8\pi$.

In analogy to the current amplitude we find that the flux dependence of the current interpolates smoothly between a regime above T_c, where it is strongly dominated by the first harmonic in $(4\pi\varphi)$, and a regime below T_c, where it is linear with discontinuous jumps at $|\varphi|=0.25$ as obtained from mean-field theory. For $T=T_c$ the flux dependence does not become singular, in contrast to Ref. 1. Although the thermodynamic response that we obtain from the free energy is flux periodic, experiments may not show this at temperatures too far below T_c, where fluxoid states become increasingly (meta)stable. We give an estimate of the interval below T_c, in which the thermal phase-slip rate[14-16] is sufficiently large so that a thermodynamic response may be observed and find that it can be of order E_c for typical parameters. Quantum phase slips[16,17] persist to zero temperature but their rate is significant only for systems with a very small cross-sectional area.

There is an interesting connection between the fluctuation-induced current considered in this paper and the mesoscopic persistent current.[1] The collective contribution to the mesoscopic persistent current[6,7] is induced by electron-electron interactions and for attractive electron coupling it may also be thought of as arising from Cooper pair correlations above T_c. However, inelastic scattering suppresses the mesoscopic persistent current exponentially, while it affects the fluctuation-induced current only weakly through a downward shift[18] of T_c. The amplitude of the fluctuation-induced current is of order[1,10] T_c/ϕ_0 for typical mesoscopic parameters, compared to the mesoscopic persistent current with typical amplitude[6,7,19] E_c/ϕ_0; this should simplify experimental observation greatly.

The results of this paper are relevant to the disorder-averaged response in a single-ring experiment. Multiring experiments involve the additional complication of a sample-specific T_c.

Calculation. We consider a disordered, superconducting ring of circumference L, transverse size $L_\perp$, and $M=k_F^2L_\perp^2/4\pi$ channels. A convenient disorder parameter containing the elastic mean free path l_{el} is the effective number of channels $M_{eff}=Ml_{el}/L$, which is proportional to the dimensionless conductance in the normal state.[20] The correlation energy $E_c=\pi^2\hbar D/L^2$ (here $D=v_Fl_{el}/3$ is the diffusion constant) can be expressed in terms of the effective number of channels as[19] $E_c\propto M_{eff}\Delta_M$, where Δ_M denotes the average level spacing at the Fermi energy. The superconducting coherence length $\xi(T)$ is given by $\xi^2=\pi\hbar D/(8|T-T_c|)$. In the presence of a magnetic flux the GL functional in reduced variables

$$E[\Psi(x)]=E_0T_c\int_{-\Lambda/2}^{\Lambda/2}dx\left\{\left|\left(\nabla-\frac{4\pi i}{\Lambda}\varphi\right)\Psi(x)\right|^2+\eta|\Psi(x)|^2+\frac{1}{2}|\Psi(x)|^4\right\} \quad (1)$$

contains the two parameters Λ and E_0T_c. Here Ψ denotes the order parameter in reduced units and η takes on the values $+1$ (-1) for temperatures above (below) the zero-flux mean-field transition temperature T_c. The flux dependence of $E[\Psi]$ is controlled by the reduced circumference of the ring $\Lambda=L/\xi=(8\pi|T-T_c|/E_c)^{1/2}$. In terms of standard parameters of superconductivity, E_0T_c is proportional to the condensation energy of a ring section of length ξ, $E_0T_c\propto(H_c^2/8\pi)L_\perp^2\xi$, where H_c denotes the critical magnetic field. Using the mesoscopic parameters introduced above, we may also write

$$E_0=\frac{(2\pi)^{5/2}}{21\zeta(3)}\left(\frac{|T-T_c|}{E_c}\right)^{3/2}\frac{E_c}{T_c}M_{eff}. \quad (2)$$

