Classification of Abelian quantum Hall states and matrix formulation of topological fluids

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We give a simple and unified treatment of quantum topological fluids such as the quantum Hall fluid. We show that the order in such fluids can be characterized by a symmetric matrix K, in terms of which various physical quantities can be determined. We construct K by a matrix iteration procedure which may be decomposed into two simple elementary steps. The hierarchy construction is shown to be contained in our matrix iteration construction. The relationship between the vortex basis and the dual electron basis is clarified. We also show that under certain mild assumptions the generalized hierarchy construction exhausts all possible Abelian fractional quantum Hall states. We identify and determine the topological quantity known as the shift. Our formalism may be relevant for recent experimental data on multilayered systems.

I. INTRODUCTION

It has become increasingly clear that the fractional quantum Hall (FQH) states[1] contain an extremely rich internal structure. A long-standing problem has been how to characterize and label the orders in these states. The answer to this problem is ever more urgently needed since there are now so many different constructions[2] available in the literature. It is difficult to see whether different constructions lead to the same FQH states or not. We would like to have a unified picture of the universality classes and the ordering in these states. In this paper, we present a simple and unified description.

Traditionally, one uses broken symmetries and their associated order parameters to classify the order and universality classes of condensed-matter systems. However, this approach is not applicable to the FQH states. It was shown[3] that the orders in these states cannot be classified by broken symmetries and their associated order parameters, because the degeneracy of the ground state depends on the genus g of the two-dimensional (closed) space over which these states are defined. Thus, these states contain a different type of order called topological order.[3] We will call these topologically ordered quantum fluids topological fluids for short.

It was pointed out that the simplest topological fluids—the Laughlin states with filling factor $\nu=1/$(odd integer)—contain off-diagonal long-range order for some unphysical nonlocal operators. This leads to the Ginzburg-Landau (GL) theory of the Laughlin states. However, it appears that the concept of off-diagonal long-range order and the corresponding GL theory do not provide a general description of the topological orders in the FQH states. In particular, it appears that one cannot use the off-diagonal long-range orders to describe the internal structures of the non-Abelian FQH states.

Recently, a general mathematical approach to the long-distance physics of quantum topological fluids was formulated based on the Chern-Simons theory,[4,5] building upon earlier work by a number of authors.[6–8] Examples of quantum topological fluids include the quantum Hall fluid,[1] the chiral spin fluid,[9] and the anyon superfluid.[10] Here we will focus on the Hall fluid. Similar discussions may be given for the other fluids. This approach has the advantage of making clear which properties of these topological fluids are general and independent of the detailed physics at short distances. It also provides a unified description of both Abelian and non-Abelian FQH states. (The Abelian states are defined as those in which all quaisparticles have Abelian statistics, while the non-Abelian states are those in which some quasiparticles have non-Abelian statistics.)

In this paper, we will give a complete classification of the topological orders (or the universality classes) of the Abelian FQH states. We will show that all possible Abelian FQH states are labeled by a symmetric integer valued matrix K with odd diagonal elements. We will show that the generalized hierarchical construction discussed in Refs. 5 and 11 exhausts all possible Abelian FQH states. (Here we ignore the possibility of pairing between electrons. See the forthcoming discussion.) Another purpose of this paper is to extend, clarify, and elaborate on our earlier discussion.

Let us begin by reviewing briefly some basic facts of this subject. The long-distance physics of the quantum topological fluids is described in general by the Lagrangian in $(2+1)$-dimensional space time[7]

$$\mathcal{L}=\frac{1}{4\pi}\sum_{I,J}\alpha_I K_{IJ}\varepsilon\partial\alpha_J+\cdots\ . \tag{1.1}$$

We use the compact notation $\alpha\varepsilon\partial\beta\equiv\varepsilon^{\mu\nu\lambda}\partial_\nu\beta_\lambda$ for two gauge potentials α_μ and β_μ. The ellipses in (1.1) represent short-distance physics about which this formalism has nothing to say. If the matrix K does not have any zero eigenvalue, the gauge potentials α_I's are all massive[12] and

the fluid is incompressible; for example, the Hall fluid and the chiral spin fluid. On the other hand, if K has a zero eigenvalue, some linear combination of the gauge potentials is massless and the fluid is a compressible superfluid; for example, the anyon superfluid. The ground states of these theories are described by nontrivial global configurations of the gauge potentials α_I's. The degeneracy of the ground state is given by[3]

$$D=(\det K)^g , \tag{1.2}$$

where g is the genus of the two-dimensional closed surface on which the theory is defined. This shows clearly the topological character of the theory. It also follows that the determinant of K must be an integer. [In writing (1.2) we have, of course, assumed an appropriate normalization for the gauge potentials, such that the currents coupling to α_I carry integer charges. See (2.1) below.]

In the simplest cases, the matrix K is 1×1 and reduces to a number k, an integer. For $k=1,3,\ldots$ an odd integer we have the Hall fluid with the classic odd denominator filling fraction $\nu=1/k$. For $k=2,4,\ldots$ an even integer we have the chiral spin fluid with statistics parameter $\theta/\pi=1/k$. Thus, satisfyingly, the chiral spin fluid fills the gaps left open by the classic Hall fluid.

As we will explain, the discussion can now proceed either in a vortex basis, in which vortices play a "primary" role, or in an electron basis, in which electrons (or holes) play a "primary" role. We hope to make clear the connection between these two complementary bases. Let us first turn to the vortex basis, following earlier work.[7]

II. VORTEX BASIS

The vortices, or quasiparticles, in these fluids are introduced as (massive) sources for the gauge potentials α_I's. We add to the Lagrangian in (1.1) the term

$$\Delta_V\mathcal{L}=\sum_I \alpha_I j^I , \tag{2.1}$$

where j^L_μ denotes the current of the vortices of the Ith type. Thus, it is vortex quantization that fixes the normalization of α and hence of the matrix K in (1.1).

These various topological fluids are, of course, also distinguished by their couplings to the external electromagnetic gauge potential A_μ. The elementary excitations in the chiral spin fluid (the spinons) are electrically neutral. On the other hand, the elementary excitations in the Hall fluid are electrically charged, with a coupling to A_μ described by

$$\Delta_A\mathcal{L}=\frac{1}{2\pi}A\sum_I t_I\varepsilon\partial\alpha_I . \tag{2.2}$$

In previous works, the charges t_I's were taken to be equal to 1. The possibility of setting t_I not equal to 1 was mentioned by us[7] but not pursued in detail. We consider this possible generalization here.

The total long-distance Lagrangian of the Hall fluid is thus the sum of (1.1), (2.1), and (2.2):

$$\mathcal{L}=\frac{1}{4\pi}\alpha K\varepsilon\partial\alpha+\frac{1}{2\pi}At\varepsilon\partial\alpha+\alpha j . \tag{2.3}$$

As we have already discussed in detail in an earlier work,[7] we are free to integrate out the gauge potentials α_I's to obtain the nonlocal (matrix) Hopf Lagrangian[13]

$$\mathcal{L}=\pi\tilde{j}K^{-1}\frac{\varepsilon\partial}{\partial^2}\tilde{j} , \tag{2.4}$$

with the modified currents

$$\tilde{j}^I_\mu=j^I_\mu-\frac{1}{2\pi}t_I\varepsilon_{\mu\nu\lambda}\partial^\nu A^\lambda . \tag{2.5}$$

The effective Lagrangian (2.4) contains three types of terms, of the form AA, Aj_I, and j_Ij_J. These determine, respectively, the conductance or the filling factor of the Hall fluid

$$\sigma=\nu=\sum_{I,J} t_I(K^{-1})_{IJ}t_J , \tag{2.6}$$

the charge of a conglomerate consisting of l_I vortices of the Ith type

$$q=\sum_{I,J} t_I(K^{-1})_{IJ}l_J , \tag{2.7}$$

and the statistics of such a conglomerate

$$\frac{\theta}{\pi}=\sum_{I,J} l_I(K^{-1})_{IJ}l_J . \tag{2.8}$$

Earlier we derived[7] these formulas with t_I all set equal to 1. The formulas written here are more symmetric looking, with the t_I's and the l_I's appearing on equal footing. The physics involved is also made clearer: the conductance has to do only with the t_I's, the statistics only with the l_I's, the charge with both.

