Spectral functions for the Tomonaga-Luttinger model

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Results for the one-particle Green's function for the Tomonaga-Luttinger model of one-dimensional interacting fermions take a simple form using space and time variables. To obtain the corresponding spectral functions which determine photoemission and inverse-photoemission spectra, a double Fourier transform is necessary. For the model including spin we present analytical and numerical results. They show drastic differences from the spinless case. Expressions for the critical indices which determine the singularities of the spectra are given.

I. INTRODUCTION

Long before any experimental realization of quasi-one-dimensional metals, Tomonaga[1] made a pioneering step towards the theoretical understanding of one-dimensional interacting fermions. In contrast to three-dimensional systems which have an excitation spectrum consisting of single-particle excitations as well as of collective modes, there are only collective long-wavelength low-energy modes in the one-dimensional (1D) case, which approximately behave as bosons. To obtain his results, Tomonaga linearized the energy dispersion around the two Fermi points $\pm k_f$. In the Luttinger model,[2] which is closely related to the massless Thirring model,[3] an exactly linear dispersion is assumed. An exact solution for the Luttinger model was presented by Mattis and Lieb.[4] The original Tomonaga model and Luttinger model were compared by Gutfreund and Schick,[5] who showed that the low-energy physics in both models is the same for long-range interactions with a rather weak restriction on the interaction strength. As we are interested in the low-energy spectral functions of 1D interacting electrons, we mostly refer to the model as the Tomonaga-Luttinger (TL) model in the following.

The one-particle Green's function for the TL model can be calculated exactly. Three different approaches were proposed: the use of Ward identities,[6-8] the generalization of the method of Mattis and Lieb[4] to calculate time-independent correlation functions,[9,10] and the elegant method[12] which uses the bosonization of the fermion field operators.[11-16] Apart from cutoff problems which were treated on different levels of sophistication, all approaches yield explicit results for the Green's function in space and time variables, when simplifying assumptions about the form of the interaction are made.[12]

Originally the model was treated for spinless fermions. The inclusion of spin still leads to an exactly solvable model as spin and charge degrees decouple completely. The calculation of the one-particle Green's function is then easily generalized to the model including spin.

To obtain the spectral functions, which determine the photoemission and inverse-photoemission spectra, a double Fourier transform is necessary. For the spinless model this can be performed quite easily using the simplified form of the interaction.[9,11,12,17] This is shortly reviewed in Sec. II. For the model including spin, the one-particle Green's function $G(x,t)$ has a more complicated structure and no results for the physically interesting spectral functions have been published so far. In Sec. III we generalize the methods used for the spinless model to obtain general analytical results for the critical indices which determine the singularities of the spectra for the model with spin. For the case of spin-independent interaction exact numerical results for the complete low-energy spectra are presented. The details of how one of the integrations is performed analytically are given in the Appendix.

II. SPINLESS LUTTINGER MODEL

To clarify our notation we briefly review the Luttinger model for spinless fermions on a line of length L. Assuming periodic boundary conditions the field operators $\hat{\psi}_\alpha(x)$ with $\alpha=+\ (-)$ for right- (left-) moving particles can be decomposed as

$$\hat{\psi}_\alpha(x)=\frac{1}{\sqrt{L}}\sum_k e^{ikx}\hat{a}_{k,\alpha}\ , \tag{1}$$

where $\hat{a}_{k,\alpha}$ is the annihilation operator for a particle of type α with momentum $k=2\pi n/L$, $n\in\mathbb{Z}$, and kinetic energy $\alpha v_f k$. The density operators $\hat{\rho}_\alpha(x)\equiv\hat{\psi}^\dagger_\alpha(x)\hat{\psi}_\alpha(x)$ are expanded in a Fourier series,

$$\hat{\rho}_\alpha(x)=\frac{1}{L}\sum_k \hat{\rho}_{k,\alpha}e^{ikx}\ , \tag{2}$$

$$\hat{\rho}_{k,\alpha}=\int_0^L \hat{\rho}_\alpha(x)e^{-ikx}dx\ . \tag{3}$$

Using Eq. (1) this yields[18]

$$\hat{\rho}_{q,\alpha}=\sum_k \hat{a}^\dagger_{k,\alpha}\hat{a}_{k+q,\alpha}\ . \tag{4}$$

With proper normalization the $\hat{\rho}_{q,\alpha}$ obey Bose commutation relations. If one defines

$$\hat{b}_q\equiv\left[\frac{2\pi}{|q|L}\right]^{1/2}\begin{cases}\hat{\rho}_{q,+} & \text{for } q>0\\ \hat{\rho}_{q,-} & \text{for } q<0\end{cases} \tag{5}$$

the commutation relations read as[4]

$$[\hat{b}_q,\hat{b}^\dagger_{q'}]=\delta_{q,q'},\quad [\hat{b}_q,\hat{b}_{q'}]=0\ . \tag{6}$$

The basic clue to the solution of the interacting model lies in the fact that (up to an infinite constant) the kinetic energy can be expressed in terms of the Bose operators and (normal ordered) particle number operators $\hat{B}_\alpha$ for the right- and left-moving particles,[4] which commute with $\hat{b}_q$ and $\hat{b}^+_q$ as

$$\hat{H}_0=v_f\left\{\sum_{q\neq 0}|q|\hat{b}^\dagger_q\hat{b}_q+\frac{\pi}{L}\sum_{\alpha=\pm}(\hat{B}^2_\alpha-\hat{B}_\alpha)\right\}. \tag{7}$$

