# Real-space renormalization study of disordered interacting bosons

Kanwal G. Singh and Daniel S. Rokhsar
*Department of Physics, University of California, Berkeley, California 94720*
(Received 18 December 1991)

We study a real-space renormalization group for disordered, interacting bosons at zero temperature in one and two dimensions. In the absence of disorder at commensurate density we find a superfluid–Mott-insulator transition that is unaffected by the addition of weak disorder, i.e., disorder is weakly irrelevant at the Mott-superfluid transition. Above a threshold disorder a gapless insulating phase—the "Bose glass"—intervenes.

## I. INTRODUCTION

How does disorder affect the phase transition between the superfluid and Mott-insulating ground states of interacting bosons in a periodic potential? Debate related to this question concerns the "Bose glass," an insulator in which bosons are localized by disorder. The Mott and Bose-glass states are distingushed by their zero-temperature excitation spectra: while the incompressible Mott insulator has a gap to charged excitations, the compressible Bose glass is gapless. (Both insulators are distinguished from the superfluid phase by their lack of off-diagonal long-range order.) The occurrence of a Bose glass for strong disorder is well established,[1] but it has been argued[2] that even arbitrarily weak disorder should stabilize a Bose glass near the Mott-superfluid phase boundary. The transition from a Mott insulator to a superfluid would then necessarily proceed through an (insulating) Bose glass, and the collapse of the Mott gap would precede the appearance of a condensate [Fig. 5(c)]. This effect should be most dramatic when the density of bosons is commensurate with a periodic potential. Numerical studies[3,4] have found an intervening Bose-glass phase at *incommensurate* densities for weak disorder, but not for commensurate densities. We present here the phase diagram (Fig. 3) predicted by a simple real-space renormalization group for interacting bosons at *commensurate* density on disordered lattices in one and two dimensions.

Granular superconductors[6,7] and $^4$He adsorbed on porous media[8–10] provide experimental realizations of disordered interacting Bose systems. Previous theoretical approaches to the problem have included scaling arguments,[2,11] quantum Monte Carlo calculations on one- and two-dimensional models,[3–5,12] and a perturbative renormalization-group calculation for the one-dimensional problem.[13]

We present a real-space renormalization-group calculation for interacting bosons at zero temperature subjected to a random potential. The density is fixed at an average of one boson per site and we monitor the flow of parameters corresponding to the relative strengths of disorder, interactions, and hopping. Stable fixed points of the flow correspond to the superfluid, Mott-insulator, and Bose-glass phases; unstable fixed points represent zero-temperature critical points. The insulator-superfluid transition for the pure system at commensurate density is known to be in the universality class of the $(d+1)$-dimensional $XY$ model.[2] Our real-space renormalization-group calculation (which entails certain approximations detailed below) does not reproduce the expected Kosterlitz-Thouless behavior in one dimension, but instead gives an ordinary critical point with an encouragingly large correlation-length exponent $\nu \sim 5$. In two dimensions, we find $\nu \sim 1.2$, in contrast with the known value of $\sim \frac{2}{3}$ for the three-dimensional $XY$ model. When weak disorder is introduced, we find that the transition still proceeds directly from the Mott insulator to the superfluid, so that infinitesimally weak disorder does *not* stabilize a Bose glass at commensurate density. The Bose glass is found beyond a threshold disorder. Our renormalization procedure is described in Sec. II, and the results for the pure and disordered cases are discussed in Secs. III and IV, respectively. In Sec. V we put these results in the context of other work on this problem.

## II. REAL-SPACE RENORMALIZATION SCHEME

Our starting point is the Hubbard Hamiltonian for spinless interacting bosons,

$$\mathcal{H} = -T \sum_{\langle i,j \rangle} [b_i^\dagger b_j + b_j^\dagger b_i] + \sum_i [\epsilon n_i + V(n_i)], \qquad (1)$$

where $b_i^\dagger$ and $b_i$ create and annihilate bosons at site $i$, and $n_i \equiv b_i^\dagger b_i$ is the boson number operator at site $i$; the first sum is over nearest-neighbor pairs of sites. The parameters entering the model are the hopping matrix element $T$, the on-site energy $\epsilon$, and the on-site repulsion $V(n)$. Although it is common to consider the pair repulsion energy $V(n) = Un(n-1)/2$, it will be convenient for us to consider instead a simpler model which assigns an energy cost $U$ to double occupancy of a site

[i.e., $V(2) = U$] but *completely* forbids triple or higher occupancy; i.e., $V(n) \equiv \infty$ for $n \geq 3$. [There is of course no interaction energy associated with empty or singly occupied sites, so $V(0) = V(1) = 0$.] The Hilbert space for a single site is then spanned by the three states $|0\rangle$, $|1\rangle$, and $|2\rangle$, which are labeled by occupation number. This restriction to zero, one, or two bosons per site makes the calculation tractable. The universality classes of the various transitions of interest, however, should not be affected since our model still has only local interactions.

To study (1), we use a real-space renormalization scheme similar to that developed by Hirsch[14] and Ma[15] in their studies of interacting fermions. The blocking procedure is illustrated in Fig. 1.

Consider first a one-dimensional chain, which we divide into nonoverlapping cells containing two sites each. The normalized basis states for the two-site cell are written $|n_L\, n_R\rangle$, indicating $n_L$ bosons on the left and $n_R$ bosons on the right. (The adjoint of $|n_L\, n_R\rangle$ will be denoted $\langle n_L\, n_R|$, so that sites will always be ordered from left to right in both bras and kets.) An isolated two-site cell has a total of nine energy eigenstates, each with a definite total particle number ranging from zero to four. As in the real-space schemes of Refs. 14 and 15 we select three of these cell states to be our "block" states $|0'\rangle$, $|1'\rangle$, and $|2'\rangle$. Since we are interested in systems with an average density of one boson per site, we choose the block state $|1'\rangle$ to be the lowest energy state with unit density, i.e., with two particles on the two-site cell. The block state $|0'\rangle$ is the lowest energy state with one boson per cell, corresponding to a *deficit* of one particle, while $|2'\rangle$ is the lowest energy state with three particles per cell, an *excess* of one particle.