Let us pause from our general discussion to explain why we may wish to consider t_I's not equal to 1. One possibility is that the short-distance physics we have consistently ignored, and which we are unable to treat, may produce bound states. For instance, two electrons may bind into a charge-2 boson. The filling factor $\nu=2\pi n/eB$ then scales to $\nu=\nu_{\text{old}}\times\frac{1}{2}\times\frac{1}{2}=\nu_{\text{old}}/4$, since the number density is half of what it was and the charge is doubled. Thus the $\nu=\frac{1}{2}$ state, for example, becomes a $\nu=\frac{1}{8}$ state for which a Laughlin-type wave function may be written down readily:

$$\Psi\sim\prod_{i>j}(z_i-z_j)^8 e^{-\sum|z|^2} . \tag{2.9}$$

This was discussed long ago by Halperin and more recently by us in the language of effective Lagrangians.[7] The effective Lagrangian (2.3) is then

$$\mathcal{L}=\frac{1}{4\pi}8\alpha\varepsilon\partial\alpha+\frac{1}{2\pi}A(2\varepsilon\partial\alpha)+\alpha j . \tag{2.10}$$

(Note that we cannot simply scale the gauge potential α because of its coupling to the vortices.) From (2.7) and (2.8), we see that the vortex has charge $t/k=2/8=1/4$ and statistics $\theta/\pi=1/k=1/8$. Thus, in general, we may have t_I assuming any (integral, presumably) values.

In this basis, which we will call the vortex basis, a vortex of type L is represented by

$$l_J^{(L)}=\delta_{JL} \tag{2.11}$$

by definition. We see from (2.7) and (2.8) that its charge is

$$q^{(L)}=\sum_I t_I(K^{-1})_{IL} \ , \tag{2.12}$$

and when it moves all the way around a vortex of type L' the wave function acquires the phase angle $\phi^{(LL')}$:

$$\frac{\phi^{(LL')}}{2\pi}=(K^{-1})_{LL'} \ . \tag{2.13}$$

For $L=L'$, the statistics angle is defined as the phase angle acquired by the wave function when we move one vortex halfway around the other, namely

$$\frac{\theta^{(L)}}{\pi}=(K^{-1})_{LL} \ . \tag{2.13'}$$

The classic Laughlin results that for filling factor $\nu=1/k$ with k an odd integer the vortex has charge $1/k$ and statistic π/k appear as the simplest special cases of (2.12) and (2.13').

In earlier works,[4,14] we argued that we have to impose on K the condition that in the excitation spectrum there is one conglomerate of vortices with the quantum number of the hole (or the electron). Here we generalize this discussion.

Given the Lagrangian in (2.4), we are, of course, free to pick any l_I we please and compute the charge and statistics of that particular conglomerate of vortices. Here we note that we can make a particularly judicious choice, namely

$$l_J^{(L)}=K_{JL} \ . \tag{2.14}$$

Different choices are labeled by L. In other words, we choose the columns in the matrix K. [Note that, by definition (1.1), K is a symmetric matrix, and thus it does not matter whether we take the columns or rows in K.]

This choice is judicious because we obtain immediately from (2.7) and (2.8) that the charge of this conglomerate is

$$q^{(L)}=t_L \ , \tag{2.15}$$

and that when this conglomerate moves all the way around another conglomerate of type L' the wave function acquires the phase [compare with (2.13) and (2.13')]

$$\frac{\phi^{(LL')}}{2\pi}=\sum_{I,J} K_{IL'}K_{IJ}^{-1}K_{JL}=K_{LL'} \ . \tag{2.16}$$

These formulas indicate the physical meaning of t and K.

We see immediately that there is a hole in the excitation spectrum if some t_L is equal to 1 and if K_{LL} is an odd integer. (This discussion is an improvement over that in Ref. 4. There we focused on some specific K matrices and were obliged to solve various number theoretic equations to determine whether or not the hole exists.)

In this paper we will not always impose the requirement that the hole or electron exists. Unless specified otherwise, we will consider more general K matrices in which only some of the t_L's are equal to 1.

Setting $l_J^{(L)}=K_{JL}$ and $l_J^{(L')}=\delta_{JL'}$, we see that when an "electron" of the Lth type moves all the way around a vortex of the L'th type, the wave function acquires a phase

$$\frac{\phi^{(LL')}}{2\pi}=\delta_{LL'} \ . \tag{2.17}$$

(We put quotation marks around the word "electron" because the excitation considered would be a hole or an electron only if $t_L=1$ and K_{LL} is odd.) This indicates that the electron wave function is single valued even in the presence of a vortex.

III. ELECTRON BASIS AND DUALITY

In general, we are free to go to another basis by writing $\alpha_I=\sum_J W_{IJ}\alpha'_J$. In the transformed basis $K'=W^TKW$, $t'=tW$, and so on. We may be tempted to go to a basis defined by $W=K^{-1/2}$ if it exists. In this basis, K' would be just I but t' becomes complicated. In fact, physics tells us about another natural basis, namely the "electron" basis. Now that we have learned that an "electron" of the Lth type consists of a bound state of K_{JL} vortices of the Jth type, we can go to another basis. Write

$$j^J_{\text{vortex}}=\sum_L K_{JL}j^L_{\text{"electron"}} \ . \tag{3.1}$$

Then the coupling of the gauge potentials to the vortice may be written as

$$\sum_J \alpha_J j^J_{\text{vortex}}=\sum_{J,L}\alpha_J K_{JL}j^L_{\text{"electron"}} \ . \tag{3.2}$$

In other words, we should use the gauge potentials

$$\beta_{\mu L}=\sum_J K_{JL}\alpha_{\mu J} \ . \tag{3.3}$$

In this basis, the Chern-Simons Lagrangian governing the gauge interaction (1.1)

$$4\pi\mathcal{L}=\alpha K\varepsilon\partial\alpha \tag{3.4}$$

becomes

$$4\pi\mathcal{L}=(\beta K^{-1})K\varepsilon\partial(K^{-1}\beta)=\beta K^{-1}\varepsilon\partial\beta \ . \tag{3.5}$$

The matrix K has exchanged places with its own inverse K^{-1}. We have discovered the duality[15] between the vortex basis and the "electron" basis.

In this basis the "electron" is represented by $m_J^{(L)}=\delta_{JL}$, where $m_J^{(L)}$ corresponds to $l_J^{(L)}$ in the vortex basis. Note, however, that the coupling to the electromagnetic gauge potential A_μ becomes complicated:

$$At\varepsilon\partial\alpha=At\varepsilon\partial K^{-1}\beta=As\varepsilon\partial\beta \ , \tag{3.6}$$

with $s=K^{-1}t$.

