Electron hopping in the presence of random flux

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(Received 15 January 1992; revised manuscript received 30 March 1992)

We calculate the distribution of single-particle states for a spinless electron hopping on a two-dimensional square lattice in the presence of random magnetic flux. The flux is taken to be uniformly random between 0 and 1 flux quantum, taking on either continuous or rational values. We consider nearest-neighbor and next-nearest-neighbor hopping. Compared with the density of states without flux, the allowed energies span a smaller range, and the distribution is relatively flat. If states are filled with noninteracting fermions, the random flux lowers the energy for a wide range of filling fractions and next-nearest-neighbor couplings. Examination of the wave functions shows that most of the states are extended with a tail of localized states near the band edge.

The problem of a charged particle hopping on a lattice in the presence of a magnetic field has been of interest for a long time. In spite of the problem's apparent simplicity, the allowed energies have an intricate structure which depends on the magnetic flux per plaquette.[1,2] In recent years this problem, commonly known as the Hofstadter problem, has taken on added importance in the quantum Hall effect,[3-5] and in mean-field treatments of highly correlated electron systems.[6-8] There have been a number of attempts to generalize the discussion.[9] In this paper, we would like to consider the problem of a particle hopping on a lattice in the presence of random flux.

If we include a diagonal hopping term the Hamiltonian is

$$H=(1-J)\sum_{\mathrm{NN}} U_{ij}c_i^{\dagger}c_j+\sum_{\mathrm{NNN}} U_{ij}c_i^{\dagger}c_j \ , \tag{1}$$

where $c_i^{\dagger}$ and c_i are the creation and annihilation operators, respectively, for a particle at site i, and U_{ij} is the phase acquired by hopping from site i to site j. The first sum is over nearest neighbors and the second is over next-nearest neighbors. The flux through a plaquette k is given by

$$\exp(i2\pi\phi_k)=\prod U_{ij} \ , \tag{2}$$

where the product is around the plaquette k and ϕ_k is in units of flux quanta. The coefficient of the first term was chosen so that for both $J=0$ and 1 there is only nearest-neighbor hopping. As $J\rightarrow 1$, one obtains nearest-neighbor hopping on two decoupled rotated lattices.

In the problem considered by Hofstadter, the flux per plaquette is uniform and given by p/q for integers p and q. A natural generalization is to allow the flux to fluctuate from plaquette to plaquette. Besides being of interest in its own right, flux phases of correlated electron systems are subject to fluctuations which alter some of the mean-field results. Thus, the random flux model may provide insight into the effect of these fluctuations, since it is a model consisting of nothing but fluctuations.

For simplicity we will let the ϕ_k's take on random values uniformly distributed between 0 and 1. (Other distributions would require the introduction of more parameters.) From Eq. (2) we see that such random ϕ_k's are equivalent to choosing each U_{ij} as a random phase. As is well known, without loss of generality one can always make a gauge choice of setting $U_{i,i+\hat{y}}=1$. This allows us to make a simple check on our program. The randomness destroys translation invariance, meaning that Bloch's theorem is not available, and we must consider a finite system with n sites. These complications suggest a brute force approach to the problem. We generate some random configuration of ϕ_k's and construct the Hamiltonian given by Eq. (1), imposing periodic boundary conditions in both directions. The quantity of interest is computed using this Hamiltonian and then an average is taken over flux configurations.

The randomness prevents us from calculating a true density of states, however, we can compute a closely related quantity, which we denote by $p(E)$. To compute $p(E)$ the randomly generated Hamiltonians are diagonalized using standard numerical techniques and the energies are counted in bins, normalized so that $\int dE\, p(E)=1$. Rather than the number of states in an energy interval, $p(E)dE$ is the probability that there is a state between E and $E+dE$. Nonetheless, $p(E)$ tells us how the states are distributed, and it may be useful to compare $p(E)$ with the density of states. The density of states for an infinite lattice with no flux will be denoted by $\rho(E)$. (We choose an infinite lattice because for a finite lattice we would simply obtain a sum of δ functions.) Because $p(E)$ is calculated by the Monte Carlo method it is subject to statistical fluctuations. We used 300 bins and averaged enough configurations so as to make the statistical errors only a few percent. For the sake of clarity we have not included error bars on all 300 data points.

