Effects of spin-orbit interactions in disordered conductors: A random-matrix approach

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A description in terms of random matrices is used to study the effects of strong spin-orbit coupling in disordered conductors. It is shown that the ensemble of transfer matrices can be conveniently parametrized using quaternions. A diffusion equation is derived for the evolution, with sample length, of the transfer-matrix distribution. In the insulating regime, a uniform density of Lyapunov exponents is obtained, and the expected universal multiplication factors in the localization length are derived when time-reversal symmetry is broken. Weak antilocalization, backscattering depletion, and universal conductance fluctuations are obtained in the metallic regime.

I. INTRODUCTION

Interference effects in quantum-electronic transport in mesoscopic systems have been the subject of many recent investigations. Experimentally, interference effects provide new methods of measuring characteristic times of the conduction electrons, such as the inelastic lifetime and spin-orbit coupling time.[1] Theoretically, interference effects give rise to weak localization, a precursor to the Anderson transition. In addition, they are responsible for striking phenomena, particularly universal conductance fluctuations (UCF) observed in small metallic samples at low temperatures.[2–5] The measured conductance exhibits time-independent, reproducible stochastic variations as a function of magnetic field or Fermi energy, with a variance always of order $(e^2/h)^2$ independent of sample size and degree of disorder.

This behavior has been studied theoretically, both using microscopic, diagrammatic calculations[6–8] and via macroscopic random-matrix theory.[9–19] There is agreement between microscopic and macroscopic approaches, which underlines the possibility of a generalized central-limit theorem[10] governing the behavior of the system, and motivates theories that do not depend on the details of the model Hamiltonian. Two such theories have recently been developed. One theory[9–13] takes a local approach based on a maximum entropy ansatz for the probability density of the transfer matrix of small slices of the conductor. The other[14–19] is based on a global approach, and proposes a maximum entropy ansatz for the probability density of the transfer matrix of the whole sample. These two theories have been proved[20] to be equivalent in the limit of large numbers of channels. In this paper we extend the local maximum entropy approach to treat conductors with strong spin-orbit scattering.

Spin-orbit scattering influences quantum interference effects in a particularly interesting way. Bergmann[1] has shown in a series of experiments on thin films of Mg with different coverages of the strong spin-orbit coupler Au, that the main effect of spin-orbit coupling is to change weak localization into weak antilocalization. This is due to the fact that, in the presence of a spin-orbit interaction, each scattering event slightly rotates the spin of the electron. According to quantum mechanics, the sign of the electron spin state is reversed after a rotation by 2π. This sign change yields a destructive interference in the backscattering direction, which in turn implies weak antilocalization.

Theoretically, the effects of spin-orbit scattering on random conductors have been studied recently by Pichard and co-workers[16,18] using random-transfer-matrix theory in the global maximum-entropy approach. This formulation, which is built on earlier work by Imry[14] and Muttalib, Pichard, and Stone,[15] emphasizes the Dyson conjecture[21] that the local statistical properties of the eigenvalue distribution in the ensemble are universal, and only the global properties contain nonuniversal physics. In order to introduce these global properties into the theory, the average eigenvalue density is taken as input. A maximum-entropy ansatz is invoked to generate from this the joint eigenvalue distribution. The influence of spin-dependent hopping is studied[16] by defining an ensemble for the transfer matrices with the appropriate (symplectic) symmetry. Local statistics agree with standard random-matrix theory and numerical simulations show a uniform density of Lyapunov exponents. Zanon and Pichard also argue, based on the results of Dyson and Mehta[22] for the variance of a quantity which is a linear statistic of the spectrum of a random matrix, that the presence of spin-orbit coupling should reduce the conductance fluctuations by a factor of 4, in agreement with microscopic calculations.[8]

The local maximum-entropy approach,[9–13] on the other hand, divides the sample into thin slices, assumes Ohm's law to be valid for each slice, and combines them by means of a convolution requirement. Very recently, using this approach, Mello and Stone[13] have obtained, for various quantities, impressive quantitative agreement with elaborate microscopic calculations. These authors, however, did not consider the effects of spin-orbit scattering. Motivated by the striking experimental consequences of spin-orbit scattering, indicated above, we provide such a treatment in the present paper.

Our work is complementary to a recent discussion of the same problem by Hüffmann,[23] from a coordinate-free, geometric viewpoint. We recover this author's expressions for several quantities, in particular the amplitude of conductance fluctuations. We also find that explicit introduction of suitable coordinates makes it possible to obtain a number of additional results, most importantly, the values of the Lyapunov exponents.