We note also that in this basis the vortex is represented by

$$m_J^{(L)}=K_{JL}^{-1} \ . \tag{3.7}$$

The vortex of the Lth type may be thought of as a conglomerate or K_{JL}^{-1} "electrons" of the Jth type. Evi-

dently, we have from (2.14) and (3.7) the orthonormality relation

$$\mathbf{m}^{(L)}_{\text{vortex}} \cdot \boldsymbol{l}^{(L')}_{\text{"electron"}} = \delta^{LL'} \ . \tag{3.8}$$

IV. COMBINING INCOMPRESSIBLE FLUIDS

One of the messages conveyed in Ref. 4 is that, given a bunch of quantum topological fluids, we can put them together and construct another topological fluid. Suppose we have W topological fluids, each of which is described by

$$4\pi \mathcal{L}_s = \alpha^s K_s \varepsilon \partial \alpha^s \ , \tag{4.1}$$

with K_s an $m_s \times m_s$ matrix, for $s = 1,2,\ldots,W$. Each fluid contains m_s gauge potentials $\alpha^s_{I_s}$, $I_s = 1,2,\ldots,m_s$. The physical assumption is that the long-distance interaction between these fluids involves only the electromagnetic interaction and thus only the total electromagnetic current

$$J^\mu = \frac{1}{4\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu \sum_{s=1}^{W} \sum_{I_s=1}^{m_s} t_{I_s} \alpha^s_{I_{s_\lambda}} \ . \tag{4.2}$$

(In some contexts we may wish to relax this assumption.) We then argue on general grounds that the effect of the interaction on the long-distance physics is uniquely determined to be that of a Chern-Simons terms involving the "total" gauge potential $\alpha_{\text{tot}} = \sum_s \sum_{I_s} \alpha^s_{I_s}$. (This follows from gauge and scale invariances.) Thus, the composite fluid is described by the Lagrangian

$$4\pi \mathcal{L} = \sum_s \alpha^s K_s \varepsilon \partial \alpha^s + p \alpha_{\text{tot}} \varepsilon \partial \alpha_{\text{tot}} \ . \tag{4.3}$$

Here p is an unknown real number characterizing the interaction.

Lest we get overwhelmed by notational complexity, let us note the simple case in which each component fluid is just a classic 1/(odd denominator) Hall fluid. In that case, $m_s = 1$ and K_s is an odd integer k_s. This case, with the further simplification that all the $t_{I_s} = 1$, was studied in Ref. 4.

We can write the Lagrangian (4.3) as

$$4\pi \mathcal{L} = \alpha K \varepsilon \partial \alpha \ , \tag{4.4}$$

with the composite matrix K given by

$$K = K_0 + pT \ . \tag{4.5}$$

Here K_0 is a block diagonal matrix with the blocks K_s, $s = 1,2,\ldots,W$. To avoid having indices on indices let us replace the index $\{I_s\}$ by $\{C\}$. (Thus we write, for example, t_C, with C taking on $\sum_{s=1}^{W} m_s$ different values.) With this notation, we have explicitly

$$K_{CD} = K_{0CD} + p t_C t_D \ . \tag{4.6}$$

This defines the matrix T.

Thus far in this paper we were able to proceed without having to use an explicit form for the matrix K. Equation (4.5) should now be regarded as defining an iterative construction of the matrix K. Given some matrices K_s, we can construct a different matrix K. To start this iteration, we can choose K_s to be simply 1×1 and equal to odd integers. This construction may be referred to as the matrix iteration construction.

V. EXAMPLES OF INCOMPRESSIBLE TOPOLOGICAL FLUIDS

Before proceeding with a general discussion, we will give some specific examples of our construction. For the sake of simplicity, let us now take all the t_I's to be equal to 1. It turns out that even with this dramatic simplification, we can reach most of the FQH states discussed in the literature. The matrix T collapses to the pseudo-identity matrix

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} , \tag{5.1}$$

namely the matrix in which every entry is equal to 1. Sometimes we will find it necessary to write C_κ to indicate the κ by κ pseudo-identity matrix.

Our iterative construction in (4.5) may be decomposed into two steps as follows. In what we will call step A, we put two matrices together to form a new matrix

$$K_1 \oplus K_2 \rightarrow K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} . \tag{5.2}$$

In step B, we add the pseudo-identity to a given matrix

$$K \rightarrow K + C \ . \tag{5.3}$$

According to (2.6), the filling factor is given by $\nu = \sum_{I,J} (K^{-1})_{IJ}$. Thus, in step A, we have trivially

$$\nu = \nu_1 + \nu_2 \ , \tag{5.4}$$

where ν_1 and ν_2 are the filling factors corresponding to the fluid defined by K_1 and K_2, respectively. In step B, we have to invert the matrix $(K + C)$. This can be done easily,[7] with

$$(K+C)^{-1} = K^{-1} - (1+\nu_K)^{-1} K^{-1} C K^{-1} \ , \tag{5.5}$$

as the reader can verify immediately using the properties of C. Here ν_K denotes the filling factor corresponding to K. The new filling factor ν is given by $\sum_{I,J} (K+C)^{-1}_{IJ}$. We find easily that

$$\nu^{-1} = \nu_K^{-1} + 1 \ . \tag{5.6}$$

These two exceedingly simple formulas (5.4) and (5.6) tell us that in step A we add the filling factors and that in step B we add to the inverse filing factor. (Thus, we may wish to refer to step A as combining incompressible fluids "in series" and to step B as combining incompressible fluids "in parallel.")

We mention in passing that we can always go from K to $(-K)$ by time reversal or parity. Combining this with step A, we can obtain a matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -K \end{bmatrix},$$

with the filling factor $\nu = 1 - \nu_K$.

Let us start with the 1×1 matrix K equal to 1 and apply step B repeatedly. After p steps we reach

$$K = 1 + p \tag{5.7}$$

corresponding to

$$\nu^{-1} = 1 + p \ . \tag{5.8}$$

As mentioned earlier, for p even we have an excitation with the quantum numbers of the electron. These are the classic Laughlin odd-denominator fluids.

Alternatively, we can also start with $K = 1$ and first apply step A repeatedly until we get the $m \times m$ identity matrix before applying step B repeatedly. We obtain the $m \times m$ matrix[7]

$$K = I + pC \ . \tag{5.9}$$

From (5.3) and (5.6) we see that $\nu^{-1} = m^{-1} + p$ or the hierarchy state with filling fraction $\nu = m/(mp+1)$ (including $\frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$ for $p = 2$).

To describe the hiearachy construction, let us start with the lowest level. Start with a 1×1 matrix $K = p_2 + 1$ and use step A to construct

$$\begin{bmatrix} -1 & 0 \\ 0 & p_2+1 \end{bmatrix}, \tag{5.10}$$

with filling factor

$$\nu = -1 + (p_2+1)^{-1} = -\frac{p_2}{p_2+1} \ , \tag{5.11}$$

and then step B to construct

$$K = \begin{bmatrix} -1 & 0 \\ 0 & p_2+1 \end{bmatrix} + (p_1+1)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_1+1 \\ p_1+1 & p_1+p_2+2 \end{bmatrix}, \tag{5.12}$$

with filling factor

$$\nu^{-1} = \left[-\frac{p_2}{p_2+1} \right]^{-1} + p_1 + 1 = p_1 - \frac{1}{p_2} \tag{5.13}$$

or

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2}} \ . \tag{5.14}$$

(For p_1 odd and p_2 even, these states include $\nu = \frac{2}{3}, \frac{2}{7}, \ldots .$)

As an illustration of the power of our method, we now see how to continue this construction. We describe the construction (in an obvious notation) starting with a matrix K as follows:

$$\begin{aligned} K &\overset{B}{\to} K + C \\ &\overset{A}{\to} \begin{bmatrix} -1 & 0 \\ 0 & K+C \end{bmatrix} \\ &\overset{B}{\to} \begin{bmatrix} -1 & 0 \\ 0 & K+C \end{bmatrix} + (p+1)C \ . \end{aligned} \tag{5.15}$$

(In the last step, the two C matrices differ in their dimensions by one.)