The *excited* states with one, two, or three bosons per cell are not considered further. These cell states cannot contribute to the ground state of the chain because the ground state must be nodeless (since $T$ is positive), while the excited states have nodes. A more serious truncation of the Hilbert space comes from the unjustified neglect of the two states with larger density fluctuations, i.e., those with zero or four bosons per cell.

If we restrict the state of each cell to be $|0'\rangle$, $|1'\rangle$, or $|2'\rangle$, then we can construct a renormalized Hamiltonian $\mathcal{H}'$ whose matrix elements between the states of this truncated Hilbert space are the same as those of the original Hamiltonian. The renormalized Hamiltonian is roughly

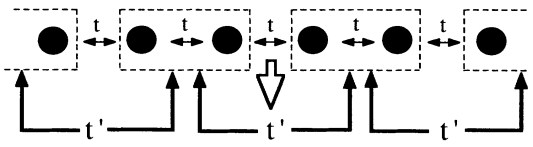


FIG. 1. Blocking scheme. The chain is divided into nonoverlapping cells consisting of two sites each. Recursion relations for the intracell parameters $\epsilon'$ and $U'$ are determined by the energies of the cell states with one, two, and three bosons [Eqs. (7) and (8)]. The intercell couplings ($t'$) are determined by calculating the intersite matrix elements ($t$) between sites on the edges of two cells.

of the same form as (1), and the procedure can be iterated. By repeated blockings we construct a hierarchical, truncated Hilbert space within which the Hamiltonian (1) is diagonalized exactly. Our calculation therefore provides a variational bound on the ground-state energy. After $N$ blockings, the truncated Hilbert space is spanned by three states that correspond to a fixed number of bosons on a chain $2^N$ sites long. The state $|1^{(N)}\rangle$ always has an average density corresponding to one boson per site, and therefore contains $2^N$ bosons. The other two states, $|0^{(N)}\rangle$ and $|2^{(N)}\rangle$, have $2^N - 1$ and $2^N + 1$ bosons, respectively. Thus our procedure seriously curtails particle-number fluctuations by limiting them to $\pm 1$ boson on a region of size $L^d$, rather than $L^{(d-1)/2}$ as expected for a condensate. Since our calculation only permits very small density fluctuations, our approximations are evidently best for large Hubbard $U$ and low dimensionality.

To proceed further it is convenient to write the kinetic energy for hopping between two adjacent sites in the expanded form

$$\begin{aligned}\mathcal{H}_{\rm kin} = &- t_h\Big\{|0\,1\rangle\langle 1\,0| + |1\,0\rangle\langle 0\,1|\Big\} \\ &- t_o\Big\{|1\,1\rangle\langle 2\,0| + |1\,1\rangle\langle 0\,2| + |2\,0\rangle\langle 1\,1| + |0\,2\rangle\langle 1\,1|\Big\} \\ &- t_p\Big\{|2\,1\rangle\langle 1\,2| + |1\,2\rangle\langle 2\,1|\Big\}, \end{aligned} \tag{2}$$

which displays the three distinct hopping processes. These have simple physical interpretations if we take as a reference state the ideal Mott insulator with precisely one particle per site. Then $t_h$ and $t_p$ specify the hopping amplitude for a hole and an added particle, respectively; $t_o$ gives the amplitude for the creation or annihilation of a particle-hole pair. For the Hubbard Hamiltonian (1), we have $t_h = t_o/\sqrt{2} = t_p/2 = T$. The Bose-Hubbard model (1) therefore does *not* possess particle-hole symmetry; $t_h$ and $t_p$ differ due to factors analogous to those appearing in stimulated absorption and emission. Such differences are also expected in granular superconductors, due to factors of the square root of the number of Cooper pairs in a given grain. Nevertheless, we will see that particle-hole symmetry is restored by blocking so that for long-wavelength, low-energy phenomena we need only consider the particle-hole symmetric case $t_h = t_p$.

For a system with no disorder, the normalized block states with one, two, and three particles per cell can be simply written down by symmetry:

$$\begin{aligned} |0'\rangle &= \tfrac{1}{\sqrt{2}}\{|0\,1\rangle + |1\,0\rangle\}, \\ |1'\rangle &= \cos\theta|1\,1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta\{|2\,0\rangle + |0\,2\rangle\}, \\ |2'\rangle &= \tfrac{1}{\sqrt{2}}\{|1\,2\rangle + |2\,1\rangle\}, \end{aligned} \tag{3}$$

with energies

$$\begin{aligned} E_{0'} &= \epsilon - t_h, \\ E_{1'} &= 2\epsilon - \sqrt{2}t_o\sin 2\theta + U\sin^2\theta, \\ E_{2'} &= 3\epsilon - t_p + U, \end{aligned} \tag{4}$$

respectively. The parameter $\theta$ is determined by solving the problem of a pair of bosons on a single cell, which yields

$$\tan 2\theta = \frac{2\sqrt{2}t_o}{U}. \quad (5)$$

The renormalized Hamiltonian $\mathcal{H}'$ acts on the (primed) states of a chain of *cells*, and contains both diagonal (intracell) and off-diagonal (intercell) contributions. The *intra*cell part of the Hamiltonian can be simply written as

$$\mathcal{H}_{\rm pot} = \sum_i \Big( E_{0'} + n_i'(E_{1'} - E_{0'}) + \frac{n_i'(n_i'-1)}{2}(E_{2'} + E_{0'} - 2E_{1'}) \Big), \quad (6)$$

from which we deduce the renormalized potential and interaction energies

$$\epsilon' \equiv E_{1'} - E_{0'} = \epsilon + t_h + U\sin\theta, \quad (7)$$

$$U' \equiv E_{2'} + E_{0'} - 2E_{1'} = U\sec 2\theta - t_p - t_h. \quad (8)$$

[We have used (5) in determining the expression for $U'$. The overall added constant $E_{0'}$ in (6) is unimportant.]