The normal ordering is with respect to a state where all one-particle states with negative energy are filled. The interaction term can naturally be expressed in terms of the density fluctuation operators as well as the $\hat{B}_\alpha$. As the term involving the particle number operators has no influence on the dynamics, but only enters the value of the chemical potential μ, we drop it and take as the interaction

$$\hat{V}=\tfrac{1}{2}\sum_{q\neq 0}|q|\frac{v(q)}{2\pi}(\hat{b}^\dagger_q+\hat{b}_{-q})(\hat{b}_q+\hat{b}^\dagger_{-q})\ , \tag{8}$$

where $v(q)$ is the Fourier transform of the two-body potential, e.g., a screened Coulomb potential. As shown by Mattis and Lieb[4] the total Hamiltonian $\hat{H}=\hat{H}_0+\hat{V}$ can be diagonalized by new boson operators $\hat{\alpha}_q\equiv\cosh\Theta_q\hat{b}_q-\sinh\Theta_q\hat{b}^\dagger_{-q}$,

$$\hat{H}=\sum_{q\neq 0}|q|\tilde{v}_f(q)\hat{\alpha}^\dagger_q\hat{\alpha}_q+\hat{H}_2(\{\hat{B}_\alpha\}) \tag{9}$$

with $\tilde{v}_f(q)\equiv v_f\sqrt{1+v(q)/(\pi v_f)}$ and Θ_q determined by the equation

$$\tanh(2\Theta_q)\equiv-\frac{v(q)}{v(q)+2\pi v_f}\ . \tag{10}$$

$\hat{H}_2(\{\hat{B}_\alpha\})$ contains all terms with particle number operators. To calculate the $2n$ point functions, one can use the bosonization of the fermion field operators $\hat{\psi}_\alpha(x)$.[12–16] We use the procedure by Haldane[16] that properly incorporates an operator $\hat{U}_\alpha$ which applied to the noninteracting ground state produces the noninteracting ground state with one additional particle of type α and commutes with the $\hat{b}_q$ and $\hat{b}^\dagger_q$,

$$\hat{\psi}^\dagger_\alpha(x)=\frac{e^{i\alpha(\pi/L)x}}{\sqrt{L}}e^{-i\hat{\phi}^\dagger_\alpha(x)}\hat{U}_\alpha e^{-i\hat{\phi}_\alpha(x)} \tag{11}$$

with

$$\hat{\phi}_\alpha(x)=\alpha\frac{\pi}{L}\hat{B}_\alpha x-i\sum_{q\neq 0}\Theta(\alpha q)e^{iqx}\left[\frac{2\pi}{L|q|}\right]^{1/2}\hat{b}_q\ . \tag{12}$$

If one defines

$$iG^>_\alpha(x,t)\equiv\langle\hat{\psi}_\alpha(x,t)\hat{\psi}^\dagger_\alpha(0,0)\rangle\ , \tag{13}$$

$$iG^<_\alpha(x,t)\equiv\langle\hat{\psi}^\dagger_\alpha(0,0)\hat{\psi}_\alpha(x,t)\rangle\ , \tag{14}$$

the calculation of the right-hand side (rhs) of Eqs. (13) and (14) is simple if one expresses $\hat{b}_q(t)$ in terms of $\hat{\alpha}_q(t)$ and $\hat{\alpha}^\dagger_{-q}(t)$ and uses the Baker-Hausdorff formula,

$$iG^{\gtrless}_\alpha(x,t)e^{i\mu t}=\frac{1}{L}e^{i\alpha k_f x}\exp\left\{\frac{2\pi}{L}\sum_{q>0}\frac{1}{q}[e^{\pm i\alpha qx}e^{\mp i\omega_q t}+2s^2(q)[\cos(qx)e^{\mp i\omega_q t}-1]]\right\}, \tag{15}$$

where $s^2(q)\equiv\sinh^2(\Theta_q)$ and $\omega_q\equiv|q|\tilde{v}_f(q)$. Following Luther and Peschel,[12] we assume a potential that leads to

$$\sinh^2(\Theta_q)=\gamma e^{-r|q|}\ . \tag{16}$$

In the large x and t limit, one obtains, after taking the limit $L\to\infty$ with $\tilde{v}_f\equiv\tilde{v}_f(q=0)$,

$$iG^{\gtrless}_\alpha(x,t)e^{i\mu t}=\underset{(-)}{+}\frac{i\alpha}{2\pi}\frac{e^{i\alpha k_f x}}{x-\alpha\tilde{v}_f t\underset{(-)}{+}i\alpha 0}\left[\frac{r^2}{(x-\tilde{v}_f t\underset{(-)}{+}ir)(x+\tilde{v}_f t\underset{(+)}{-}ir)}\right]^\gamma. \tag{17}$$

This is the basic result for the spinless model mentioned in the Introduction. It agrees with expressions given by Luther and Peschel[12] who used a not quite correct form of the bosonized field operator. This is not surprising as the proper inclusion of $\hat{U}_\alpha$ only modifies the value of the chemical potential, which in the following we take as our reference energy.

To understand the physical content of Eq. (17) we define the spectral functions

$$\rho^>_\alpha(k,\omega)\equiv\langle\phi^N_0|\hat{a}_{k,\alpha}\delta[\omega-(\hat{H}-E^{N+1}_0)]\hat{a}^\dagger_{k,\alpha}|\phi^N_0\rangle =\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int_{-\infty}^{\infty}dx\,e^{-ikx}e^{i\mu t}iG^>_\alpha(x,t)\ , \tag{18}$$

$$\rho^<_\alpha(k,\omega)\equiv\langle\phi^N_0|\hat{a}^\dagger_{k,\alpha}\delta[\omega+(\hat{H}-E^{N-1}_0)]\hat{a}_{k,\alpha}|\phi^N_0\rangle =\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int_{-\infty}^{\infty}dx\,e^{-ikx}e^{i\mu t}iG^<_\alpha(x,t)\ . \tag{19}$$