According to (5.4) and (5.6), the inverse filling factor changes to

$$\nu^{-1} = [(\nu_K^{-1}+1)^{-1} - 1]^{-1} + (p+1) = p - \nu_K \ . \tag{5.16}$$

(Note that in particular for $p = 0$, we have $\nu = -\nu_K^{-1}$. In other words, we have solved the following problem: given an incompressible fluid characterized by a matrix K and a filling factor ν_K, find an incompressible fluid with filling factor equal to the inverse of the filling factor of the given fluid.)

By iterating, we reach the most general hierarchical states with filling factor

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2 - \cfrac{1}{\ddots - \cfrac{1}{p_\kappa}}}} \ . \tag{5.17}$$

Explicitly, starting with the 2×2 K matrix in (5.13) we obtain on the next step

$$K = \begin{bmatrix} p_1 & p_1+1 & p_1+1 \\ p_1+1 & p_1+p_2+2 & p_1+p_2+3 \\ p_1+1 & p_1+p_2+3 & p_1+p_2+p_3+4 \end{bmatrix} . \tag{5.18}$$

If one insists, one can also write down the general K matrix we reach with the iteration in (5.16):

$$K = (p_1+1)C_\kappa^{(1)} + \sum_{A=2}^{\kappa} (p_A+2)C_\kappa^{(A)} - I \ , \tag{5.19}$$

where $C_\kappa^{(A)}$ is a $\kappa\times\kappa$ matrix with

$$(C_\kappa^{(A)})_{IJ} = \begin{cases} 1, & I,J = A, A+1, \ldots, \kappa \\ 0, & I,J = \text{others} \ . \end{cases} \tag{5.20}$$

In other words,

$$C_\kappa^{(A)} = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}, \tag{5.21}$$

with C the $(\kappa - A + 1)\times(\kappa - A + 1)$ pseudo-identity matrix. [Here we have only shown that the matrix K gives the filling fraction of the hierarchical states. At the end of the next section we will show that the effective Chern-Simons theory with K in (5.20) is identical (after a field redefinition) to the effective Chern-Simons theory for the standard hierarchical states derived in Ref. 5.] In this way, we have demonstrated that the hierarchy construction is contained within the matrix iteration construction.

The simplest FQH state which does not belong to the hierarchy construction is given by

$$K=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$$

with filling factor $\nu=\frac{1}{2}$. This state may be obtained by having quasiparticle pairs condense on top of the $\nu=\frac{1}{3}$ Laughlin state.[16]

VI. CLASSIFICATION OF ABELIAN HALL STATES

In Ref. 5, a class of generalized hierarchical states was constructed. The effective theory is given by (2.3) with K and t satisfying

$$\begin{aligned} &K^h_{11}=\text{odd integer}\ ,\\ &K^h_{II}|_{I>1}=\text{even integer}\ ,\\ &K^h_{IJ}=\text{integer for}\ I\neq J\ ,\\ &t^h_I=\delta_{1I}\ . \end{aligned} \tag{6.1}$$

Notice that all the charges t^h_I vanish except for t^h_1. We will call this basis the hierarchical basis, as indicated by the superscript h. We will refer to the previous basis, in which all the t_I's are equal to 1, as the symmetric basis.

Even though the K given here is almost arbitrary, it is still not clear whether these generalized hierarchical states cover all possible Abelian fractional quantum Hall states or not. Here we will derive the effective theory of the most general Abelian fractional quantum Hall states. We will show that the generalized hierarchical states indeed represent the most general Abelian FQH states.

Our first working assumption is that (a) the effective theory of Abelian FQH states is described by a Lagrangian of the form in (2.3). However, this assumption alone is not enough. For arbitrary choices of K and t the Lagrangian in (2.3) may not describe an electron system. So the problem we are facing is not how to derive the effective theory of an electron system, but the reverse, how to determine whether an effective theory is consistent with the underlying electron system or not. This leads us to our second working assumption: (b) In order for the effective theory to describe an electron system, the effective theory must contain κ independent electron operators, where κ denotes the rank of K. (Here we ignore the possibility of electron pairing.) This requirement is a more stringent version of the requirement discussed in Ref. 4. We will discuss why we need κ electron operators later.

To implement (b), it is convenient to work in the electron basis as discussed above:

$$\mathcal{L}=\frac{1}{4\pi}\tilde{\alpha}\tilde{K}\varepsilon\partial\tilde{\alpha}+\frac{1}{2\pi}A\tilde{t}\varepsilon\partial\tilde{\alpha}+\tilde{\alpha}\tilde{j}\ . \tag{6.2}$$

We have used the tilde to remind ourselves that we are in the electron basis and that $\tilde{j}_I$ represents the current of the Ith electron. As discussed above, we have the dual relation $\tilde{K}=K^{-1}$.

In order for the electrons to have Fermi statistics, we must have

$$(\tilde{K}^{-1})_{IJ}\equiv K_{IJ}=\begin{cases}\text{odd integer} & \text{for } I=J\\ \text{integer} & \text{for } I\neq J\ .\end{cases} \tag{6.3}$$

Furthermore, in order for the electrons to carry unit charge, $\tilde{t}$ must satisfy

$$\sum_J(\tilde{K}^{-1})_{IJ}\tilde{t}_J=1\quad\text{for all } I\ . \tag{6.4}$$

This equation implies that

$$\tilde{t}_J=\sum_L\tilde{K}_{JL}\ . \tag{6.5}$$

Given the effective theory in the electron basis, we now need to determine the allowed quasiparticle excitations. A generic quasiparticle is described by a current of the form

$$j_\mu=\sum_I c_I\tilde{j}_{I\mu}\ . \tag{6.6}$$

(The c_I's are, of course, the analogs of the l_I's in this dual basis.) Since the electron wave function must be single valued (even in the presence of the quasiparticle), the phase induced by moving an electron around the quasiparticle must be a multiple of 2π. According to the analog of (2.8), this requires c_I to satisfy

$$\sum_{J=1}^{\kappa}(\tilde{K}^{-1})_{IJ}c_J=\text{integer for all } I\ . \tag{6.7}$$

The set of all c_J's satisfying (6.7) forms a κ-dimensional lattice, with the basis vectors of the lattice given by $\mathbf{c}^{(L)}$:

$$c^{(L)}_J=\tilde{K}_{JL}\ . \tag{6.8}$$

Correspondingly, there are κ fundamental quasiparticles whose currents are given by

$$j_L=\sum_I c^{(L)}_I\tilde{j}_I\quad\text{for } L=1,\ldots,\kappa\ . \tag{6.9}$$

The currents of any other excitations, including the electron excitations, are linear combinations of the currents of these fundamental quasiparticles with integer coefficients.

Now let us change back to the vortex or quasiparticle basis. According to (3.3), we substitute $\alpha_I=\sum_J\tilde{K}_{IJ}\tilde{\alpha}_J$ in (6.2) and obtain, using (6.4),

$$\mathcal{L}=\frac{1}{4\pi}\alpha K\varepsilon\partial\alpha+\frac{1}{2\pi}A\varepsilon\partial\alpha+\alpha j\ . \tag{6.10}$$

This is the same as (2.3) except that K satisfies (6.3) and the t_I's are all equal to 1. In an earlier work,[7] we assumed $t_I=1$.