The *inter*cell couplings arise from hopping matrix elements between adjacent cells, and are given by

$$\begin{aligned} -t_h' &= \langle 0'1'|\mathcal{H}_{\rm kin}|1'0'\rangle, \\ -t_o' &= \langle 1'1'|\mathcal{H}_{\rm kin}|2'0'\rangle \equiv \langle 1'1'|\mathcal{H}_{\rm kin}|0'2'\rangle, \\ -t_p' &= \langle 2'1'|\mathcal{H}_{\rm kin}|1'2'\rangle, \end{aligned} \quad (9)$$

where $\mathcal{H}_{\rm kin}$ is the hopping operator (2) between the rightmost site of the left cell and the leftmost site of the right cell (Fig. 1).

## III. PURE SYSTEM

For $\langle n\rangle = 1$, in both one and two dimensions, we find that as long as $t_o$ is initially nonzero the flows (7)–(9) always head toward the plane $t_h = t_p$, so that all interesting fixed points for the pure system possess particle-hole symmetry. (There are also pathological fixed points of our recursion relations with $t_o = 0$.) Indeed, we find that in one dimension the differences between hopping parameters disappear, so that we may study the flows along the line $t_h = t_o = t_p \equiv t$. The clean system is thus a one-parameter problem whose behavior is determined entirely by the flow of $t/U$, or equivalently, $\theta$. The recursion relations (7)–(9) then collapse to the single equation

$$\tan 2\theta' = \left( \frac{1 + \cos^2\theta + \sqrt{2}\sin 2\theta}{4\csc 2\theta - 2\sqrt{2}} \right). \quad (10)$$

(The uniform site potential $\epsilon$ is of course physically unimportant.)

The two stable fixed points of (10) correspond to a Mott insulator ($t/U = 0$) and a superfluid ($t/U = 0.5165$). An unstable fixed point appears at $(t/U)_* = 0.3309$, corresponding to the second-order transition between the two zero temperature phases. This translates to the critical point $(T/U)_c = 0.215$ of the Hubbard model (1) which matches the value of $(T/U)_c = 0.215 \pm 0.01$ obtained by Batrouni *et al.* via Monte Carlo simulations.[12] Note that as $t/U$ becomes larger, our truncation becomes a worse approximation. As a consequence of our approximations, the superfluid fixed point occurs at a finite $t/U$.

The universality class of the Mott-superfluid transition at commensurate density is that of the $(d+1)$-dimensional $XY$ model.[2,11] For a two-dimensional $XY$ model the correlation length diverges as

$$\xi \sim e^{\text{const}/\sqrt{(t/U)-(t/U)_*}}, \quad (11)$$

which corresponds to an *infinite* correlation length exponent $\nu$. We can determine $\nu$ for our (ordinary) critical point by linearizing the recursion relation (1) near $(t/U)_*$:

$$\left(\frac{t}{U}\right)' \approx \left(\frac{t}{U}\right)_* + \left[\left(\frac{t}{U}\right) - \left(\frac{t}{U}\right)_*\right] b^{1/\nu}, \quad (12)$$

where $b$ is the scaling factor (here $b$=2) and $\nu$ is the correlation length exponent. We find $\nu = 4.87$. This uncommonly large value of $\nu$ suggests that our calculation is close to capturing the correct Kosterlitz-Thouless behavior. (In contrast, mean-field theory gives a mere $\nu = \frac{1}{2}$.)

At the transition the characteristic energies (e.g., the Mott gap of the insulator or the inverse compressibility of the superfluid) disappear like

$$U' = Ub^{-z}, \quad (13)$$

where $z$ is the dynamical exponent. We find $z = 0.4985$, in contrast to scaling arguments[2] which predict $z = 1$ for a system at commensurate filling in any dimension. This discrepancy is not surprising since in our approximation we only allow the cell occupation to vary by one "bare" particle, thus artificially constraining density fluctuations. (This deficiency of our scheme cannot be corrected by retaining more states per site in any simple way, because such an approach would still only permit a finite number fluctuation on each block. A better calculation would permit density fluctuations with proper scaling behavior.)

In two dimensions we use a square lattice and block into four-site (two-by-two) cells. The procedure for determining the recursion relations for the various parameters is the same as described for the one-dimensional problem. We again fix the average density at $\langle n\rangle = 1$, so that after $N$ blockings the block states $|0^{(N)}\rangle$, $|1^{(N)}\rangle$, and $|2^{(N)}\rangle$ contain $4^N - 1$, $4^N$, and $4^N + 1$ bosons, respectively, on a $4^N$ site cell. Within the accuracy of our calculation, we again find that flows tend to the line $t_h = t_o = t_p \equiv t$; again, we have a one-parameter problem. The two stable fixed points are $t/U = 0$ (Mott) and $t/U = 1.794$ (superfluid). The transition between the two phases takes place at $(t/U)_* = 0.0830$ and the critical exponents are $\nu = 1.2$ and $z = 0.51$. Putting our result in terms of the Hubbard model (1) gives $(T/U)_c = 0.0564$ which is within 5% of the critical value 0.059 obtained by Krauth *et al.* using path-integral Monte Carlo simulations on a two-dimensional lattice.[4] Once again, we expect $z = 1$ for commensurate densities, and for a three-dimensional $XY$ model $\nu \sim \frac{2}{3}$. Real-space renormalization is notori-

ously worse in higher dimensions and our calculation is no exception.