In a celebrated paper,[24] Dyson introduced three possible universality classes for random Hamiltonians. His terminology is now commonplace in weak-localization theory.[25] The *orthogonal* class is appropriate to systems with time-reversal symmetry and in the absence of spin-orbit scattering. If sufficiently strong spin-orbit coupling is present, and time-reversal symmetry is still preserved, one has the *symplectic* class. When time-reversal symmetry is destroyed by an applied magnetic field, both classes reduce to the *unitary* class. Each class is closely related to one of the three types of division algebra with real coefficients: the real numbers, the quaternions, and the complex numbers, respectively.

This work is concerned with the symplectic case. In Sec. II, an ensemble of quasi-one-dimensional disordered conductors with transverse width L_t and length L is described by means of an ensemble of random transfer matrices M. It is shown that a representation in terms of quaternions is a natural consequence of flux conservation and time-reversal symmetry in the presence of spin-orbit scattering. This representation greatly facilitates the subsequent calculations. The invariant (Haar) measure $d\mu(M)$ is constructed for transfer matrices with the symmetry resulting from flux conservation. In Sec. III, following the local approach,[9–13] a statistical distribution for each slice is chosen on the basis of a maximum-entropy criterion, and a combination law for the slices is formulated. The resulting diffusion equation is derived and compared with earlier results for the orthogonal and unitary cases.[9,13] Reflection and transmission coefficients are defined in a form appropriate for performing averages on the quaternion unitary group. The conductance is derived as a sum of the transmission coefficients that relate incident and outgoing fluxes, in accordance with the two-probe Landauer formula.[26–28] The two-probe geometry, although not appropriate for quantitative comparison with many experiments, serves to demonstrate the essential phenomena. In Sec. IV we apply the theory to samples much longer than the localization length. In this regime one expects from Oseledec's theorem[29] that the transfer matrix of the sample, which is the product of a large number of transfer matrices associated with each slice, will be characterized by its Lyapunov exponents. We show how this behavior arises from the diffusion equation obtained in Sec. III. The localization length is calculated and the Lyapunov exponents are shown to have a uniform distribution when $N \to \infty$. In Sec. V, an expansion in inverse powers of the average conductance[11,13] is used to obtain the weak antilocalization correction to the conductance, and the UCF in the metallic regime. The expected backscattering depletion[1] is found as a natural consequence of the quaternion algebra and the assumption of uniform distribution of phase factors in the transfer-matrix representation. Our conclusions are summarized briefly in Sec. VI.

II. THE TRANSFER MATRIX: A QUATERNION PARAMETRIZATION

The simplest measurement geometry is a two-probe system consisting of a finite disordered section of length L and transverse width L_t, to which current is supplied by two semi-infinite ordered leads. The multichannel scattering process can be completely described by the corresponding scattering matrix,

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix} , \tag{2.1}$$

which relates the incoming fluxes I, I' to the outgoing fluxes O, O' through

$$S \begin{bmatrix} I \\ I' \end{bmatrix} = \begin{bmatrix} O \\ O' \end{bmatrix} , \tag{2.2}$$

where r, r' and t, t' are, respectively, reflection and transmission matrices. Although more familiar in scattering problems, the S matrix obeys a rather complex composition law and a description in terms of a transfer matrix, defined as

$$M \begin{bmatrix} I \\ O \end{bmatrix} = \begin{bmatrix} O' \\ I' \end{bmatrix} , \tag{2.3}$$

turns out to be more convenient. The presence of spin adds another degree of freedom, which implies that I, O, I', O' are $2N$-component vectors containing the wave amplitudes, where N is the number of quantized transverse momenta and the factor of 2 is due to the additional spin degree of freedom. Therefore, M is a $4N \times 4N$ matrix, which we shall divide into four $N \times N$ quaternion blocks. Flux conservation and time-reversal symmetry imply that M must satisfy the following requirements:[30]

$$M^{\dagger}\Sigma_z M = \Sigma_z \tag{2.4}$$

and

$$M^* = \Sigma_x M \Sigma_x , \tag{2.5}$$

where Σ_z and Σ_x denote, respectively,

$$\Sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.6}$$

and

$$\Sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \tag{2.7}$$

with 1 and 0 designating the $N \times N$ quaternion unit and zero matrix. The matrix operations in (2.4) and (2.5) are defined as follows:

$$(M^{\dagger})_{ij} \equiv M_{ji}^{\dagger} \tag{2.8}$$

and

$$(M^*)_{ij} \equiv M_{ij}^* , \tag{2.9}$$

where $M_{ji}^{\dagger}$ and M_{ji}^* are, respectively, the Hermitian and the complex conjugate of the quaternion M_{ji}, which are defined as

$$M_{ji} = M_{ji}^{(0)} e_0 + \mathbf{M}_{ji} \cdot \mathbf{e} , \tag{2.10}$$

$$M_{ji}^{\dagger} = M_{ji}^{(0)*} e_0 - \mathbf{M}_{ji}^* \cdot \mathbf{e} , \tag{2.11}$$

$$M_{ji}^* = M_{ji}^{(0)*} e_0 + \mathbf{M}_{ji}^* \cdot \mathbf{e} . \tag{2.12}$$