This agrees with the usual definition if the chemical potential is put to zero. As we have calculated $G^>_\alpha$ and $G^<_\alpha$ in the large x and t limit and as $G^{\gtrless}_\alpha$ is proportional to $\exp(i\alpha k_f x)$, our results for the spectral function will be exact only in the limit $k\approx\alpha k_f$ and $\omega\approx 0$. In the following we evaluate the functions only for right movers ($\alpha=+$) and define $\tilde{k}\equiv k-k_f$. As we have the relation

$$\rho^>_+(k_f+\tilde{k},\omega)=\rho^<_+(k_f-\tilde{k},-\omega)\ , \tag{20}$$

it is sufficient to calculate $\rho_+^>$. Using (17) and (18) and the variable substitutions $s=x-\tilde{v}_f t$, $s'=x+\tilde{v}_f t$ we obtain

$$\rho_+^>(k_f+\tilde{k},\omega)=\frac{1}{2\tilde{v}_f}\int_{-\infty}^{\infty}\frac{ds'}{2\pi}\frac{e^{iu_-s'}r^\gamma}{(s'-ir)^\gamma}\times\int_{-\infty}^{\infty}\frac{ids}{2\pi}\frac{e^{-iu_+s}r^\gamma}{(s+i0)(s+ir)^\gamma}\equiv\frac{1}{2\tilde{v}_f}I_1(u_-)I_2(u_+) \tag{21}$$

with $u_{\substack{+\\(-)}}\equiv(\omega_{\substack{+\\(-)}}\tilde{v}_f k)/(2\tilde{v}_f)$. As the first (second) integrand is analytic in the lower (upper) complex integration plane $\rho_+^>(k_f+\tilde{k},\omega)$ vanishes for $\omega<\tilde{v}_f|\tilde{k}|$. The u_- dependence of the first integral is simple and follows from the substitution $\tilde{s}=s'-ir$,

$$I_1(u_-)=\Theta(u_-)u_-^{\gamma-1}e^{-ru_-}\frac{i^\gamma}{\Gamma(\gamma)}\ . \tag{22}$$

The second integral can be expressed in terms of the confluent hypergeometric function $\Phi(a,b,z)$,[19]

$$I_2(u_+)=\Theta(u_+)u_+^\gamma e^{-ru_+}\frac{(-i)^\gamma}{\Gamma(\gamma+1)}\Phi[1,1+\gamma,(r-0)u_+]. \tag{23}$$

We therefore obtain the final result

$$\rho_+^>(k_f+\tilde{k},\omega)=\Theta(\omega-\tilde{v}_f|\tilde{k}|)\left[\frac{r}{2\tilde{v}_f}\right]^{2\gamma}\times\frac{\gamma}{[\Gamma(1+\gamma)]^2}(\omega-\tilde{v}_f\tilde{k})^{\gamma-1}\times(\omega+\tilde{v}_f\tilde{k})^\gamma e^{-r\omega/\tilde{v}_f}\times\Phi\left[1,1+\gamma,\frac{r}{2}\left[\frac{\omega}{\tilde{v}_f}+\tilde{k}\right]\right]. \tag{24}$$

As the confluent hypergeometric function $\Phi(a,b,z)$ is an entire function of z, the factor Φ ( ) in the above equation is irrelevant for the singular behavior of $\rho_+^>(k_f+\tilde{k},\omega)$. In the limit when both ω and $\tilde{k}$ tend to zero it can be replaced by $\Phi(a,b,0)=1$. For $0<\gamma<1$ and $\tilde{k}>0$, $\rho_+^>$ has an algebraic singularity $(\omega-\tilde{k}\tilde{v}_f)^{\gamma-1}$ when ω approaches $\tilde{v}_f\tilde{k}$ from above. This is a weaker singularity than the δ function in the noninteracting case. Using Eq. (20) we see that $\rho_+^<$ behaves as $(\omega+\tilde{k}\tilde{v}_f)^\gamma$ when ω approaches $-\tilde{k}\tilde{v}_f$ from below. The total spectral function

$$\rho_+(k,\omega)\equiv\rho_+^>(k,\omega)+\rho_+^<(k,\omega) \tag{25}$$

is shown in Fig. 1(a) for $\gamma=0.1$. Replacing $\tilde{k}$ by $-\tilde{k}$ leads to the curve of Fig. 1(a) mirror reflected at $\omega=0$. For the special value $\tilde{k}=0$, one obtains a symmetric function $\rho_+(k_f,\omega)\sim|\omega|^{2\gamma-1}$, i.e., a divergence for $\gamma<\frac{1}{2}$. There is no sharp quasiparticle peak. This so-called Luttinger liquid behavior is shown in Fig. 1(b).

Before we examine the model including the spin we want to point out that the critical exponents for the singularities of the spectrum can be read off Eq. (21)