We have reached one of the central results of this paper: the most general Abelian FQH states of unpaired electrons are described by the effective theory (6.10) with K satisfying (6.3). In other words, the topological orders in the Abelian FQH states are labeled by integer valued symmetric matrices with odd diagonal elements, up to an equivalency condition $K\sim W^TKW$, with W an element of

$SL(\kappa,\mathbb{Z})$ which leaves the vector $\mathbf{t}=(1,1,\ldots,1)$ invariant. The determinant of W must be equal to 1 in order to preserve the ground-state degeneracy.

Next we would like to show that this condition is not only necessary for the effective theory to describe an electron system, but also sufficient. We will show that all possible effective theories of the Abelian FQH states describing an electron system could be realized through the generalized hierarchical construction discussed in Ref. 5 and described above. We simply make a field redefinition to transform from the symmetric basis:

$$\alpha_I \to \alpha'_I = \sum_j W_{IJ}\alpha_J \ , \tag{6.11}$$

with $W_{IJ}=\delta_{IJ}-\delta_{I+1,J}$ an element of $SL(\kappa,\mathbb{Z})$, a transformation that also appears in Ref. 4. In other words,

$$\begin{aligned} &\alpha_1 \to \alpha'_1=\alpha_1-\alpha_2, \quad \alpha_2 \to \alpha'_2=\alpha_2-\alpha_3, \ldots , \\ &\alpha_\kappa \to \alpha'_\kappa=\alpha_k \ . \end{aligned} \tag{6.12}$$

Under the transformation, $K \to K^h = W^T K W$ and $t^h = tW$, with

$$K^h_{IJ}=K_{IJ}+K_{I-1,J-1}-K_{I-1,J}-K_{I,J-1} \ . \tag{6.13}$$

Evidently, a K matrix in the symmetric basis (6.3) is transformed into one in the hierarchical basis. Similarly, the charge vector with $t_I=1$ for all I is transformed into one with $t^h_1=\delta_{I1}$. We can, of course, transform in the opposite direction. We have thus demonstrated our statement above regarding how the topological orders in the Abelian FQH states are classified.

The physics behind this transformation was already stated in, for example, Ref. 4 (and was made particularly clear in the derivation given in Ref. 7). In the hierarchical basis, the gauge potentials $\alpha_1,\ldots,\alpha_{\kappa-1}$, are differences of the gauge potential in the symmetric basis and thus correspond to electrically neutral excitations.

This hierarchical basis, in which only t_1 is nonzero, is clearly convenient for certain purposes. Given a K, the transformed K, namely K^h as defined in (6.13), can be constructed readily: for every entry in K, we add to it its neighbor to the "northwest" and subtract from it its neighbors to the "north" and to the "west." Notice that this operation when applied to the identity matrix I produces

$$\begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} , \tag{6.14}$$

a matrix close to the Cartan root matrix[11,5,4] for the simple unitary Lie algebras, and when applied to the (generalized) pseudo-identity matrix $C^{(A)}_\kappa$ defined in (5.22) produces

$$\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & 0 \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix} . \tag{6.15}$$

Let us now apply this operation to the matrix K in (5.19), or (5.12), and (5.18). We obtain

$$K=\begin{pmatrix} p_1 & 1 & & & & & \\ 1 & p_2 & 1 & & & & \\ & 1 & p_3 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & p_\kappa \end{pmatrix} , \tag{6.16}$$

precisely the matrix found in Ref. 5, and the matrix which defines the standard hierarchical construction.

We would next like to say a few words on why we need κ independent electron operators. If we had less than κ electron operators, the condition (6.7) would become $\sum_{J=1}^{\kappa}(\tilde{K}^{-1})_{IJ}c_J=$integer only for $I=1,\ldots,\kappa'$ for some $\kappa'<\kappa$. This is not enough to fix c_J on a lattice. In this case, excitations with arbitrarily small charge would be allowed. Such excitations may be continuously connected to the ground state and would be, we believe, gapless. This argument suggests that in order for the effective theory to have finite energy gap, we require the presence of κ different electron operators. (Of course, we also require $\det K \neq 0$ in order for the gauge fluctuations to have finite gaps.)

More generally, we may have pairing and perhaps even charge-3 bound states and so on. In that case, with κ_1 electron operators, κ_2 pair operators, and so on, with $\kappa_1+\kappa_2+\cdots=\kappa$, we will have a set of conditions generalizing (6.7).

VII. CHARGE AND STATISTICS

We saw earlier that the physical properties of the Hall fluid, namely the conductance, and the charge and statistics of the vortices are all determined directly by K^{-1}. Let us now continue the general discussion in Sec. IV, where we showed that given the matrices K_s, $s=1,2,\ldots,W$, we can construct the matrix $K=K_0+pT$. Because T is a simple projection, we can, in fact, invert K explicitly. We note that

$$(TK_0^{-1})(TK_0^{-1})=\sigma_T(TK_0^{-1}) \ , \tag{7.1}$$

where we defined

$$\sigma_T \equiv \mathrm{tr}(TK_0^{-1})=\sum_{C,D} t_C K^{-1}_{0CD} t_D = \mathbf{t}\cdot\mathbf{w} \ , \tag{7.2}$$

which we recognize from (2.6) as $\sum_s \sigma_s$, the sum of the conductances of the component fluids. Here

$$\omega_C = \sum_D K^{-1}_{0CD} t_D \tag{7.3}$$

[in some sense the dual of the charge vector t_D, and, as the astute reader would recognize from (2.7), the charge of the "elementary" vortices in each of the component fluids]. Using (7.1) we see that the inverse of K is given by

$$K^{-1} = K_0^{-1} - \left[\frac{p}{1+p\sigma_T}\right] K_0^{-1} T K_0^{-1} , \tag{7.4}$$

or, more explicitly,

$$K^{-1}_{CD} = K^{-1}_{0CD} - \left[\frac{p}{1+p\sigma_T}\right] \omega_C \omega_D . \tag{7.5}$$

Given (7.4) we can now compute various physical properties of the composite fluid. Using (2.6) and (7.1) we obtain the conductance

$$\begin{aligned}\sigma &= \sum t_C K^{-1}_{CD} t_D \\ &= \mathrm{tr} T K^{-1} \\ &= \sigma_T - \left[\frac{p}{1+p\sigma_T}\right] \sigma_T^2 \\ &= \frac{1}{p + (1/\sum_s \sigma_s)} = \sigma_T \left\{\frac{1}{1+p\sigma_T}\right\} .\end{aligned} \tag{7.6}$$

The charge of a conglomerate of vortices made up to l_D vortices of the Dth type comes out to be

$$\begin{aligned}q &= \sum t_C K^{-1}_{CD} l_D \\ &= \sum t_C K^{-1}_{0CD} l_D - \left[\frac{p}{1+p\sigma_T}\right] (\mathbf{t}\cdot\mathbf{w})(\boldsymbol{l}\cdot\mathbf{w}) \\ &= \left[\sum_s q_s(l)\right] \left\{\frac{1}{1+p\sigma_T}\right\} .\end{aligned} \tag{7.7}$$

We have recognized

$$\boldsymbol{l}\cdot\mathbf{w} = \sum t_C K^{-1}_{0CD} l_D = \sum_s q_s(l) \tag{7.8}$$

to be the sum of the charges of the vortices in each of the component fluids. Finally, the statistics of this conglomerate of vortices may be computed:

$$\begin{aligned}\frac{\theta}{\pi} &= \sum l_c K^{-1}_{CD} l_D \\ &= \sum l_C K^{-1}_{0CD} l_D - \left[\frac{p}{1+p\sigma_T}\right] (\boldsymbol{l}\cdot\mathbf{w})^2 \\ &= \frac{1}{\pi} \sum_s \theta_s(l) - \left[\frac{p}{1+p\sigma_T}\right] \left[\sum_s q_s^{(l)}\right]^2 \\ &= \frac{1}{\pi} \sum_s \theta_s(l) - p(1+p\sigma_T) q^2 .\end{aligned} \tag{7.9}$$

In deriving (7.7) and (7.9) we have used the fact that K_0^{-1} is block diagonal.