## IV. DISORDERED SYSTEM

We introduce disorder as a site dependent, uniformly distributed, on-site potential, $-\Delta < \epsilon(i) < \Delta$. Although we begin with a uniform hopping strength $[t_h(i) = t_o(i) = t_p(i) \equiv t]$ and repulsion $U(i) = U$, these quantities immediately develop distributions since the recursion relations couple all parameters. Thus for the blocked disordered system all parameters in the Hamiltonian become site dependent and we follow the evolution of their distributions.

We begin with a chain of 1000 sites with $\epsilon(i)$ chosen from a uniform distribution of width $2\Delta$, centered at zero. This chain is then blocked into a 500-cell chain, and the intracell Hamiltonians (for each cell) are diagonalized to find $\epsilon'(i)$ and $U'(i)$. The hopping parameters $t'_h(i)$, $t'_o(i)$, and $t'_p(i)$ between cells are determined as previously described for the pure case. Although the three hopping parameters are no longer equal on any given site, their distributions are similar. For simplicity, we merge the distributions of $t'_h$, $t'_o$, and $t'_p$ into a single distribution with mean $\overline{t}'$ and standard deviation $\sigma(t')$. We also calculate the averages $\overline{\epsilon}'$ and $\overline{U}'$, and define the half-widths of these distributions:

$$\begin{aligned} \Delta' &= \sqrt{3}\sigma(\epsilon'), \\ \delta U' &= \sqrt{3}\sigma(U'), \\ \delta t' &= \sqrt{3}\sigma(t'). \end{aligned} \tag{14}$$

Then a new 1000-site chain is generated with parameters $\epsilon(i)$ and $U(i)$ chosen from uniform distributions of half-widths $\Delta'$ and $\delta U'$ centered at $\overline{\epsilon}'$ and $\overline{U}'$, respectively. The new hopping parameters $t_h(i)$, $t_o(i)$, and $t_p(i)$ are all selected from the same uniform distribution of half width $\delta t'$ centered at $\overline{t}'$. We ignore any induced correlations between $\epsilon$, $U$, and $t$ on a given cell, which we confirmed to be a good approximation for small disorder. Also, since we only keep track of the mean and standard deviation of the distributions, our calculation is insensitive to any effects of rare but large fluctuations in the parameters.

Figure 2 shows the first renormalization-group iteration projected onto the plane $\delta t = \delta U = 0$. (Since the "flow" is discrete and is projected onto a plane, the arrows can cross even though the blocking procedure is well defined.) The Mott and superfluid fixed points both lie in this plane, as does the fixed point corresponding to the Mott-superfluid transition; all other fixed points have nonzero $\delta t$ and/or $\delta U$. With no disorder ($\Delta = 0$), the flow of $t/U$ is indicated on the vertical axis, showing the transition at $(t/U)_* = 0.3309$. The phase diagram of the Hubbard model with site disorder and particle-hole symmetric hopping $t_h = t_o = t_p \equiv t$ is shown in Fig. 3, as determined by the basins of attraction of the three stable fixed points. Note that while the *initial* flows for $\Delta/U$ between $\sim 0.2$ and $\sim 0.3$ are toward *smaller* site randomness (Fig. 2), the system eventually ends up at the Bose-glass fixed point (Fig. 3). This is due to the generation of disorder $\delta U$ by the renormalization procedure, which moves successive systems out of the plane shown in Fig. 2. Figure 3 indicates that the insulator-superfluid transition occurs directly from the Mott insulator below a threshold disorder $\Delta/U \sim 0.1$.

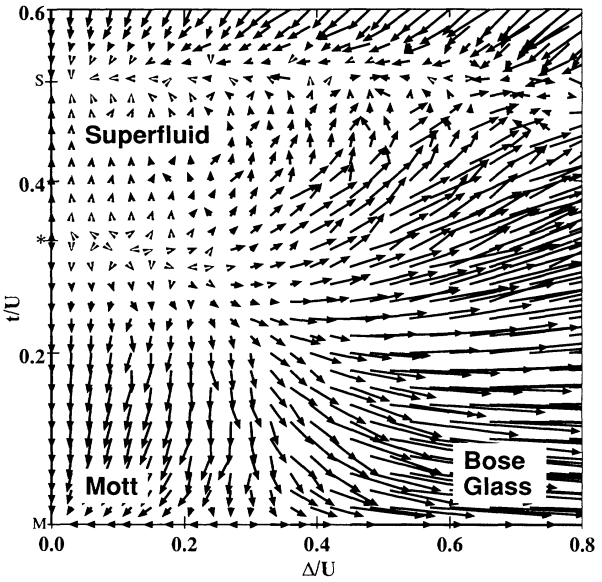


FIG. 2. First renormalization-group iteration for the particle-hole symmetric Bose-Hubbard model ($t_h = t_o = t_p = t$) with site disorder $\Delta$, projected onto the $\delta t = \delta U = 0$ plane. Results for the pure system are illustrated on the vertical axis. Since the "flows" are discrete and are projected onto a plane, the arrows can cross even though the blocking procedure is well defined.