We first consider systems with only nearest-neighbor couplings ($J=0$). In Fig. 1 we see $p(E)$ plotted for several different lattice sizes. The figures include $\rho(E)$ for comparison. Because of the singularity in $\rho(E)$ it cannot be normalized the same as $p(E)$. The scale of $\rho(E)$ was

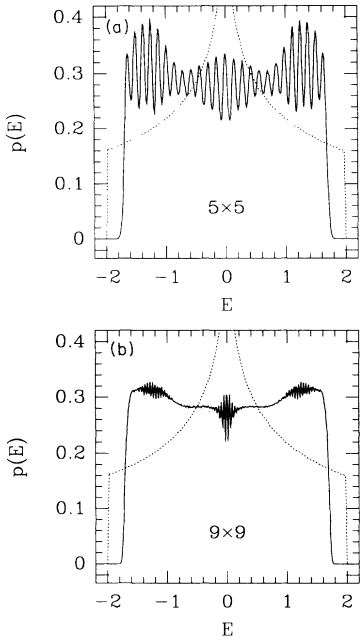


FIG. 1. The probability distribution for single-particle states, $p(E)$, on two different lattices. The dotted lines indicate the density of states without flux with an arbitrary normalization. The lattice sizes are (a) 5×5, (b) 9×9.

simply chosen to make it fit the figures. One feature worthy of note is that $p(E)$ is restricted to a smaller energy range than $\rho(E)$. With flux $p(E)$ falls off rapidly at $E\approx\pm1.7$ instead of $E=\pm2$, although the drop is not as sharp as for $\rho(E)$. The same range of E is seen for lattices from $n=9$ to $n=100$, including those with different numbers of sites in the x and y directions. Note that since $\phi_k=0$ is one particular configuration of random flux there are states in the region $1.7<|E|<2.0$, but they are extremely improbable.

Another interesting feature of $p(E)$ is the oscillatory behavior, which is most prominent in the smaller lattices, such as the 5×5 lattice shown in Fig. 1(a). The number of maxima is equal to the number of lattice sites, and the distance between maxima is constant across the range of E. Since the width of $p(E)$ is approximately 3.4 independent of lattice size the spacing between maxima is $3.4/n$. Therefore, the oscillations are a finite-size effect.

The amplitude of the oscillations also varies with lattice size, decreasing with the number of sites. For the larger lattices, such as 9×9 in Fig. 1(b), $p(E)$ is fairly smooth, except around $E=0$ and ±1.4. In these regions the oscillations are still present, though reduced, with the same $3.4/n$ spacing between maxima.[10] Note that $p(E)$ is a constant to within about 10%, which resembles the two-dimensional continuum density of states. The random flux effectively smears out the lattice and eliminates the singularity, although the lattice still provides a momentum cutoff that makes E bounded.

An interesting variation is to restrict the hopping phases to a discrete $\mathbb{Z}_N$ subgroup of the original U(1) group. That is, we choose $\phi_k=\exp(i2\pi q_k/N)$ where q_k and N are integers and q_k is random from 0 to $N-1$. For $N=2$ there is a sharp peak at $E=0$, as seen in Fig. 2.

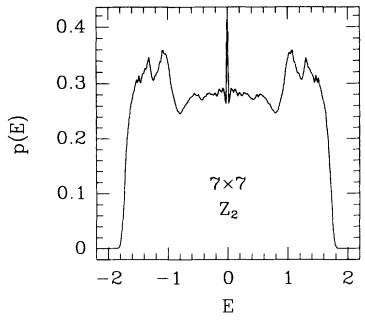


FIG. 2. The probability distribution for a 7×7 lattice with hopping phases restricted to the elements of $\mathbb{Z}_2$.

This is a finite-size effect since the amplitude decreases with increasing lattice size. Aside from the peak, however, $p(E)$ in Fig. 2 is insensitive to further increases in n. Note that $p(E)$ is approximately constant for $|E|<1.7$, although the deviations from a constant are different from those of the U(1) model. On the other hand, the $\mathbb{Z}_N$ and U(1) models give virtually identical $p(E)$'s for $N>2$. While this is to be expected for large N, it is somewhat surprising for $N=3$.