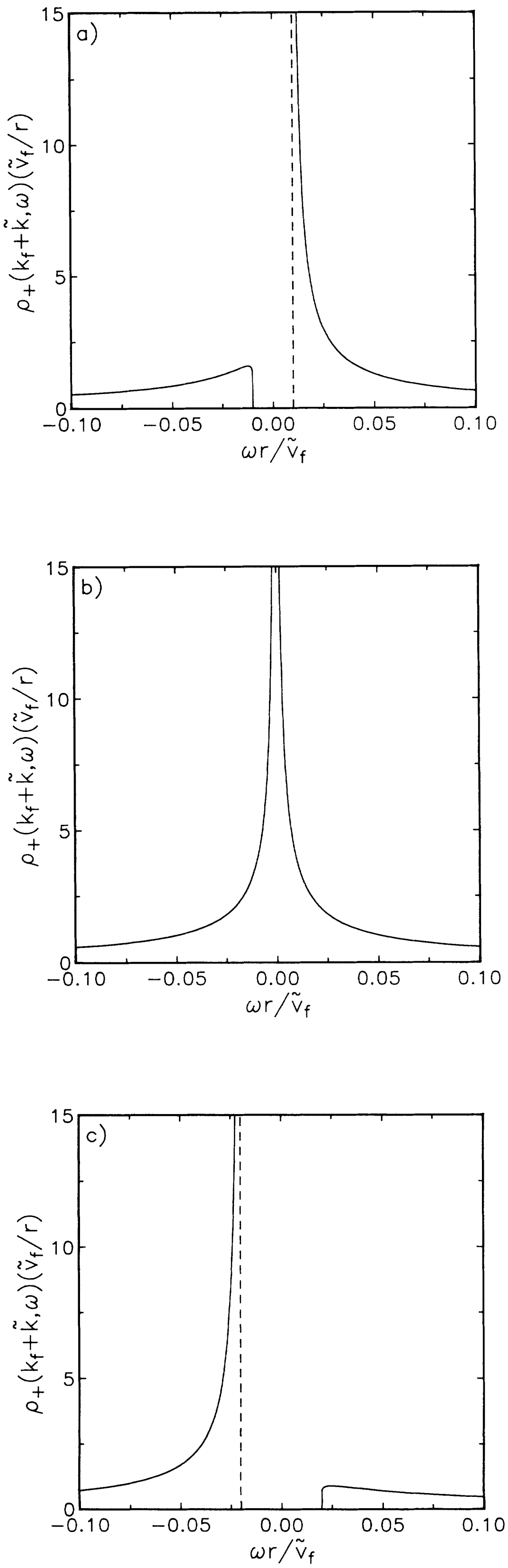

FIG. 1. Total spectral function $\rho_+(k_f+\tilde{k},\omega)$ of the spinless TL model as a function of $\omega r/\tilde{v}_f$ in the low-frequency regime for $\gamma=0.1$ and different momenta k. (a) $r\tilde{k}=0.01$. (b) $\tilde{k}=0$. (c) $r\tilde{k}=-0.02$.

without really performing the complicated integral $I_2(u_+)$. Using $u_+=u_-+\tilde{k}$, we can write $I_1(u_-)I_2(u_+)=I_1(u_-)I_2(u_-+\tilde{k})=I_1(u_+-\tilde{k})I_2(u_+)$. For fixed $\tilde{k}\neq 0$ the function I_2 (I_1) is regular near $\tilde{k}$ $(-\tilde{k})$. For $\tilde{k}>0$ $(\tilde{k}<0)$ the nonanalytic threshold behavior of $\rho_+^{\gtrless}(k_f+\tilde{k},\omega)$ is therefore determined by $I_1(u_-)$ $[I_2(u_+)]$. The small u_+ behavior of I_2 is as simple as the behavior of I_1 for small u_- because it is determined by the large s behavior of the integrand. Then $+i0$ can be replaced by $+ir$ and I_2 becomes an integral of the same type as I_1.

III. MODEL INCLUDING SPIN

To describe interacting electrons in (quasi-) one-dimensional systems the spin degree of freedom has to be taken into account. In the framework of the Tomonaga model one writes an (effective) interaction term

$$\hat{V}=\frac{1}{2}\int_0^L dx\int_0^L dx'\sum_{\sigma,\sigma'}\hat{\psi}_\sigma^\dagger(x)\hat{\psi}_{\sigma'}^\dagger(x')\times V_{\sigma\sigma'}(|x-x'|)\hat{\psi}_{\sigma'}(x')\hat{\psi}_\sigma(x)\ , \tag{26}$$

where the two-body potential is allowed to be spin dependent,

$$V_{\sigma\sigma'}(|x-x'|)=V(|x-x'|)+\sigma\sigma' U(|x-x'|)\ . \tag{27}$$

In Eq. (26) the field operator $\hat{\psi}_\sigma^\dagger(x)$ creates an electron with spin component σ in the Fock space corresponding to the Tomonaga model. Only after the transition to the Luttinger model does one obtain field operators $\hat{\psi}_{\alpha,\sigma}^\dagger(x)$ where α as in Eq. (1) distinguishes right- $(\alpha=+)$ and left- $(\alpha=-)$ moving electrons and the corresponding density operators $\hat{\rho}_{\alpha,\sigma}(x)$. It is useful to define charge- and spin-density operators as[20]

$$\hat{\rho}_{q,\alpha}\equiv\hat{\rho}_{q,\alpha,\uparrow}+\hat{\rho}_{q,\alpha,\downarrow}\ , \tag{28}$$

$$\hat{\sigma}_{q,\alpha}\equiv\hat{\rho}_{q,\alpha,\uparrow}-\hat{\rho}_{q,\alpha\downarrow}\ , \tag{29}$$

and with a normalization which differs by a factor of $\sqrt{2}$, the analogous definition to Eq. (5) reads as

$$\hat{b}_{q,c}\equiv\left[\frac{\pi}{|q|L}\right]^{1/2}\begin{cases}\hat{\rho}_{q,+} & \text{for } q>0\\ \hat{\rho}_{q,-} & \text{for } q<0\ ,\end{cases}$$
$$\hat{b}_{q,s}\equiv\left[\frac{\pi}{|q|L}\right]^{1/2}\begin{cases}\hat{\sigma}_{q,+} & \text{for } q>0\\ \hat{\sigma}_{q,-} & \text{for } q<0\ .\end{cases} \tag{30}$$

These operators describe independent boson degrees of freedom. Again the kinetic energy can be expressed in terms of the boson operators and particle number operators

$$\hat{H}_0=v_f\left\{\sum_{q\neq 0}|q|(\hat{b}_{q,c}^\dagger\hat{b}_{q,c}+\hat{b}_{q,s}^\dagger\hat{b}_{q,s})+\frac{\pi}{2L}\sum_{\alpha=\pm}(\hat{B}_{\alpha,c}^2-2\hat{B}_{\alpha,c}+\hat{B}_{\alpha,s}^2-2\hat{B}_{\alpha,s})\right\}\ . \tag{31}$$