Here $\zeta(z)$ denotes the Riemann zeta function. To exhibit the geometry and disorder dependence explicitly, we do not introduce a special symbol for the "reduced" transition temperature $(T_c/E_c)(1/M_{eff})$. The current is obtained from the free energy by differentiation,[21]

$$\langle I\rangle=\frac{T_c}{\phi_0}\frac{\partial}{\partial\varphi}\ln Z(\varphi)=\frac{T_c}{\phi_0}\frac{\partial}{\partial\varphi}\ln\int[d\Psi(x)][d\Psi^*(x)]\exp\{-E[\Psi(x)]/T_c\}. \quad (3)$$

We assumed that for T near T_c the important temperature dependence is contained in $E[\Psi]$. By using the transfer operator technique[11] and interpreting the spatial variable x as an imaginary time variable τ, the one-dimensional GL theory can be transformed into a quantum-mechanical eigenvalue problem. The quantum Hamiltonian can be conveniently obtained by the following correspondences. We interpret the order parameter as a spatial coordinate, $[\mathrm{Re}\Psi(x),\mathrm{Im}\Psi(x)]\leftrightarrow\mathbf{r}(\tau)$, define a vector potential $\mathbf{A}=(4\pi i/\Lambda)\varphi[r_2,-r_1]$, where r_i denotes the components of $\mathbf{r}$ (we shall also employ the notation $\partial_i=\partial/\partial r_i$), and use $2E_0\leftrightarrow1/\hbar$. Then the functional integral for $Z(\varphi)$ is equivalent to that for the partition function of a (quantum) particle in the presence of an anharmonic (scalar) potential $V(\mathbf{r})=\frac{1}{2}[\eta+(4\pi/\Lambda)^2\varphi^2]\mathbf{r}^2+\frac{1}{4}\mathbf{r}^4$ and the vector potential $\mathbf{A}$,

$$Z(\varphi)=\int[d\mathbf{r}(\tau)]\exp\left\{-\frac{1}{\hbar}\int_{-\Lambda/2}^{\Lambda/2}d\tau\left[\frac{1}{2}\left(\frac{d\mathbf{r}}{d\tau}\right)^2-i\left(\frac{d\mathbf{r}}{d\tau}\right)\mathbf{A}(\mathbf{r})+V(\mathbf{r})\right]\right\}. \quad (4)$$

The free energy may now be written as $Z(\varphi)=\sum_n\exp(-2E_0\Lambda\mathcal{E}_n)$ in terms of the eigenvalues $\mathcal{E}_n$ of the corresponding Hamiltonian $H=\frac{1}{2}(i\hbar\nabla+\mathbf{A})^2+V(\mathbf{r})$. Explicitly the eigenvalue equation is

$$\left\{-\frac{1}{8E_0^2}\nabla^2-\frac{1}{2E_0}\left(\frac{4\pi}{\Lambda}\right)\varphi(r_2\partial_1-r_1\partial_2)+\frac{1}{2}\eta\mathbf{r}^2+\frac{1}{4}\mathbf{r}^4\right\}\phi_n(\mathbf{r})=\mathcal{E}_n\phi_n(\mathbf{r}). \quad (5)$$

It is an important feature of the Hamiltonian that the vector potential $\mathbf{A}$ is imaginary and consequently H is non-Hermitian. The anti-Hermitian flux-dependent term is proportional to the angular momentum operator and thus commutes with the remaining part of the Hamiltonian because of rotational invariance. Therefore it is convenient to choose $\phi_n(\mathbf{r})$ as a simultaneous eigenfunction of the angular momentum operator, $-i(r_1\partial_2-r_2\partial_1)\phi_{n,l}(\mathbf{r})=l\phi_{n,l}(\mathbf{r})$. This choice splits off explicitly the imaginary part of the eigenvalues, $\mathrm{Im}\mathcal{E}_{n,l}=(1/2E_0)(4\pi/\Lambda)l\varphi$, which secures the flux periodicity of the free energy. The angular momentum l plays the role of the harmonic index in the Fourier expansion of the partition function,

$$Z(\varphi)=\sum_{l=-\infty}^{\infty}Z_l\exp(-i4\pi l\varphi)=\sum_{l=-\infty}^{\infty}\exp(-i4\pi l\varphi)\sum_n\exp(-2E_0\Lambda\mathrm{Re}\mathcal{E}_{n,l}). \quad (6)$$

The real part of $\mathcal{E}_{n,l}$ is flux independent and satisfies the eigenvalue equation of a two-dimensional quartic oscillator.