These three rather compact formulas (7.6), (7.7), and (7.9) tell us how to sum conductance, charge, and statistics when we put topological fluids together. In the absence of interactions, that is, with $p=0$, we see that the conductance, charge, and statistics of the composite fluid are just equal to the sum of the conductance, charge, and statistics of the component fluids, of course. In (7.6) and (7.7), the effect of the interaction is clearly summarized by a multiplicative factor indicated by the quantity in the curly brackets. Notice that statistics, being a binary or quadratic quantity in contrast to charge, is corrected by the interaction with a term proportional to the square of the total charge.

It is useful to have these formulas in a basis-independent form and written for arbitrary t. In particular, in the basis in which only $t_1 = 1$ is nonzero we have $\nu = \sigma = K^{-1}_{11}$. For example, for the K given in (6.16) we see immediately that ν is given by the ratio of two quantities of the same form, namely the determinant of K with its first row and first column deleted, and the determinant of K itself.[5] We obtain (5.17) readily.

It is also useful to specialize these general formulas [(7.6), (7.7), and (7.9)] to our two elementary iterative steps A and B and with the t_I's all taken to be $=1$. In step A, with

$$K_1 \oplus K_2 \to K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} , \tag{5.2}$$

we have, trivially,

$$\nu = \nu_1 + \nu_2, \quad q = q_1 + q_2 , \quad \frac{\theta}{\pi} = \frac{\theta_1}{\pi} + \frac{\theta_2}{\pi} . \tag{7.10}$$

In step B, with

$$K \to K + C , \tag{5.3}$$

we have

$$\nu \to \frac{\nu}{1+\nu}, \quad q \to \frac{q}{1+\nu} , \quad \frac{\theta}{\pi} \to \frac{\theta}{\pi} - \frac{q^2}{1+\nu} . \tag{7.11}$$

It is interesting to note what relationships between q, θ, and ν are preserved by these steps. Thus, for example, if we have excitations whose charge q and statistics θ/π are equal to the filling factor ν, this is preserved by step A (trivially) and by step B. In particular, since the hierarchy states may be reached using steps A and B, we conclude that in the hierarchy state there is always at least one excitation whose charge q and statistics θ/π are equal to the filling factor ν. As another example, we note that step B also preserves the relationship $\theta/\pi = q^2/\nu$ (mod 2).

Using (7.10) and (7.11) and referring to (5.16), we can easily write down how the charges and statistics of excitations in the hierarchy construction iterate:

$$q = -\nu(q_K - 1 - \nu_K) \tag{7.12}$$

and

$$\frac{\theta}{\pi} = \frac{\theta_K}{\pi} - 1 + [\nu^{-1}(p+1)q^2 - q_K^2]/(1+\nu_K) . \tag{7.13}$$

VIII. WAVE FUNCTION

Thus far, we have studied the internal topological structures in the Abelian FQH using the effective Chern-Simons theory. While one advantage of the effective theory approach is that we do not have to deal with specific variation wave functions, it is important for certain purposes to study actual wave functions. In the following we will discuss the representative wave functions of the universality classes of the FQH states labeled by the matrices K.

The representative wave function for the FQH state labeled by K can be constructed using the generalized hierarchical construction discussed in Refs. 5 and 11. Here we will present a simpler and more transparent wave function. We simply ask what the two steps, A and B, in our matrix iterative construction correspond to in the language of wave functions. In combining different topological fluids, we may think of the electrons in the different fluids as carrying different "flavors." In the end, of course, we may wish to antisymmetrize in all the electron coordinates. In reality, the flavor quantum numbers may arise from different subbands, spins, and Landau levels.

Let ψ_K be the wave function of the FQH states labeled by K. Then corresponding to step A the electron wave function described by the "sum" of K_1 and K_2,

$$K_1 \oplus K_2 = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$$

is given by

$$\psi_{K_1 \oplus K_2}(z_i) = \psi_{K_1}(z_i^{(1)}) \psi_{K_2}(z_i^{(2)}) , \tag{8.1}$$

where the coordinate $\{z_i\}$ is the union of $\{z_i^{(1)}\}$ and $\{z_i^{(2)}\}$. We have assumed that the electrons in ψ_{K_1} and ψ_{K_2} carry different flavors so that we do not antisymmetrize between $z_i^{(1)}$ and $z_i^{(2)}$.

The operation $K \to K + pC$ corresponds to adding p unit of flux to the electrons. Therefore

$$\psi_{K+pC}(z_i) = \left[\prod_{i<j} (z_i - z_j)^p \right] \psi_K(z_i) . \tag{8.2}$$

In the state labeled by $K + pC$, when an electron z_i moves around z_j, the wave function acquires an additional phase $e^{i2\pi p}$ compared to what would have been the case were we discussing the state labeled by K.

The two operations A and B in (5.2) and (5.3) can generate a rather general set of matrices starting from the simple Laughlin states. From (8.1) and (8.2) we see that the wave function of the FQH state described by K can be written in compact form:

$$\begin{aligned} \psi_K(z_i^{(I)}) = & \left[\prod_{I=1}^{\kappa} \prod_{i<j}^{N_I} (z_i^{(I)} - z_j^{(I)})^{K_{II}} \right] \\ & \times \left[\prod_{I<J}^{\kappa} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} (z_i^{(I)} - z_j^{(J)})^{K_{IJ}} \right] \\ & \times e^{(1/4)\sum_{I,i} |z_i^{(I)}|^2} . \end{aligned} \tag{8.3}$$

Note the electrons in (8.3) have κ different flavors. For instance, the wave function for any 2×2 matrix

$$\begin{aligned} K &= \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \\ &= \begin{bmatrix} K_{11} - K_{12} & 0 \\ 0 & K_{22} - K_{12} \end{bmatrix} + K_{12} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned} \tag{8.4}$$

is constructed by combining (8.1) and (8.2). For general κ, we can verify that the quantum numbers of the excitations derived from the wave function in (8.3) agree with the quantum numbers determined directly from K.

The FQH state described by K can be viewed as containing κ components of incompressible fluid described by the Laughlin wave function $\prod_{i<j}(z_i^{(I)} - z_j^{(I)})^{K_{II}}$. The off-diagonal elements of K describe the coupling between the different fluids. The coupling is of the charge flux coupling type; i.e., the electrons in the Ith fluid behave like flux tubes with K_{IJ} units of flux to the electrons in the Jth fluids. When $z_i^{(I)}$ moves around $z_j^{(I)}$ the wave function acquires the phase factor $e^{i2\pi K_{IJ}}$. This gives us a simple and direct picture of the internal structure of the general FQH states.