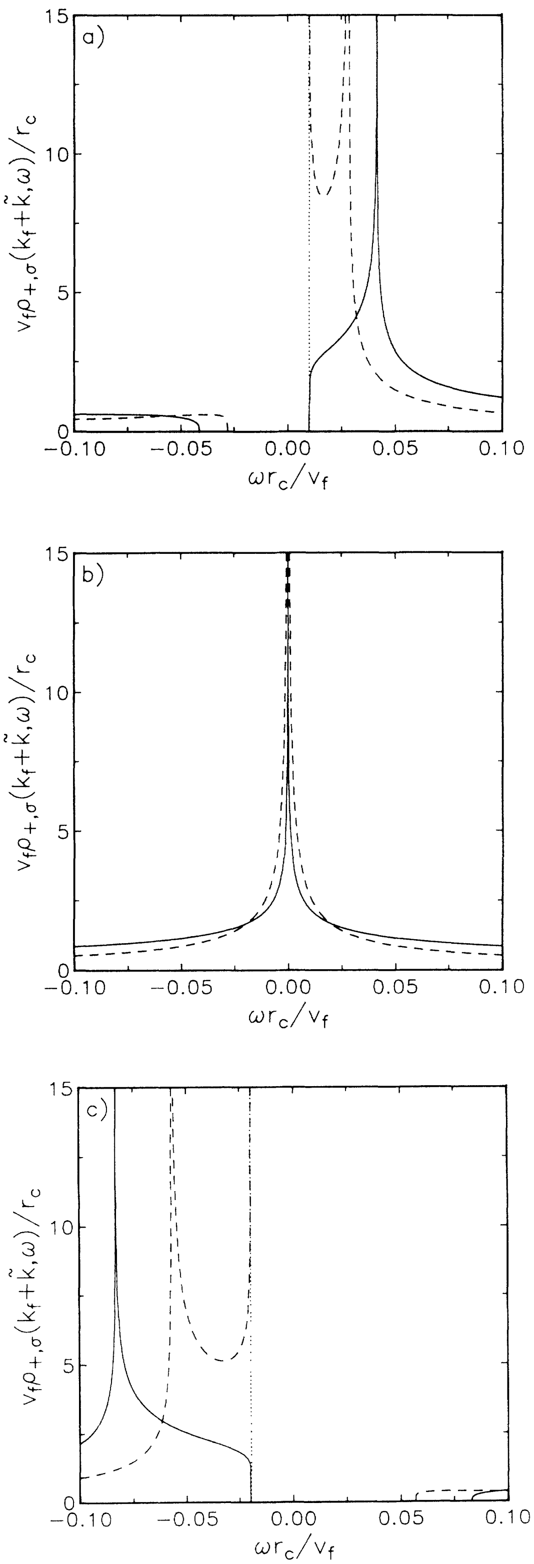

FIG. 2. Total spectral function $\rho_{+,\sigma}(k_f+\tilde{k},\omega)$ of the TL model including spin as a function of $\omega r_c/v_f$ in the low-frequency regime for $\gamma_c=0.3$ (dotted curve), $\gamma_c=0.6$ (full curve), and different momenta k. The interaction is taken spin independent and repulsive. (a) $r_c\tilde{k}=0.01$. (b) $\tilde{k}=0$. (c) $r_c\tilde{k}=-0.02$. The curve for $\gamma_c=0.6$ is scaled up by a factor of 3.

The interaction term naturally decomposes into a sum of a charge-density and spin-density part which both are of type (8). The Hamiltonian for the interacting Luttinger model including spin then reads as

$$\hat{H}=\sum_{q\neq 0}|q|\left\{\left[v_f+\frac{v(q)}{\pi}\right]\hat{b}^\dagger_{q,c}\hat{b}_{q,c}+\frac{v(q)}{2\pi}(\hat{b}^\dagger_{q,c}\hat{b}^\dagger_{-q,c}+\hat{b}_{+q,c}\hat{b}_{q,c}+1)\right\}$$
$$+\sum_{q\neq 0}|q|\left\{\left[v_f+\frac{u(q)}{\pi}\right]\hat{b}^\dagger_{q,s}\hat{b}_{q,s}+\frac{u(q)}{2\pi}(\hat{b}^\dagger_{q,s}\hat{b}^\dagger_{-q,s}+\hat{b}_{-q,s}\hat{b}_{q,s}+1)\right\}+\hat{H}_2(\{\hat{B}_{\alpha,\sigma}\})\ . \tag{32}$$

This shows the complete spin and charge separation. Therefore the results of Sec. II can be used to obtain an expression for the function $G^{\gtrless}_{\alpha,\sigma}(x,t)$. Apart from a factor $\frac{1}{2}$ which is due to the factor $1/\sqrt{2}$ in the spin-dependent boson operator $\hat{b}_{q,\sigma}=(\hat{b}_{q,c}+\sigma\hat{b}_{q,s})/\sqrt{2}$ and the sum over the charge and spin part in the exponent in the second term on the rhs of Eq. (15), this result can be used also for the model with spin. Assuming the special form (16) for $v(q)$ and $u(q)$, one obtains in the large x and t limit