Before discussing the full problem, it is instructive to consider the Gaussian (GA), Hartree[12] (HA), and Hartree-Fock[13] (HFA) approximations. The quartic term in the GL functional is replaced by $|\Psi|^4\to\gamma\langle|\Psi|^2\rangle|\Psi|^2$, where $\gamma^{(GA)}=0$, $\gamma^{(HA)}=2$, and $\gamma^{(HFA)}=4$. For the case considered in this paper $(T_c>E_c)$ the GA gives the same results as a recent diagrammatic calculation.[1] Both HA and HFA provide insight into the limitations of the Gaussian result. The self-consistency condition for $\langle|\Psi|^2\rangle$ is conveniently written in terms of the parameter $y^2=(\Lambda/2\pi)^2(\eta+\gamma\langle|\psi|^2\rangle/2)$,

$$y^3-\frac{2}{\pi}\frac{T-T_c}{E_c}y=\frac{21\zeta(3)}{8\pi^4}\gamma\left[\frac{T_c}{E_c}\frac{1}{M_{\rm eff}}\right]\times\frac{\sinh 2\pi y}{\cosh 2\pi y-\cos 4\pi\varphi}. \quad (7)$$

This shows that above T_c the quartic term in (1), which was neglected in previous work,[1] becomes increasingly important with increasing $(T_c/E_c)(1/M_{\rm eff})$.[22] With these approximations the eigenvalue problem reduces to that of a two-dimensional harmonic oscillator, and from Eqs. (6) and (3) one then obtains for the current

$$\langle I_\gamma\rangle=-\frac{8\pi T_c}{\phi_0}\sum_{l=1}^{\infty}\sin(4\pi l\varphi)\exp(-2\pi l y)=-\frac{4\pi T_c}{\phi_0}\frac{\sin 4\pi\varphi}{\cosh 2\pi y-\cos 4\pi\varphi}. \quad (8)$$

In the Gaussian approximation ($\gamma=0$) one recovers the results of Ref. 1 for the case $T_c > E_c$. In particular one finds $y\rightarrow 0$ for $T\rightarrow T_c$, i.e., all harmonics have the same amplitude in this limit, cf. Eq. (8). Below the zero-flux T_c the solution for y becomes imaginary and the (Gaussian) current diverges at the flux-dependent transition temperature[8] $T_c(\varphi)=T_c-2\pi E_c\varphi^2$ (as obtained from $y=2i\varphi$ with $\gamma=0$). These conclusions do not carry over to the HA and HFA approximations, where the new parameter $(T_c/E_c)(1/M_{\rm eff})$ enters and y^2 contains an additional term. Both temperature and flux dependences remain nonsingular in qualitative agreement with the exact numerical result.

Since only the first few eigenvalues $\mathrm{Re}\mathcal{E}_{n,l}$ contribute significantly to the partition function (6), we truncate the Hamiltonian (5) in the harmonic oscillator basis provided by the HFA. The truncated Hamiltonian is diagonalized numerically.

Results. Here we show results for the two representative values $(T_c/E_c)(1/M_{\rm eff})=1.0$ and 0.1. The first value could be realized by choosing a system with, e.g. $E_c=50$ mK, $M_{\rm eff}=100$, and $T_c=5$ K. The values for E_c and $M_{\rm eff}$ are close to those of the copper rings employed in the first successful experiment on persistent currents in normal metals by Lévy *et al.*[23] The value $(T_c/E_c)(1/M_{\rm eff})=0.1$ could be realized by using a "low-T_c" material with, e.g., $T_c=0.5$ K.