To include the effects of next-nearest-neighbor hopping the diagonal U_{ij}'s were chosen randomly in the same manner as the nearest-neighbor U_{ij}'s. In terms of flux this is equivalent to placing an independent ϕ_k in each of the $4n$ triangles and choosing it randomly between 0 and 1. Figure 3 shows $p(E)$ for $J=0.2$ and 0.5 on a 7×7 lattice. For reference the appropriate no-flux densities of states are also shown. Again, the normalization of the density of states (DOS) was chosen arbitrarily. While $p(E)$ changes slightly due to the diagonal coupling, the

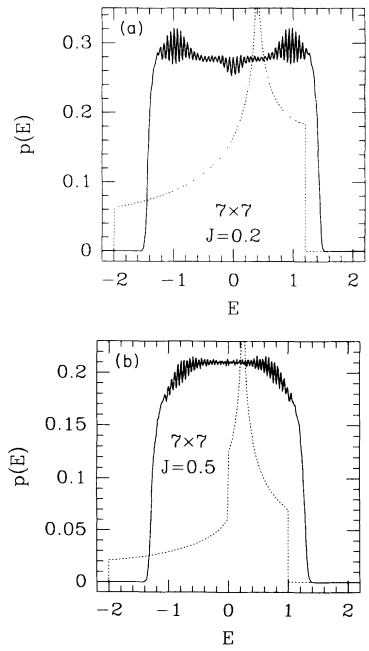


FIG. 3. The probability distributions for a 7×7 lattice with two different values of the next-nearest-neighbor hopping constant J. (a) $J=0.2$, (b) $J=0.5$.

smallness of the change is striking. With $J \neq 0$, the amplitude of the finite-size oscillations is decreased and the width is increased slightly. This indicates that the overall structure of $p(E)$ is independent of the details of the underlying lattice. In contrast, the no-flux density of states changes significantly as J increases, and is asymmetric about $E=0$. Note that for $E<0$ the flux has made a range of energies highly improbable, while for $E>0$ the flux has extended the range of allowed energies.

Let us now consider filling a fraction ν of the available states with noninteracting fermions. Averaging over flux configurations gives a probability distribution for the total energy. The randomness broadens the distribution, although it is sharply peaked about the mean value. (With uniform or no flux the distribution would be a δ function.) The position of the peak is insensitive to lattice size, and remains unchanged as the lattice size is varied from $n=4\times4$ to $n=10\times10$. The width, however, goes to zero as $n\rightarrow\infty$. Therefore, we can compute accurate energies for the infinite system using finite lattices. For a given ν and J the distribution for the infinite system is a δ function located at the average energy computed on a small lattice. We denote the energy per particle on an infinite lattice with random flux as $E_R(\nu,J)$.

In Fig. 4 we see E_R as a function of filling fraction for $J=0$. This is nearly linear in ν, as is to be expected from a flat DOS. For comparison the energy per particle without flux, $E_N(\nu,J)$, is also shown. Note that at small ν the random system has a much higher energy, while at large ν the energy of the random system is only slightly higher. For $0.3<\nu<0.7$ the random flux lowers the energy. We compare E_R with a no-flux system because the random flux configurations are clustered around total flux $\Phi_{\text{tot}}=0$, and the probability of a configuration with $\Phi_{\text{tot}}\neq0$ goes to zero as $n\rightarrow\infty$. On a finite lattice it is possible to select only those rare configurations with $\Phi_{\text{tot}}\neq0$, however, the total energies obtained are the same as those found by including all flux configurations. Adding a constant flux per plaquette also leaves the energy unchanged since the random flux per plaquette goes from 0 to 2π. Note that this means the magnetic susceptibility is zero.

For $J>0$, E_R is still linear in ν, although the intercept

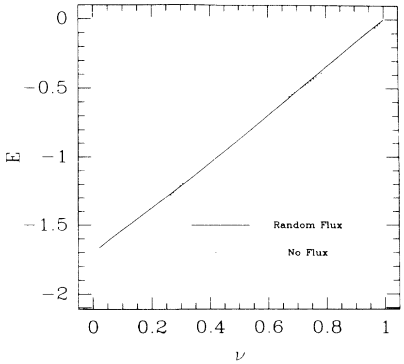


FIG. 4. Energy per particle on an infinite lattice as a function of filling fraction for $J=0$. The solid curve is for random flux and the dotted curve is for no flux.