$$iG^{\gtrless}_{\alpha,\sigma}(x,t)e^{i\mu t}=\underset{(-)}{+}\frac{i\alpha}{2\pi}e^{i\alpha k_f x}\frac{1}{(x-\alpha\tilde{v}_{f,c}t\underset{(-)}{+}i\alpha 0)^{1/2}}\frac{1}{(x-\alpha\tilde{v}_{f,s}t\underset{(-)}{+}i\alpha 0)^{1/2}}$$
$$\times\left[\frac{r_c^2}{(x-\tilde{v}_{f,c}t\underset{(-)}{+}ir_c)(x+\tilde{v}_{f,c}t\underset{(+)}{-}ir_c)}\right]^{\gamma_c/2}\left[\frac{r_s^2}{(x-\tilde{v}_{f,s}t\underset{(-)}{+}ir_s)(x+\tilde{v}_{f,s}t\underset{(+)}{-}ir_s)}\right]^{\gamma_s/2}, \tag{33}$$

where the renormalized Fermi velocities are given by the $q=0$ limit for the velocities in the interacting system

$$\tilde{v}_{f,c}\equiv v_f\sqrt{1+2v(0)/(\pi v_f)}\ , \tag{34}$$

$$\tilde{v}_{f,s}\equiv v_f\sqrt{1+2u(0)/(\pi v_f)}\ . \tag{35}$$

The spin wave velocity $\tilde{v}_{f,s}$ equals the unrenormalized Fermi velocity v_f in the case of a spin-independent interaction.

The spectral functions $\rho^{>}_{\alpha,\sigma}(k,\omega)$ and $\rho^{<}_{\alpha,\sigma}(k,\omega)$ corresponding to $G^{\gtrless}_{\alpha,\sigma}(x,t)$ are defined as in Eqs. (18) and (19). It has been pointed out previously that the double Fourier transform to obtain the spectral functions in the model with spin poses a difficult problem.[7,21]

In the following we discuss only $\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)$ as the relation (20) also holds for the spin components of the spectral functions. In our attempt to perform the x and t integration we use variable substitutions $s=x-\tilde{v}_{f,c}t$ and $s'=x+\tilde{v}_{f,c}t$ as in Sec. II. This yields with $\beta\equiv\tilde{v}_{f,s}/\tilde{v}_{f,c}$ and $u^c_\pm\equiv(\omega\pm\tilde{v}_{f,c}\tilde{k})/(2\tilde{v}_{f,c})$,

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)=\frac{r_c^{\gamma_c}r_s^{\gamma_s}}{2\tilde{v}_{f,c}}\int_{-\infty}^{\infty}\frac{ds'}{2\pi}\frac{e^{iu^c_- s'}}{(s'-ir_c)^{\gamma_c/2}}F(s') \tag{36}$$

with

$$F(s')=\int_{-\infty}^{\infty}\frac{ids}{2\pi}\frac{e^{-iu^c_+ s}}{(s+i0)^{1/2}(s+ir_c)^{\gamma_c/2}[s(1+\beta)/2+s'(1-\beta)/2+ir_s]^{\gamma_s/2}}$$
$$\times\frac{1}{[s'(1+\beta)/2+s(1-\beta)/2-ir_s]^{\gamma_s/2}[s(1+\beta)/2+s'(1-\beta)/2+i0]^{1/2}}\ . \tag{37}$$

In contrast to the spinless case, $\rho^{>}_{+,\sigma}$ is not given as a product of two independent integrals. Nevertheless, it is possible to read off the critical indices for the singularities of the spectrum from Eqs. (36) and (37) and to determine the threshold of the spectrum. For $\beta>1$, i.e., $\tilde{v}_{f,c}<\tilde{v}_{f,s}$, the singularities of the integration variable s' (s) are in the upper (lower) complex integration plane. Therefore $\rho^{>}_{+,\sigma}$ vanishes for $\omega<|\tilde{k}|\tilde{v}_{f,c}$. For $\beta<1$ the roles of $\tilde{v}_{f,c}$ and $\tilde{v}_{f,s}$ are interchanged and $\rho^{>}_{+,\sigma}$ vanishes for $\omega<|\tilde{k}|\tilde{v}_{f,s}$.

For a given value of $\tilde{k}>0$ we expect singularities in $\rho^{>}_{+,\sigma}$ at $\omega=\tilde{v}_{f,c}\tilde{k}$ *and* $\omega=\tilde{v}_{f,s}\tilde{k}$. This can already be seen for the special case $\gamma_c=\gamma_s=0$ but $\tilde{v}_{f,c}\neq\tilde{v}_{f,s}$, where the integrations can be performed and one obtains square-root singularities at both frequencies.[7,22] For $\omega\approx\tilde{v}_{f,c}\tilde{k}$, i.e., $u^c_-\approx 0$, we have $u^c_+\approx\tilde{k}$ and the main contribution to the s integration in $F(s')$ comes from finite $|s|<C(\tilde{k})$. Therefore, for large s', we have $F(s')\sim s'^{-(\gamma_s+1/2)}$. As the small u^c_- behavior of the integral in (36) is determined by the large s' behavior of the integrand, we obtain for $\beta>1$ the threshold behavior

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)\sim\Theta(\omega-\tilde{v}_{f,c}\tilde{k})(\omega-\tilde{v}_{f,c}\tilde{k})^{(2\gamma_s+\gamma_c-1)/2},\quad \tilde{k}>0\ . \tag{38}$$

For $\beta<1$ the roles of $\tilde{v}_{f,c}$ and $\tilde{v}_{f,s}$ are interchanged and the threshold behavior is

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)\sim\Theta(\omega-\tilde{v}_{f,s}\tilde{k})(\omega-\tilde{v}_{f,s}\tilde{k})^{(2\gamma_c+\gamma_s-1)/2}\,,\quad \tilde{k}>0\,. \tag{39}$$

For $\beta<1$ there is an additional singularity at $\omega=\tilde{v}_{f,c}\tilde{k}$ with the exponent given in (38) but $\omega-\tilde{v}_{f,c}\tilde{k}$ replaced by $|\omega-\tilde{v}_{f,c}\tilde{k}|$ while for $\beta>1$ the additional singularity is at $\omega=\tilde{v}_{f,s}\tilde{k}$ with the exponent given in (39).