As a first step towards estimating the regime below T_c, in which one might expect a periodic flux response, one needs to know the temperature range in which fluxoid states are metastable in the sense that they minimize the GL functional. The pertinent calculation has been performed in Ref. 14. We deduce that there exist no metastable states for $T > T_c(\varphi=0.25)=T_c-\pi E_c/8$; this implies that a periodic flux response should be obtained in this regime. For $\varphi=0.25$, the $n=0$ and $n=1$ fluxoid states (here $\Delta\chi=2\pi n$ is the total phase shift of the order parameter around the ring) are degenerate and a free-energy barrier exists between the two states when $T < T_c(\varphi=0.25)$. The height of the barrier increases with decreasing temperature. The phase-slip rate across this barrier is the relevant relaxation rate in the temperature range $T_c-5\pi E_c/4 < T < T_c(\varphi=0.25)$, in which there is at most one metastable fluxoid state. Unfortunately this phase-slip rate is difficult to estimate since the standard expression[14–16] for the thermal phase-slip rate Γ is valid only for $\varphi=0$ and $L\gg\xi(T)$. Although these conditions are hardly satisfied here we use the expression[14,15]

$$\Gamma\approx\frac{L}{\xi}\left[\frac{4\sqrt{2}E_0}{3}\right]^{1/2}\frac{1}{\tau_s}\exp\{-4\sqrt{2}E_0/3\} \quad (9)$$

with $\tau_s^{-1}=8(T_c-T)/\pi\hbar$, to obtain a rough estimate of the phase-slip rate below $T_c(\varphi=0.25)$. The prefactor in Eq. (9) is typically of the order 10^{10}–10^{11} s^{-1}. For $(T_c/E_c)(1/M_{\rm eff})=1.0$ the exponent is $7[(T_c-T)/E_c]^{3/2}$, yielding a significant phase-slip rate for temperatures within a few E_c below T_c. This interval shrinks as $(T_c/E_c)(1/M_{\rm eff})$ decreases. For $(T_c/E_c)(1/M_{\rm eff})=0.1$ the exponent equals $70[(T_c-T)/E_c]^{3/2}$, so that the temperature range over which phase slips are relevant below T_c becomes a fraction of E_c.

The temperature dependence of the current is shown in Fig. 1 for $(T_c/E_c)(1/M_{\rm eff})=1.0$. The qualitative features are independent of this particular choice. As expected from the physical argument given above, the current exhibits a behavior qualitatively similar to that of the order-parameter fluctuations $\langle|\Delta|^2\rangle$ (whose temperature dependence is exhibited in Refs. 9 and 11). The curve for the exact current interpolates smoothly between that for the Gaussian current far above T_c (see inset in Fig. 1) and the mean-field current below T_c. The exact current approaches the mean-field current from below in analogy with the corresponding curves[9,11] for $\langle|\Delta|^2\rangle$. Hartree and Hartree-Fock currents are also shown. Whereas the Hartree result shows qualitatively the same behavior as the exact result over the full temperature range, the Hartree-Fock result provides a much better approximation in the "critical regime."

One of the striking results of the Gaussian approximation was that all harmonics contribute equally at T_c, thus leading to a flux dependence $\langle I\rangle\propto(T_c/\phi_0)\cot 2\pi\varphi$.[1] By contrast, the exact solution for $(T_c/E_c)(1/M_{\rm eff})=1.0$