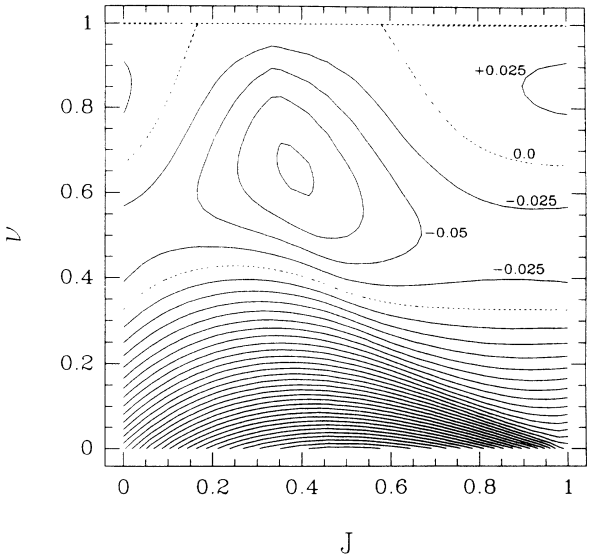
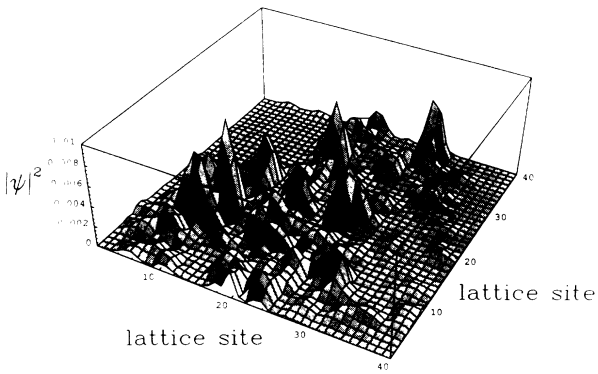
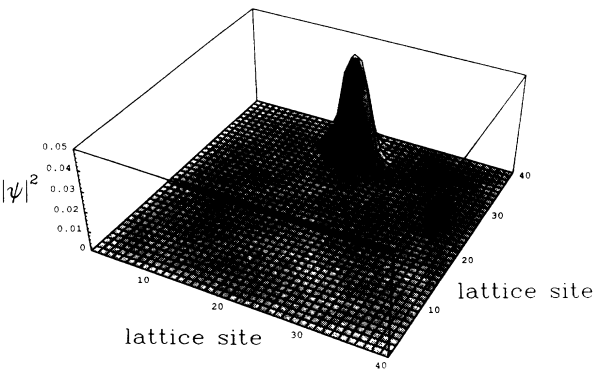


FIG. 5. The energy shift due to random flux, $\Delta E(\nu,J)=E_N(\nu,J)-E_R(\nu,J)$. The contours are 0.025 apart and the dotted curve is the $\Delta E=0$ contour.



(a)



(b)

FIG. 6. $|\psi(x)|^2$ plotted for two different states with a single (random) flux configuration on a 40×40 lattice with $J=0$. (a) $E=1.5843$, (b) $E=1.7201$.

at $\nu=0$ increases with J. This reflects the narrower range of $p(E)$ as J increases. Rather than plotting Fig. 4 for each value of J it is simpler to look at the energy shift $\Delta E(\nu,J)=E_N(\nu,J)-E_R(\nu,J)$ as a function of ν and J. As shown in Fig. 5 there is a large region of ν-J space in which the random flux gives a lower energy. For $0.45<\nu<0.65$, the random system has a lower energy for all J, and for $\nu<0.3$, the random energy is always higher. The largest (downward) shift of 0.1 occurs at $J\approx 0.4$, $\nu\approx 0.6$. As a consistency check we note that the results are the same for $J=0$ and 1 since both have nearest-neighbor hopping only. Figure 5 is not symmetric about $J=\frac{1}{2}$ because the nearest-neighbor hopping for $J=1$ is on two decoupled rotated lattices.