For $\tilde{k}<0$, as in the spinless case, there is no singularity in $\rho^{>}_{+,\sigma}$ itself but only in its derivative with respect to ω. Interchanging the order of the s and s' integration we can argue as for $\tilde{k}>0$ and obtain for $\beta>1$ the threshold behavior

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)\sim\Theta(\omega-\tilde{v}_{f,c}|\tilde{k}|)(\omega-\tilde{v}_{f,c}|\tilde{k}|)^{\gamma_s+\gamma_c/2}\,,\quad \tilde{k}<0\,. \tag{40}$$

while for $\beta<1$ one has

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)\sim\Theta(\omega-\tilde{v}_{f,s}|\tilde{k}|)(\omega-\tilde{v}_{f,s}|\tilde{k}|)^{\gamma_c+\gamma_s/2}\,,\quad \tilde{k}<0 \tag{41}$$

For $\beta<1$ ($\beta>1$) there are additional nonanalyticities at $\omega=\tilde{v}_{f,c}|\tilde{k}|$ ($\omega=\tilde{v}_{f,s}|\tilde{k}|$) with the exponents given in (40) and (41).

The case $\tilde{k}=0$ requires special treatment as in the spinless case. It can be treated by examining the behavior of the integrals in (36) under the scaling $\omega\to\epsilon\omega$, $\tilde{k}\to\epsilon\tilde{k}$, i.e., $u^c_\pm\to\epsilon u^c_\pm$. This leads for $\tilde{k}=0$ to

$$\rho^{>}_{+,\sigma}(k_f,\omega)\sim|\omega|^{\gamma_c+\gamma_s-1}\,. \tag{42}$$

The total spectral density per unit length

$$\rho_{\alpha,\sigma}(\omega)\equiv\int_{-\infty}^{\infty}\frac{dk}{2\pi}\rho_{\alpha,\sigma}(k,\omega) \tag{43}$$

with $\rho_{\alpha,\sigma}(k,\omega)\equiv\rho^{>}_{\alpha,\sigma}(k,\omega)+\rho^{<}_{\alpha,\sigma}(k,\omega)$ can easily be calculated, as it involves only the time Fourier transform of $G^{\gtrless}_{\alpha,\sigma}(0,t)$. Using (33) this yields for small ω

$$\rho_{\alpha,\sigma}(\omega)\sim|\omega|^{\gamma_c+\gamma_s}\,. \tag{44}$$

This completes our general analytical results for the spectral function for the model including spin.

As it is desirable to obtain the spectral function in the whole low-frequency range we discuss in the rest of this section the important special case of a *spin-independent repulsive interaction,* i.e., $u(q)\equiv 0$, $v(q)>0$. This implies $\tilde{v}_{f,s}=v_f$, $\gamma_s=0$, and $\tilde{v}_{f,c}>v_f$, i.e., $\beta<1$. In this special case the integrand in $F(s')$ is an analytic function in the upper s plane. Therefore $\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)$ is proportional to $\Theta(\omega+\tilde{v}_{f,c}\tilde{k})$. This implies that the threshold for $\tilde{k}<0$ is not given by $v_f|\tilde{k}|$ as in Eq. (41) but the additional step function shifts the threshold to $\omega=\tilde{v}_{f,c}|\tilde{k}|$. This is a specialty of the $\gamma_s=0$ case.

As described in the Appendix it is possible using the integral representation of the confluent hypergeometric function to obtain an expression for $\rho^{>}_{+,\sigma}$ which involves only a single integration[23]

$$\begin{aligned}\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)=&\,C\exp\left[-\frac{r_c}{\tilde{v}_{f,c}+v_f}[2\omega+\tilde{k}(\tilde{v}_{f,c}-v_f)]\right]\Theta(\omega-v_f\tilde{k})(\omega-v_f\tilde{k})^{(2\gamma_c-1)/2}\\&\times\int_0^1 ds(1-s)^{(\gamma_c-2)/2}s^{(\gamma_c-1)/2}[(\omega+\tilde{v}_{f,c}\tilde{k})(1-\beta)-(\omega-v_f\tilde{k})2s]^{-1/2}\\&\times\exp\left[2r_cs\frac{\omega-v_f\tilde{k}}{\tilde{v}_{f,c}+v_f}\right]\Theta[(\omega+\tilde{v}_{f,c}\tilde{k})(1-\beta)-(\omega-v_f\tilde{k})2s]\end{aligned} \tag{45}$$

with

$$C=\left[\frac{r_c}{\tilde{v}_{f,c}}\right]^{\gamma_c}\frac{(1+\beta)^{1/2}}{(1-\beta^2)^{\gamma_c/2}}\times[\Gamma(\tfrac{1}{2})\Gamma(\gamma_c/2)\Gamma((\gamma_c+1)/2)]^{-1}\,. \tag{46}$$

The remaining integration can be performed numerically. Also starting from (45) it is possible to reproduce the above results of the threshold behavior for the special case $\gamma_s=0$ and $\tilde{v}_{f,c}>v_f$.[23] Results for the total spectral function $\rho_{+,\sigma}(k_f+\tilde{k},\omega)$ are shown in Fig. 2 for $\gamma_c=0.3$ and 0.6. The k values chosen are $r_c\tilde{k}=0.01$, $\tilde{k}=0$, and $r_c\tilde{k}=-0.02$ as in the spinless case. For $k<k_f$ the main weight lies in the photoemission part of the spectrum at $\omega<0$, while for $k>k_f$ it lies in the inverse-photoemission part at $\omega>0$. At $k=k_f$ the spectrum is a symmetric function of ω and again shows Luttinger liquid behavior, i.e., the absence of a sharp quasiparticle peak. For the large value $\gamma_c=0.6$ the spectral weight in the low-energy regime is strongly reduced compared to the spectrum for $\gamma_c=0.3$. This transfer of spectral weight to higher frequencies also happens in the spinless model, as can be inferred from Eq. (24). Large values of γ_c correspond to systems with a low electron density.