As seen in Fig. 3, disorder helps *delocalize* bosons near the Mott phase. The phase boundary between the Mott

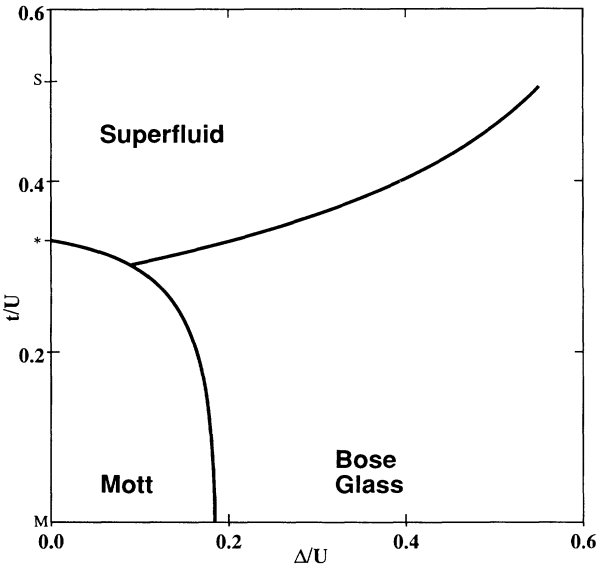


FIG. 3. Basins of attraction for the stable superfluid, Mott insulator, and Bose-glass fixed points in the $\delta t = \delta U = 0$ plane.

insulator and the superfluid curves downward, i.e., at nonzero $\Delta$, the system is superfluid at values of $t/U$ which were Mott insulating without disorder. The same effect has been noted in Monte Carlo simulations in one and two dimensions.[3,5] The phase boundary curves downward because the Mott gap is reduced by disorder, thus requiring a smaller critical value of the hopping parameter $t$ to overcome the gap and delocalize the bosons. (For $t = 0$ the Mott gap is reduced linearly as $E_G = U - 2\Delta$.)

Within our approximations, disorder is weakly irrelevant at the Mott-superfluid transition, as can be seen by linearizing the flows in the four-parameter space $(\overline{t}/\overline{U}, \Delta/\overline{U}, \delta t/\overline{U}, \delta U/\overline{U})$ around the fixed point of the pure system (0.3309,0,0,0). (This is done numerically by least-squares fitting a linear function to the flows in the vicinity of the pure-system fixed point for many realizations of disorder.) The direction (1,0,0,0) is evidently relevant, since changing $t/U$ tunes through the transition in the absence of disorder. The other three directions are all irrelevant, with the largest eigenvalue of the linearized recursion relations equal to $0.95 \pm 0.01$. Thus the scaling field corresponding to disorder, $D$ (a linear combination of $\Delta$, $\delta t$, and $\delta U$), scales as

$$\frac{D'}{\overline{U}'} = b^x \frac{D}{\overline{U}}, \tag{15}$$

where $x = -0.07$ (since $2^x = 0.95$). If $x$ were positive, disorder would be relevant; if $x$ were zero, disorder would be marginal. We find that disorder is irrelevant ($x$ is negative), but only weakly so. Due to the inherently uncontrolled nature of real-space renormalization, this result should not be regarded as conclusive. It does, however, agree with the path integral Monte Carlo study of Krauth *et al.*,[4] who found that no Bose glass intervenes at $\langle n \rangle = 1$ in two dimensions for weak disorder. (Similar calculations in one dimension are not available, presumably because of complications related to the Kosterlitz-Thouless transition in the pure system.) An intriguing possibility which reconciles our approximate calculation with the arguments of Fisher *et al.* is that weak disorder could be marginally irrelevant at the Mott-superfluid transition. In that case the superfluid–Bose-glass and Bose-glass–Mott phase boundaries would meet at $\Delta = 0$ and would be tangent to one another there, with the Bose-glass phase appearing as a thin sliver extending all the way to the $\Delta = 0$ axis in Fig. 3.

If the bosons do not hop ($t = 0$), they occupy the sites with the most-favorable energy conditions (low on-site potential, low interaction energy). In the pure system, minimizing the interaction energy leads to a Mott insulator with one particle per site. At strong disorder, we expect the bosons to be localized by the random potential, forming a Bose glass. For $t = 0$, as long as $-\Delta + U > \Delta$ it is impossible to have a site with a low enough potential to warrant its occupation by two bosons. The Mott gap is therefore $U - 2\Delta$, which collapses at $\Delta/U = 0.5$—an exact result. We find a critical point in the $t = \delta U = \delta t = 0$ plane at $\Delta/U = 0.27 \pm 0.01$, with critical exponents $\nu = 3.5 \pm 0.1$ and $z = 0.31 \pm 0.01$. This is in keeping with the linear collapse of the Mott gap $E_G = U - 2\Delta$ at the transition, which implies $z\nu = 1$. For initial disorder in the Bose-glass region of Fig. 3, the flows tend to larger disorder (Fig. 4). Since our recursion relations are untrustworthy when $\Delta/\overline{U}$ and $\delta U/\overline{U}$ become of order unity, at this point we stop and consider the system to be a Bose glass. [In addition to the stable fixed point on the $t = 0$ axis at $\Delta/U = \delta U/U = 0$ (the Mott phase), we also find one at $\Delta/\overline{U} = 1.97 \pm 0.06$, $\delta U/\overline{U} = 1.59 \pm 0.06$, which is way beyond the range of validity of our assumptions.]