Here, $M_{ji}^{(k)}$ $(k = 0, 1, 2, 3)$ are complex numbers, e_0 is the 2×2 unit matrix, and e_1, e_2, and e_3 are defined as

$$e_1=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\quad e_2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\quad e_3=\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}. \tag{2.13}$$

The simplicity of expressions (2.4) and (2.5) shows the advantage of the quaternion description.

In Appendix A, it is shown that conditions (2.4) and (2.5) imply that any transfer matrix in the symplectic ensemble can be parametrized as

$$M=\begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}\begin{bmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v^* \end{bmatrix}, \tag{2.14}$$

where λ is an $N\times N$ quaternion real, diagonal matrix with non-negative elements $\lambda_1, \lambda_2, \ldots, \lambda_N$ and u and v are arbitrary $N\times N$ quaternion unitary matrices.

We mention in passing that a $4N\times 4N$ matrix satisfying conditions (2.4) and (2.5) has $\nu=2N(4N-1)$ free parameters, but the representation (2.14) contains $8N^2+N$ parameters. The $3N$ redundant parameters are due to the fact that the parametrization (2.14) is invariant under the transformation

$$u\to us,\quad v\to \bar{s}v\ , \tag{2.15}$$

where s is a quaternion real unitary matrix, which commutes with λ, and $\bar{s}$ is its quaternion conjugate, defined as

$$(\bar{s})_{ij}\equiv\bar{s}_{ji}=s_{ji}^{(0)}e_0-\mathbf{s}_{ji}\cdot\mathbf{e}\ . \tag{2.16}$$

Such a matrix has $3N$ independent parameters, which could, in principle, be used to eliminate the $3N$ redundant parameters of (2.14). This procedure is not convenient, though, inasmuch as the averages are much simpler if M is kept in the form (2.14). The probability distribution and, therefore, all expectation values, must be invariant under (2.15). It is worth mentioning that the number of parameters, $\nu=2N(4N-1)$, is the number needed to uniquely specify a unitary and self-dual matrix of dimensionality $4N$.[24,30] Henceforth, Eq. (2.14) will be used as the most convenient parametrization.

The invariant measure of the group formed by the M matrices is worked out in Appendix B, yielding the following result:

$$d\mu(M)=J(\lambda)\prod_{a=1}^{N} d\lambda_a d\mu(u)d\mu(v)\ , \tag{2.17a}$$

$$J(\lambda)=\prod_{a<b}|\lambda_a-\lambda_b|^4\ , \tag{2.17b}$$

in which the $3N$ harmless redundant parameters are included. The fourth power obtained in Eq. (2.17b) is characteristic of the symplectic ensemble as defined in the classical theory of random matrices[31] and is, as well as the Kramers degeneracy observed in the matrix λ, a direct consequence of the quaternion algebra.

III. THE STATISTICAL ENSEMBLE OF TRANSFER MATRICES

We shall describe an ensemble of random conductors by means of an ensemble of transfer matrices, whose differential probability is defined as

$$dP_L(M)=\rho_L(M)d\mu(M)\ , \tag{3.1}$$

where $\rho_L(M)$ is the probability density function and $d\mu(M)$ is given by (2.17). A basic requirement we shall now invoke is that $\rho_L(M)$ must be reproducible under convolutions, i.e., if we combine two slices of conductors of lengths L' and L'' and transfer matrices M' and M'', the resulting probability density must be given by

$$\rho_{L''+L'}(M)=\int \rho_{L''}(MM'^{-1})\rho_{L'}(M)d\mu(M) \tag{3.2}$$

or

$$\rho_{L''+L'}=\rho_{L''}\circ\rho_{L'}\ . \tag{3.3}$$

If we now set $L''=L$ and $L'=\delta L$, where δL is considered small but still macroscopic, then (3.2) gives

$$\rho_{L+\delta L}(M)=\int \rho_L(MM'^{-1})\rho_{\delta L}(M)d\mu(M)\ , \tag{3.4}$$

which is the "Schmoluchowsky equation" for this stochastic process.