Finally, let us consider the wave functions. In particular, we are interested in seeing if the states are extended or localized. Two typical wave functions for a random flux configuration with $J=0$ are shown in Fig. 6. Note that a large lattice was used to minimize finite-size effects. Also, states with energy $\pm E$ have the same $|\psi(x)|^2$. The wave function in Fig. 6(a) is typical of those with $|E|<1.6$, and appears to be extended. On the other hand, for $|E|>1.6$ the states are localized, as seen in Fig. 6(b). There appears to be a smooth crossover from extended to localized states, with the localization length decreasing as $|E|$ increases. For $|E|<1.6$ it is difficult to distinguish extended states from those with a localization scale comparable to our finite system size. We believe that the flat part of the DOS contains extended states, while the tails of the distribution around $|E|\approx 1.7$ contain localized states.

This picture is very similar to the impurity-broadened Landau levels found in the quantum Hall effect. In the case of random flux, however, only a tiny fraction of the states are localized since the mobility edge is near the edge of the DOS. Also, the exact mechanism of localization is unclear since there is no scalar potential whose minima indicate the localization centers.

We have seen that the single-particle states are distributed quite differently in the presence of random flux. $p(E)$ does not have a singularity, and the range of available energies is narrower than without flux. The states with $1.7<|E|<2.0$ are extremely improbable. In all the cases considered the flux makes $p(E)$ approximately constant, leading to a distribution resembling the continuum density of states with a cutoff. This is also seen when the hopping phases are restricted to elements of $\mathbb{Z}_N$. The inclusion of next-nearest-neighbor hopping changes the shape of $p(E)$ only slightly, indicating that $p(E)$ is insensitive to the type of lattice. If states are filled with noninteracting fermions random flux lowers the total energy in a band around half filling. The width of this band and the amount by which the energy is lowered increase with the amount of next-nearest-neighbor coupling. Examination of the wave functions shows that across most of the band the states are extended, while near the band edge they are localized.

As we have noted, this paper represents merely a step into a potentially rich area. Many interesting questions and situations remain to be studied. The problem of localization is especially interesting. Also, we can consider the problem of a particle hopping in the presence of a dilute distribution of random fluxes.

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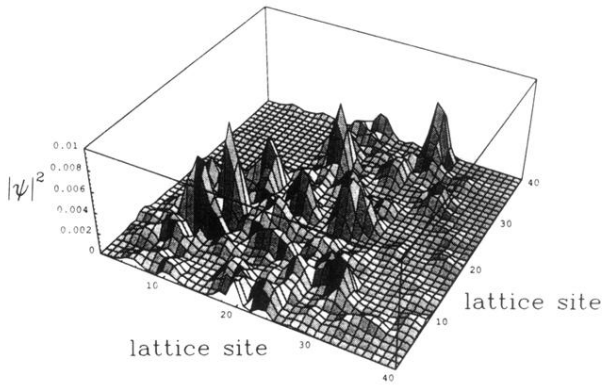
[10]When the number of lattice sites in the x and y directions are both even, $p(E=0)=0$ and $p(E)\propto|E|$ for small E. Although this feature persists as the lattice size is increased, it is a finite-size effect since the width goes to zero as $n\rightarrow\infty$.



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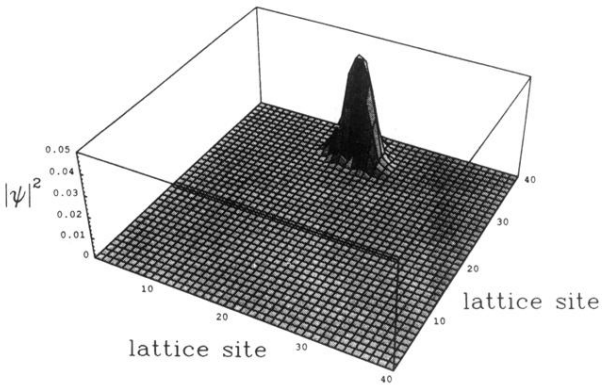

(b)

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