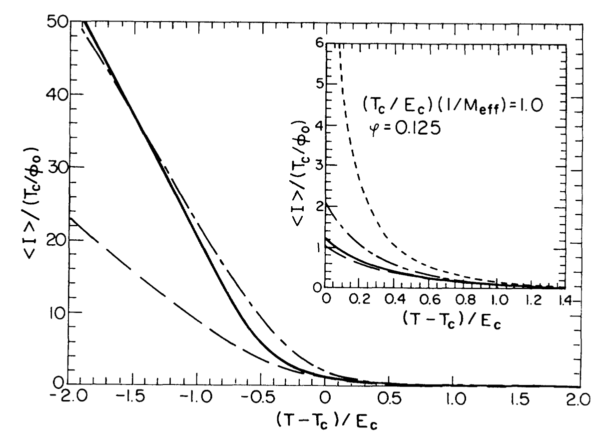

FIG. 1. Temperature dependence of the persistent current in units of T_c/ϕ_0, for $(T_c/E_c)(1/M_{\rm eff})=1.0$ and $\varphi=0.125$, based on the exact numerical result (solid curve), Hartree (long-dash–short-dashed curve), and Hartree-Fock (dashed curve) approximations. The inset shows a blowup for $T > T_c$ including the Gaussian approximation (short-dashed curve) for the current.

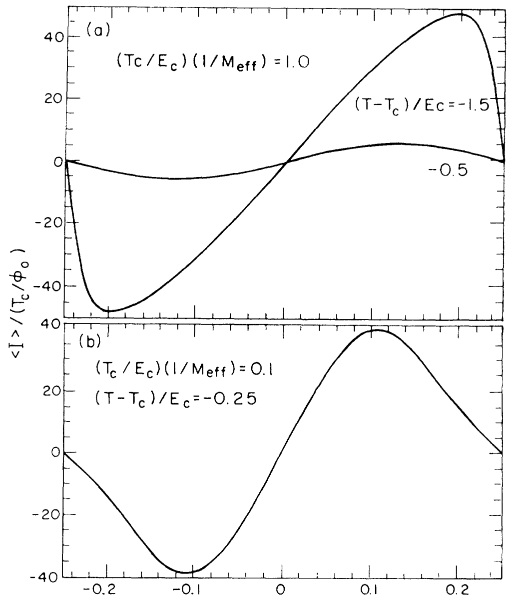

FIG. 2. Flux dependence of the persistent current in units of T_c/ϕ_0, for (a) $(T_c/E_c)(1/M_{\rm eff})=1.0$ and temperatures $(T-T_c)/E_c=-0.5$ and $(T-T_c)/E_c=-1.5$, and (b) for $(T_c/E_c)(1/M_{\rm eff})=0.1$ and $(T-T_c)/E_c=-0.25$, based on the exact solution.

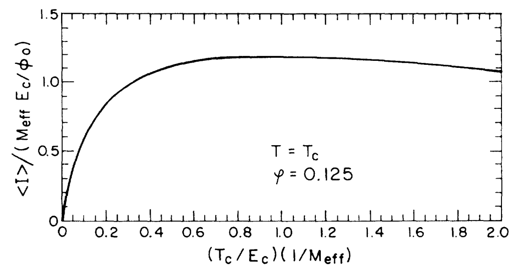

FIG. 3. Persistent current in units of $M_{\rm eff}E_c/\phi_0$ vs the scaled transition temperature $(T_c/E_c)(1/M_{\rm eff})$, for $\varphi=0.125$ and $T=T_c$. The maximum shifts towards higher (lower) values of $(T_c/E_c)(1/M_{\rm eff})$ with increasing (decreasing) temperature T.

shown in Fig. 2(a) exhibits a quite different behavior. The flux dependence of the current crosses over from behavior for $T \gtrsim T_c$, that is strongly dominated by the first harmonic, to mean-field behavior for $T \ll T_c$, that is linear when $|\varphi| < 0.25$ (i.e., the harmonics decrease as $1/l$). The flux dependence remains dominated by the first harmonic down to about $(T-T_c)/E_c=-0.5$. For lower temperatures the maximum of the current-flux characteristic continuously shifts towards larger $|\varphi|$. For $(T_c/E_c)\times(1/M_{\rm eff})=0.1$, on the other hand, one finds an intermediate regime close to T_c shown in Fig. 2(b), where the flux dependence is somewhat reminiscent of that in the Gaussian approximation at T_c. This reflects that the Gaussian approximation becomes better for decreasing $(T_c/E_c)(1/M_{\rm eff})$.