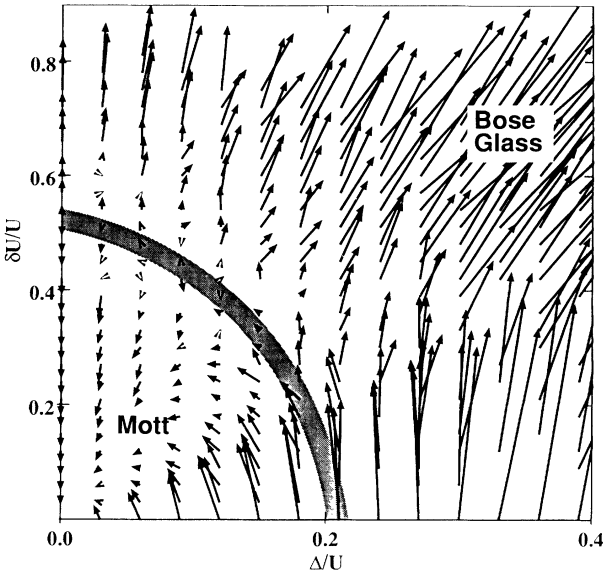


FIG. 4. Flow diagram in the $t = \delta t = 0$ plane. An approximate boundary is sketched between the Mott and Bose-glass phases.

The most complicated point is the multicritical point, which is unstable in all directions. At this point it is no longer a good approximation to lump all of the hopping parameters into one distribution. This results in an unwieldy recursion relation involving eight parameters, which is subject to significant noise. We therefore cannot characterize this point with any reasonable accuracy.

In the two-dimensional disordered system, the introduction of infinitesimal disorder causes $\delta U$ to grow until eventually $\delta U > \overline{U}$. Thus our calculation does not exhibit a stable Mott fixed point for the disordered system. The Mott phase should be stable in two dimensions, however, since as long as $\Delta$ and $t$ are small compared with the Mott gap perturbation theory should converge.

## V. DISCUSSION

The nature of the superfluid-Mott transition in the pure case depends on how the Mott phase is approached[2] [Fig. 5(a)]. If the interaction strength $U/t$ is held fixed and the number density is tuned to a commensurate value (path A), then the transition is described by mean-field theory. If, on the other hand, the boson density is held fixed at a commensurate value and the interaction strength is tuned through its critical value (path B),

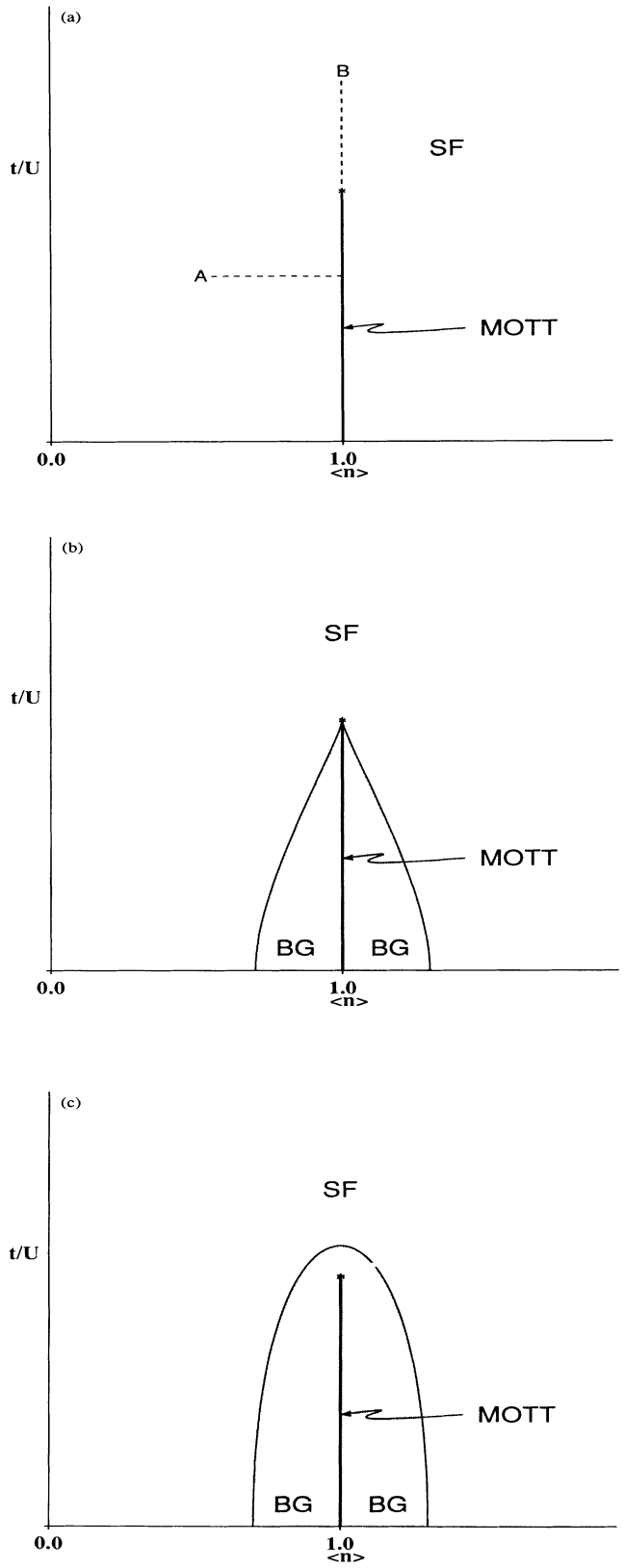


FIG. 5. Schematic phase diagram for (a) the pure Bose-Hubbard model, (b) the Bose-Hubbard model with disorder, (c) the scenario put forth by Fisher *et al.* for the disordered Bose-Hubbard model.

the transition is in the universality class of the $(d+1)$-dimensional $XY$ model. The Mott phase is characterized by a coherence length $\xi_M$ which measures the spatial extent of density fluctuations or, conversely, the range of local superfluid correlations in the insulating state. This length diverges at $(U/t)_c$.