IV. SUMMARY

While the one-particle Green's function for the Tomonaga-Luttinger model as a function of space and time variables has been known for a long time, it is surprising that the behavior of the corresponding spectral functions for the model including spin have not been discussed previously. This has also been pointed out in a recent experimental photoemission study of quasi-one-dimensional metals.[24] We have presented analytical and numerical results for the spectra, which show interesting additional structure compared to the spinless model. The critical indices for the singularities of the spectrum which depend on the coupling strength have been given analytically.

Note added in proof. After our paper had been submitted for publication we received a paper by J. Voit,[26] which for the special case of a spin-independent interaction contains results equivalent to ours.

APPENDIX

In this appendix we want to discuss the spectral function $\rho^{>}_{+,\sigma}$ for the special case of a spin-independent repulsive interaction. Starting with (18) and (33) we get for $\gamma_s=0$ and $\tilde{v}_{f,c}>v_f$,

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)=\frac{ir_c^{\gamma_c}}{4\pi^2}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int_{-\infty}^{\infty}dx\,e^{-i\tilde{k}x}\frac{1}{(x-\tilde{v}_{f,c}t+i0)^{1/2}}\frac{1}{(x-v_ft+i0)^{1/2}}\times\left[\frac{1}{(x-\tilde{v}_{f,c}t+ir_c)(x+\tilde{v}_{f,c}t-ir_c)}\right]^{\gamma_c/2}. \quad (A1)$$

With the substitution $s'=x-v_ft$ one obtains

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)=\frac{ir_c^{\gamma_c}}{4\pi^2v_f}\int_{-\infty}^{\infty}ds'\frac{e^{-i\omega s'/v_f}}{(s'+i0)^{1/2}}\int_{-\infty}^{\infty}dx\frac{e^{i(\omega-v_f\tilde{k})x/v_f}}{[x(1-\beta^{-1})+\beta^{-1}s'+i0]^{1/2}}\times\left[\frac{1}{[x(1-\beta^{-1})+\beta^{-1}s'+ir_c]}\frac{1}{[x(1+\beta^{-1})-\beta^{-1}s'-ir_c]}\right]^{\gamma_c/2}$$

$$\equiv r_c^{\gamma_c}\int_{-\infty}^{\infty}\frac{ds'}{2\pi}\frac{e^{-i\omega s'/v_f}}{(s'+i0)^{1/2}}I(s') \quad (A2)$$

with β (<1) as in Sec. III. For small ω and $\omega-v_f\tilde{k}$, the behavior of $\rho^{>}_{+,\sigma}$ is determined by the large s' and s behavior of the integrand and we can replace $+i0$ by $+ir_c$ in both integrals. This does not alter the low-energy behavior but leads to a violation of the sum rule for the total weight as the high-frequency regime is not described correctly. Then the x integration has the same structure as the integral I_2 in (21) and can also be expressed in terms of the confluent hypergeometric function[19]

$$I(s')=\sqrt{i}\,(1+\beta^{-1})^{-\gamma_c/2}(\beta^{-1}-1)^{-(\gamma_c+1)/2}\frac{v_f^{-(2\gamma_c+1)/2}}{\Gamma[(2\gamma_c+1)/2]}\exp\left\{r_c(1+\beta^{-1})^{-1}\left[\tilde{k}-\frac{\omega}{v_f}\right]\right\}\times\exp\left\{-is'(1+\beta^{-1})^{-1}\left[\tilde{k}-\frac{\omega}{v_f}\right]\right\}\Theta(\omega-v_f\tilde{k})(\omega-v_f\tilde{k})^{(2\gamma_c-1)/2}\Phi\left[\frac{\gamma_c+1}{2},\frac{2\gamma_c+1}{2},2\beta\frac{r_c\beta-is'}{1-\beta^2}\left[\tilde{k}-\frac{\omega}{v_f}\right]\right]. \quad (A3)$$

By taking the integral representation[25]

$$\Phi(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 ds\,e^{zs}s^{a-1}(1-s)^{b-a-1} \quad (A4)$$

of the confluent hypergeometric function and interchanging the remaining s' and s integrations we get

$$\rho^{>}_{+,\sigma}(k_f+\tilde{k},\omega)=\frac{r_c^{\gamma_c}}{v_f^{(2\gamma_c+1)/2}}(1+\beta^{-1})^{-\gamma_c/2}(\beta^{-1}-1)^{-(\gamma_c+1)/2}\Gamma^{-1}(\gamma_c/2)\Gamma^{-1}[(\gamma_c+1)/2]\times\exp\left\{\frac{r_c}{\tilde{v}_{f,c}+v_f}(\tilde{k}v_f-\omega)\right\}\Theta(\omega-v_f\tilde{k})(\omega-v_f\tilde{k})^{(2\gamma_c-1)/2}\times\int_0^1 ds\,s^{(\gamma_c-1)/2}(1-s)^{(\gamma_c-2)/2}\exp\left\{\frac{2sr_c\beta^2}{1-\beta^2}\left[\tilde{k}-\frac{\omega}{v_f}\right]\right\}\times\int_{-\infty}^{\infty}\frac{ds'}{2\pi}\frac{\exp\left\{-is'\left[\left[\tilde{k}+\frac{\omega}{\tilde{v}_{f,c}}\right](1-\beta)+2s\beta\left[\tilde{k}-\frac{\omega}{v_f}\right]\right]\Big/(1-\beta^2)\right\}}{(r-is')^{1/2}}. \quad (A5)$$

The s' integral can now be performed as I_1 in Sec. II (22) and we finally obtain (45) and (46).

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