The amplitudes of most superconducting fluctuation phenomena increase as a function of the sample T_c. By contrast, the T_c dependence of the fluctuation-induced current exhibits a maximum. This can be seen from the current expression (8) in HA and HFA, which contains T_c not only in the prefactor, but also implicitly in the exponent through y. The exact T_c dependence is shown in Fig. 3 for $T=T_c$. For this temperature the maximum occurs at $(T_c/E_c)(1/M_{\rm eff})\approx 1.0$. With increasing (decreasing) temperature T the position of the maximum moves towards larger (smaller) values of $(T_c/E_c)\times(1/M_{\rm eff})$. For the values of $(T_c/E_c)(1/M_{\rm eff})$ considered here, the exponential correction at $T=T_c$ is small and the current amplitude is of order T_c/ϕ_0.

We are grateful to E. A. Stern for helpful discussions. This research was supported in part by the National Science Foundation under Grants No. DMR-88-13083 and No. DMR-91-20282.

[1]V. Ambegaokar and U. Eckern, Europhys. Lett. **13**, 733 (1990); Phys. Rev. B **44**, 10358 (1991).

[2]C. Park and Y. Fu, Bull. Am. Phys. Soc. **36** (3), 659 (1991).

[3]B. Z. Spivak and A. Yu. Zyuzin, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 221 (1988) [JETP Lett. **47**, 267 (1988)].

[4]C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. **66**, 3056 (1991).

[5]M. Büttiker and T. M. Klapwijk, Phys. Rev. B **33**, 5114 (1986).

[6]V. Ambegaokar and U. Eckern, Phys. Rev. Lett. **65**, 381 (1990); U. Eckern, Z. Phys. B **82**, 393 (1991).

[7]Persistent currents are similar to mesoscopic currents in SNS junctions. The latter were discussed in B. L. Altshuler, D. E. Khmelnitskii, and B. Z. Spivak, Solid State Commun. **48**, 841 (1983); B. L. Altshuler and B. Z. Spivak, Zh. Eksp. Teor. Fiz. **92**, 609 (1987) [Sov. Phys. JETP **65**, 343 (1987)].

[8]M. Tinkham, Phys. Rev. **129**, 2413 (1963); R. D. Parks and W. A. Little, *ibid.* **133**, A97 (1964); R. P. Groff and R. D. Parks, *ibid.* **176**, 567 (1968).

[9]See W. J. Skocpol and M. Tinkham, Rep. Prog. Phys. **38**, 1049 (1975), for a review of superconducting fluctuations.

[10]Throughout this paper we use units such that $k_B=1$.

[11]D. J. Scalapino, M. Sear, and R. A. Ferrell, Phys. Rev. B **6**, 3409 (1972), and references therein.

[12]S. Marčelja, Phys. Lett. **28A**, 180 (1968).

[13]J. R. Tucker and B. I. Halperin, Phys. Rev. B **3**, 3768 (1971).

[14]J. S. Langer and V. Ambegaokar, Phys. Rev. **164**, 498 (1967).

[15]D. E. McCumber and B. I. Halperin, Phys. Rev. B **1**, 1054 (1970).

[16]N. Giordano, Phys. Rev. B **41**, 6350 (1990), and references therein.

[17]S. Saito and Y. Murayama, Phys. Lett. A **135**, 55 (1989).

[18]W. Brenig, M. Chang, E. Abrahams, and P. Wölfle, Phys. Rev. B **31**, 7001 (1985); also see Ref. 1.

[19]H. F. Cheung, E. K. Riedel, and Y. Gefen, Phys. Rev. Lett. **62**, 587 (1989).

[20]Y. Imry, Europhys. Lett. **1**, 249 (1986).

[21]N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961).

[22]The same conclusion can also be drawn directly by applying the Ginzburg criterium to Eq. (1).

[23]L. P. Lévy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. **64**, 2074 (1990).