In our real-space renormalization-group treatment of interacting bosons in one and two dimensions, we focused on the behavior at *commensurate* density as a function of interaction strength and disorder. For the pure system, we obtained ordinary critical points with correlation length exponents $\nu_{1D} = 4.8$ and $\nu_{2D} = 1.2$. Given the uncontrolled nature of real-space methods, we find the strikingly large value in one dimension (where the behavior is known to be in the Kosterlitz-Thouless universality class) encouraging.

At incommensurate densities $\langle n \rangle = N + \delta n$ where $N$ is an integer, the ground state can be either a superfluid or a Bose glass, but not a Mott insulator (if we consider only nearest-neighbor interactions). For small $\delta n$ we can describe the system as a collection of "excess" bosons of density $\delta n$ moving in the presence of a fluctuating Mott background of integer density $N$. One can identify[2] two regimes depending on whether the separation $a \sim (\delta n)^{-1/d}$ between the excess bosons is longer or shorter than $\xi_M$:

(1) If $a > \xi_M$, then the Mott background has little effect on the additional particles added to it. These excess bosons behave like a dilute, weakly interacting Bose gas.

(2) If $a < \xi_M$, then the local coherence of the Mott background will enhance the superfluid order in the fluid of excess particles.

To estimate the phase boundary between the superfluid and the Bose-glass states away from commensurate density, we must also consider the competition between the localizing effects of disorder and the delocalizing effect of interactions which tend to knock particles out of their nominally localized single-particle states. To estimate this effect it is useful to introduce a "localization length" which gives the spatial extent of these one-particle states (obtained, for example, by a mean-field approach[1]). We may then identify two regimes, depending on whether or not the interparticle spacing (of excess particles) is large or small compared with this localization length, $\xi_L$:

(3) If $a > \xi_L$, then each excess boson can occupy its own individual localized state.

(4) If $a < \xi_L$, then several bosons will occupy each localized state. Interactions between the bosons will delocalize the particles and can lead to condensation.

From these considerations, we see that only when both conditions (1) and (3) are simultaneously satisfied do we expect to find a Bose glass with Anderson-localized bosons. If *either* of these conditions is not satisfied, we expect a superfluid ground state, stabilized by either the

coherent fluctuations of the background Mott state (2), or by interactions (4). The estimated phase boundary is shown schematically in Fig. 5(b). Note that for fixed disorder the Mott coherence length $\xi_M$ can always be made larger than the (fixed) localization length by moving closer to $(U/t)_c$, so our heuristic arguments suggest that the behavior near the tip of the phase boundary is independent of the strength of the disorder.

For the disordered system at commensurate density, our real-space renormalization group suggests that the insulator-superfluid transition still takes place directly from the Mott phase. At this critical point disorder is a weakly irrelevant parameter. Our result agrees with numerical studies, which find no evidence for an intervening Bose glass for weak disorder in either one or two dimensions,[3,4] but contradicts the heuristic arguments of Fisher *et al.* They note that near the Mott transition the system can be viewed as a dilute collection of particle-hole pairs above the Mott background, whose density is of order $\xi_M^{-d}$. It is then argued that these excitations should be localized by arbitrarily weak disorder. The phase boundary for this scenario is shown in Fig. 5(c). Since the length scale $\xi_M$ also specifies the typical distance between particle and hole, it is not obvious that one may ignore the screening of disorder by these excitations.

An intriguing possibility is that disorder could be marginally relevant, which would lead to a thin sliver of Bose glass between the Mott and superfluid states in Fig. 3, extending all the way to the $\Delta = 0$ axis. This would explain the inability to resolve a Bose-glass phase in Monte Carlo studies, while remaining consistent with the arguments of Fisher *et al.*[2] We find that disorder is only slightly irrelevant, with a scaling dimension of $-0.07$. (A marginal field would have scaling dimension zero.)

The phase diagram in one dimension exhibits a localized Bose-glass phase at strong disorder. For weak disorder the Mott phase is destabilized, and the phase boundary curves downward, so that the disordered system is superfluid at interaction strengths which were Mott insulating in the pure case. Unfortunately, our method is too crude to allow investigation of the multicritical point. (According to our calculation, the Mott phase is unstable to the addition of infinitesimal disorder in two dimensions.)

We also tried to investigate systems with incommensurate densities using a real-space technique, but found no straightforward blocking scheme which holds the occupation at a constant (fractional) density but does not show a Mott insulating fixed point. Incommensurate systems of hard-core bosons have recently been studied by Zhang and Ma[16] using a real-space method.

## ACKNOWLEDGMENTS

We thank G.G. Batrouni, M.P.A. Fisher, G. Grinstein, and P.E. Lammert for useful discussions. K.G.S. was supported by the Department of Defense. Work at Berkeley was supported by the NSF under Grant No. DMR-89-14440 and PYI Grant No. DMR-91-57414. D.S.R. acknowledges a Grant from the Alfred P. Sloan Foundation.