After presenting the above simple picture, we would like to make a few remarks. In the effective theory the number of electrons in each fluid is conserved independently. However, in reality the electrons are allowed to hop from one fluid to another (e.g., transitions between different Landau levels are allowed). Although the multifluid picture is a convenient way to understand the structures in the FQH states, experimentally we may not be able to see a clear-cut signal of many distinct collective modes due to different fluids. But we believe that the concepts of the topological order and of the use of the matrix K to label different topological orders are rigorous. Several physical measurements and characterizations of K are proposed in Ref. 3 (for a review see Ref. 17) by using ground-state degeneracy and edge excitations.

The FQH states of identical electrons are also labeled by K. In this case electrons have only one flavor, of course. It is tempting to obtain the wave function labeled by K by antisymmetrizing $\psi_K(z_i^{(I)})$ in (8.3). But in some cases, upon antisymmetrization, the resulting wave function may vanish. This happens when $K = K_0$ or $K = K_0 + 2pC$, where K_0 has no negative elements and $\sum_{I,J}(K_0^{-1})_{IJ} > 1$ (filling fraction larger than 1). For example, we cannot construct the $\frac{2}{5}$ FQH state by antisymmetrizing the wave function (8.3) with

$$K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

because the wave function vanishes upon antisymmetrization. To obtain the wave function of the $\frac{2}{5}$ FQH state, we have to put some electrons in the second Landau level. We start with

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

which corresponds to two filled Landau levels with a wave function we denote by $\chi_2(z_i)$. The matrix of the $\frac{2}{5}$ FQH state

$$K=\begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix}$$

is obtained by adding $2C$ to

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

This leads to the Jain wave function $\prod_{i<j}(z_i-z_j)^2\chi_2(z_i)$ for the $\frac{2}{5}$ FQH state.

The wave functions in (8.3) are relevant to multilayered electron systems,[18,19] for which we identify the flavor index as the layer index. Our approach, developed in this paper, provides a simple way to calculate the quantum numbers of the quasiparticle excitations. For a two-layered system, the simplest wave functions are given by (8.3) and (8.4) with

$$K=\begin{bmatrix}p & l\\ l & p\end{bmatrix}$$

(where we assume the two layers to be identical, so $K_{11}=K_{22}$). Such a class of states has filling fractions $\nu=2/p+l$. A series of stable filling fractions with $\nu_{i+1}^{-1}-\nu_i^{-1}=\frac{1}{2}$ was observed in Ref. 19. This series may be described by the construction given here.

IX. SHIFT

Following Haldane[20] we will now study FQH states on a sphere. Let us recall briefly some basic facts. A magnetic monopole is placed at the center of the sphere: by the Dirac quantization condition, the number of flux quanta N_ϕ passing through the sphere must be an integer. To describe an electron moving on the sphere, we use spinor coordinates $u=\cos\theta/2e^{i\phi/2}$, $v=\sin\theta/2e^{-i\phi/2}$; the electron wave function is given in terms of the $(2N_\phi+1)$ basis functions $u^{N_\phi},u^{N_\phi-1}v,\ldots,v^{N_\phi}$. In order to put a FQH state on a sphere, the number of electrons N_e and the number of the flux quanta N_ϕ passing through the sphere must be related in a specific way.

$$N_\phi=\frac{1}{\nu}N_e-S\ . \tag{9.1}$$

S is another quantum number that characterizes the internal structures of the FQH states. We refer to S as the shift.

For the simplest $\nu=1/m$ Laughlin states $\psi\propto\prod(z_i-z_j)^m$, the corresponding wave function on the sphere is obtained by replacing (z_i-z_j) with $(u_iv_j-u_iv_j)$ and $(\xi_i-\xi_j)$ with $(\bar{u}_i\bar{v}_j-\bar{v}_i\bar{u}_j)$,

$$\psi\propto\prod_{i<j}^{N}(u_iv_j-v_iu_j)^m\ . \tag{9.2}$$

To determine N_ϕ, we simply find the combined powers with which u_1 and v_1 appear in the wave function. Clearly,

$$N_\phi=m(N-1)\ . \tag{9.3}$$

(For instance, ψ contains a term proportional to $u_1^{m(N-1)}v_1^0$.) Comparing with (9.1), we see that the shift S is equal to m.

As another example, we find that for the $\frac{2}{5}$ FQH state obtained from the standard hierarchical construction and the Jain construction, S is equal to 4. To obtain this result, it is convenient to use the hierarchical wave functions in Refs. 11 and 5. The hierarchical wave functions $\psi_{K^h}(z_i)$ corresponding to the matrix

$$K^h=\begin{bmatrix}p_1 & l\\ l & p_2\end{bmatrix}$$

are given by

$$\psi_{K^h}(z_j)=\int\left[\prod_{j=1}^{N_\xi}d^2\xi_j\right]\left[\prod_{i<j}^{N_e}(z_i-z_j)^{p_1}\right]\times\left[\prod_{i<j}^{N_\xi}(\xi_i-\xi_j)^{p_2}\right]\left[\prod_{i=1}^{N_e}\prod_{j=1}^{N_\xi}(z_i-\xi_j)^l\right]F\ , \tag{9.4}$$

where F is a positive function depending on $e^{-(1/4)|z_i|^2}$, $|\xi_i-\xi_j|$, etc. The detailed form of F does not affect the topological orders in the FQH states. Note here that K^h is the matrix in the hierarchical basis with $t_I=\delta_{I1}$. Thus p_1 is odd and p_2 is even. The wave function on the sphere has the form

$$\psi_{K^h}(u_i,v_j)=\int\left[\prod_{i=1}^{N_\xi}d\bar{u}_id\bar{v}_i\right]\left[\prod_{i<j}^{N_e}(u_iv_j-v_iu_j)^{p_1}\right]\times\left[\prod_{i<j}^{N_\xi}(\bar{u}_i\bar{v}_j-\bar{v}_i\bar{u}_j)^{p_2}\right]\times\left[\prod_{i=1}^{N_e}\prod_{j=1}^{N_\xi}(u_i\bar{v}_j-v_i\bar{u}_j)^l\right]F\ . \tag{9.5}$$

We have replaced $(\xi_i-\xi_j)$ by $(\bar{u}_i\bar{v}_j-\bar{v}_i\bar{u}_j)$, etc. In order for the integral to be nonzero, we require the total power of $(\bar{u}_i,\bar{v}_i)$ to be zero for each i:

$$p_2(N_\xi-1)+lN_e=0\ . \tag{9.6}$$

According to what we said above, the number of the flux quanta N_ϕ is the combined power of (u_i,v_i) (for fixed i):

$$N_\phi=p_1(N_e-1)+lN_\xi\ . \tag{9.7}$$

From (8.6) and (8.7) we determine the shift S:

$$S=p_1-l\ . \tag{9.8}$$

The $\frac{2}{5}$ FQH state is described by

$$K^h=\begin{bmatrix}3 & -1\\ -1 & 2\end{bmatrix}$$

in the hierarchical basis and thus we find $S=4$.

Here we would like to remark that $S=p_1-l=K_{11}-K_{12}$ is not invariant under the transforms

$K \to W^T K W$ with $W \in \mathrm{SL}(2,\mathbb{Z})$. This impiles that the FQH states described by equivalent effective Chern-Simons theories may have different shifts S. This result is expected, since, as we will now explain, the shift depends on short-distance physics not incorporated into our long-distance effective theories. Consider the $\nu=\frac{1}{2}$ FQH state formed by the electron pairs. The effective Chern-Simons theory for such a state is identical regardless of the internal angular momentum L of the electron pair. However, it is known that the shift depends on the angular momentum[21] according to

$$S=4-L \ . \tag{9.9}$$

From the above example we see that, in general, knowing K is not sufficient to determine the shift S.