[1] J.A. Hertz, L. Fleishman, and P.W. Anderson, Phys. Rev. Lett. **43**, 942 (1979).
[2] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).
[3] R.T. Scalettar, G.G. Batrouni, and G.T. Zimanyi, Phys. Rev. Lett. **66**, 3144 (1991).
[4] W. Krauth, N. Trivedi, and D. Ceperley, Phys. Rev. Lett. **67**, 2307 (1991).
[5] W. Krauth and N. Trivedi, Europhys. Lett. **14**, 627 (1991).
[6] Alice E. White, R.C. Dynes, and J.P. Garno, Phys. Rev. B **33**, 3459 (1986).
[7] H.M. Jaeger, D.B. Haviland, A.M. Goldman, and B.G. Orr, Phys. Rev. B **34**, 4920 (1986).
[8] J.D. Reppy, Physica **126B**, 335 (1984).
[9] M.H.W. Chan, K.I. Blum, S.Q. Murphy, G.K.S. Wong, and J.D. Reppy, Phys. Rev. Lett. **61**, 1950 (1988).
[10] D. Finotello, K.A. Gillis, A. Wong, and M.H.W. Chan, Phys. Rev. Lett. **61**, 1954 (1988).
[11] M. Ma, B.I. Halperin, and P.A. Lee, Phys. Rev. B **34**, 3136 (1986).
[12] G.G. Batrouni, R.T. Scalettar, and G.T. Zimanyi, Phys. Rev. Lett. **65**, 1765 (1990).
[13] T. Giamarchi and H.J. Schulz, Phys. Rev. B **37**, 325 (1988).
[14] Jorge E. Hirsch, Phys. Rev. B **22**, 5259 (1979).
[15] M. Ma, Phys Rev. B **26**, 5097 (1981).
[16] L. Zhang and M. Ma, Phys. Rev. B **45**, 4855 (1992).

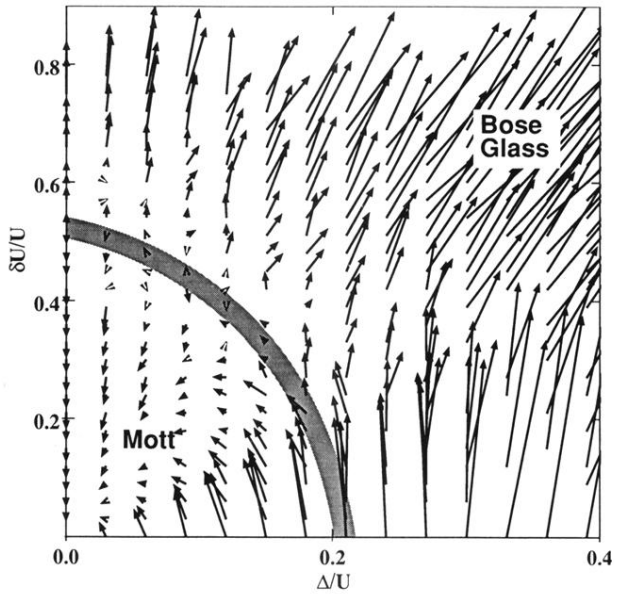


FIG. 4. Flow diagram in the $t = \delta t = 0$ plane. An approximate boundary is sketched between the Mott and Bose-glass phases.

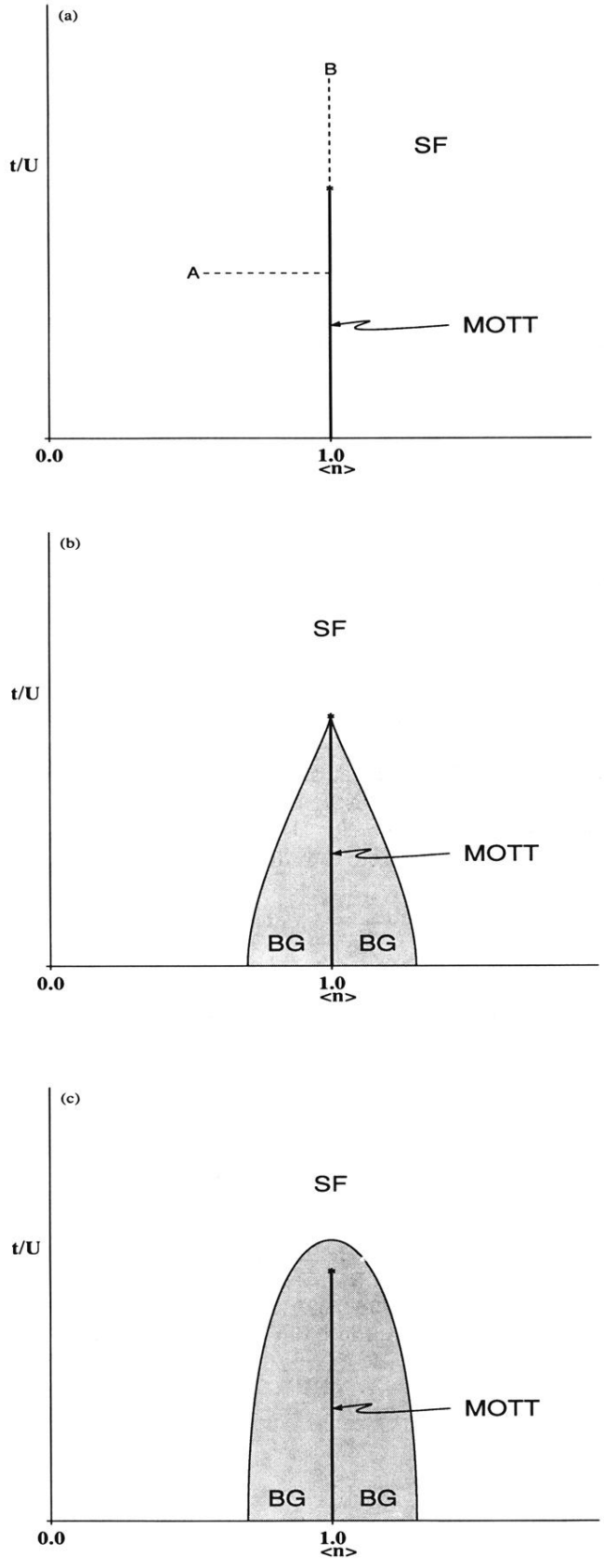


FIG. 5. Schematic phase diagram for (a) the pure Bose-Hubbard model, (b) the Bose-Hubbard model with disorder, (c) the scenario put forth by Fisher *et al.* for the disordered Bose-Hubbard model.