For general K^h, we find the generalizations of (9.6) and (9.7) to be

$$\begin{bmatrix} N_\phi + K^h_{11} \\ K^h_{22} \\ \vdots \\ K^h_{KK} \end{bmatrix} = K^h \begin{bmatrix} N_e \\ N^{(2)}_\xi \\ \vdots \\ N^{(K)}_\xi \end{bmatrix} \ . \tag{9.10}$$

This may be derived from the appropriate generalization of (9.5). Solving (9.10), we obtain the shift for hierarchical wave functions:

$$S=\frac{1}{\nu}\sum_I [(K^h)^{-1}]_{1I} K^h_{II} \ , \tag{9.11}$$

where $\nu=[(K^h)^{-1}]_{11}$ is the filling fraction.

We can also try to put the wave function in (8.3) on the sphere:

$$\psi \sim \left[\prod_{I=1}^{\kappa} \prod_{i<j}^{N_I} (u_i^{(I)} v_j^{(I)} - v_i^{(I)} u_j^{(I)})^{K_{II}} \right] \times \left[\prod_{I<J}^{\kappa} \prod_{i=1}^{N_I} \prod_{j=1}^{N_J} (u_i^{(I)} v_j^{(J)} - v_i^{(I)} u_j^{(J)})^{K_{IJ}} \right] \ . \tag{9.12}$$

To extract the number of flux quanta N_ϕ, we simply examine the combined power of $(u_i^{(I)}, v_i^{(I)})$ as explained earlier:

$$N_\phi = K_{II}(N_I - 1) + \sum_{J \neq I} K_{IJ} N_J \ . \tag{9.13}$$

Solving the κ equations contained in (9.13), we find

$$N_I = \sum_J K^{-1}_{IJ} (N_\phi + K_{JJ}) \ . \tag{9.14}$$

This may be interpreted as the number of the Ith kind of electrons. The total number of electrons is thus

$$N_e = \sum_I N_I = \sum_{I,J} K^{-1}_{IJ} (N_\phi + K_{JJ}) \ . \tag{9.15}$$

Comparing with (9.1), we see that the shift is given by

$$S=\frac{1}{\nu}\left[\sum_{I,J} (K^{-1})_{IJ} K_{JJ}\right] \ , \tag{9.16}$$

where, as always, $\nu = \sum_{I,J} K^{-1}_{IJ}$.

In general, (9.11) and (9.16) give different shifts even when K and K^h are related by the equivalence transformations (6.13). Depending on the short-distance physics, the wave function corresponding to K^h may be more appropriate than the wave function corresponding to K.

Using (9.16) we can verify that under step A of our matrix iteration scheme we have, obviously,

$$S \to (\nu_1 S_1 + \nu_2 S_2)/(\nu_1 + \nu_2) \ , \tag{9.17}$$

as is also clear from the definition (9.1) of S. Under step A we have

$$N_\phi = \frac{1}{\nu_I} N^I_e - S_I \tag{9.18}$$

for $I=1$ and 2. Using $N_e = N^1_e + N^2_e$ and combining the two equations in (9.18) we obtain (9.17) immediately. Under step B, we have, less obviously,

$$S \to S+1 \ . \tag{9.19}$$

This is evidently consistent with, for instance, the result that for the simple Laughline states $S=\nu^{-1}$. We may also derived (9.19) immediately by inspecting the explicit wave-function construction given in (8.2).

X. CONCLUSION

We have presented a simple and unified description of Abelian fractional quantum Hall states. In particular, we have answered the question raised in the beginning of this paper, namely the question of how to characterize and classify the topological orders and universality classes of these states. The answer is that they are labeled by the symmetric matrix K. The matrix K incorporates the totality of physical information contained in the long-distance physics of these topological fluids.

In particular, the conductance (or filling factor), the charge and statistics of the excitations, and the degeneracy of the ground state (on topologically nontrivial closed spaces) are all determined by K. Up to this point in this paper we have not mentioned the edge excitations[22] in a Hall fluid confined to a finite area. The number of branches is obviously given by the rank or dimension of K.

In discussions of the Hall effect, people still talk as if the filling factor ν alone characterizes the quantum Hall state. As is made clear by our formalism and as is already well known to some, this is certainly not the case: many K matrices may lead to the same ν.

As an example, let us consider what is perhaps the best-known state of the all, the $\nu=\frac{1}{3}$ state. The simplest such state is that given by $K=3$. Now consider the states defined by 2×2 matrices

$$K = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 11 & 7 \\ 7 & 5 \end{bmatrix}, \ldots \ . \tag{10.1}$$

The reader can easily verify that these states are all $\nu=\frac{1}{3}$ states and contain an excitation with charge $q=\frac{1}{3}$ and statistics $\theta/\pi=\frac{1}{3}$. However, these states are certainly not equivalent to the simple $K=3$ state. They have two branches of edge excitations rather than one.

Consider also

$$K=\begin{bmatrix}4 & 3\\ 3 & 3\end{bmatrix},\ \begin{bmatrix}7 & 5\\ 5 & 4\end{bmatrix},\ \ldots, \tag{10.2}$$

also corresponding to $\nu=\frac{1}{3}$ states. These states, however, are not reachable in the hierarchical construction. In particular, they do not satisfy (6.3) and thus do not contain two electron operators.

This example also illustrates that, in general, it is not instantly obvious whether two states are equivalent or not. In fact, the two K matrices displayed above are part of an infinite series defined by

$$K=\begin{bmatrix}r^2+2r+4 & r^2+r+3\\ r^2+r+3 & r^2+3\end{bmatrix},\quad r=0,1,2,\ldots, \tag{10.3}$$

but it is easy to verify that all these matrices are equivalent in the sense defined earlier, namely up to W^TKS with (1 1) $W=(1\ 1)$. Note also that while all the states defined by the K matrices displayed above have the same filling factor, excitations with the same charge and statistics, and the same number of edge excitation edge branches, they are not all equivalent, as we can see readily by taking their determinants and using (1.2). The ground-state degeneracy for two of the states displayed above is 3^g, while for the other two states it is 6^g. This example makes particularly clear the topological character of these fluids.

Another example is given by the $\nu=\frac{1}{2}$ state mentioned at the end of Sec. V. Using (2.7) and (2.8) we find the charge $q=(l_1+l_2)/4$ and $\theta/\pi=[3(l_1^2+l_2^2)-2l_1l_2]/8$. Thus, this $\nu=\frac{1}{2}$ state contains an excitation of charge $\frac{1}{4}$ and statistics $\theta/\pi=\frac{3}{8}$. This state is thus definitely not the same as the paired electron $\nu=\frac{1}{2}$ state mentioned in Sec. II. Furthermore, one of these $\nu=\frac{1}{2}$ states has two branches of edge excitations while the other has only one. Another $\nu=\frac{1}{2}$ state with two branches of edge excitations is described by

$$K=\begin{bmatrix}2 & 2\\ 2 & 3\end{bmatrix}$$

[this is an example of a state reachable by iterating with steps A and B but not satisfying (6.3)]. This state, however, does not contain a charge-$\frac{1}{4}$ excitation.

We have also determined the topological quantity we refer to as the shift S. It depends on short-distance physics as well as the physics contained in K.

In summary, we have shown that the topological order in the Abelian FQH states can be characterized by a symmetric matrix K. We give a matrix iterative construction for K. The iteration may be decomposed into two extremely elementary steps. In particular, the hierarchy construction is contained within this matrix iterative construction. The relation between the vortex basis and the dual electron basis is clarified. The matrix K corresponding to the generalized hierarchy construction is identified. Finally, the topological quantity known as the shift is studied.

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