The probability density $\rho_{\delta L}$ of each slice will now be chosen by maximizing Shannon's information entropy

$$S[\rho_{\delta L}]=-\int \rho_{\delta L}(M)\ln[\rho_{\delta L}(M)]d\mu(M)\ , \tag{3.5}$$

with the constraint that $\rho_{\delta L}$ is normalized. In addition, we require that

$$\frac{1}{N}\langle \mathrm{tr}\lambda\rangle_{\delta L}=\frac{\delta L}{l}\ , \tag{3.6}$$

where l is the mean free path. In Ref. 9 it is shown that these requirements ensure that Ohm's law is recovered when $\delta L\to 0$. The resulting distribution for M is isotropic, i.e., does not depend on the unitary matrices u and v in Eq. (2.14). It is also shown in Ref. 9 that the isotropic property of ρ_L is preserved under the convolution (3.4). Their proof can be easily extended to the symplectic case. It is well understood[13] that these requirements, in fact, define an ensemble of quasi-one-dimensional ($L\gg L_t$) conductors, and that additional or different[32] constraints would have to be imposed in order to describe higher-dimensional systems.

In Appendix C, the procedure devised in Ref. 9 is used to obtain a diffusion equation that describes the evolution of the probability density $w_L(\lambda)\equiv\rho_L(\lambda)J(\lambda)$ for the symplectic ensemble. The resulting equation is

$$\frac{\partial w_s(\lambda)}{\partial s}=\frac{1}{2N-1}\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left[\lambda_a(1+\lambda_a)J(\lambda)\frac{\partial}{\partial\lambda_a}\left[\frac{w_s(\lambda)}{J(\lambda)}\right]\right], \tag{3.7}$$

in which $s\equiv L/l$ is a measure of the length of the conductor in units of the mean free path, l. If we now compare this equation with the ones obtained by Mello and Stone for the orthogonal and unitary cases, we see that Eq. (3.11) of Ref. 13,

$$\frac{\partial w_s^{(\beta)}(\lambda)}{\partial s}=\frac{2}{\beta N+2-\beta}\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left[\lambda_a(1+\lambda_a)J_\beta(\lambda)\frac{\partial}{\partial\lambda_a}\times\left[\frac{w_s^{(\beta)}(\lambda)}{J_\beta(\lambda)}\right]\right], \quad (3.8)$$

$$J_\beta(\lambda)=\prod_{a<b}|\lambda_a-\lambda_b|^\beta , \quad (3.9)$$

is valid not only for $\beta=1$ (orthogonal ensemble) and $\beta=2$ (unitary ensemble), but also for $\beta=4$ (symplectic ensemble).

This equation must be solved with the initial condition $w_0^{(\beta)}(\lambda)=\delta(\lambda)$. Although we are mostly concerned with the symplectic case ($\beta=4$), we will use Eq. (3.8) instead of (3.7) in order to emphasize the effect of transitions between ensembles when certain symmetries are broken. The solution of (3.8) enables, in principle, the calculation of any expectation value, some of which will now be defined.

From (2.1)–(2.3) and (2.14) one can easily see that the reflection and transmission matrices r and t are given by

$$r=-\bar{v}\left[\frac{\lambda}{1+\lambda}\right]^{1/2}v , \quad (3.10)$$

$$t=u\left[\frac{1}{1+\lambda}\right]^{1/2}v . \quad (3.11)$$

If the channels are fed from the left with N incoherent unit fluxes, the reflection and transmission coefficients into channel a can be obtained by

$$R_a=(rr^\dagger)_{aa}=\sum_{b=1}^{N}\frac{\lambda_b}{1+\lambda_b}\bar{v}_{ba}v_{ba}^* , \quad (3.12)$$

$$T_a=(tt^\dagger)_{aa}=\sum_{b=1}^{N}\frac{1}{1+\lambda_b}u_{ab}u_{ab}^\dagger . \quad (3.13)$$

The total reflection and transmission coefficients are, by definition,

$$R=\sum_{a=1}^{N}R_a=\sum_{a=1}^{N}\frac{\lambda_a}{1+\lambda_a} , \quad (3.14)$$

$$T=\sum_{a=1}^{N}T_a=\sum_{a=1}^{N}\frac{1}{1+\lambda_a} , \quad (3.15)$$

in which we have dropped the trivial factor 2 due to Kramers degeneracy. The transmission and reflection coefficients T_{ab} and R_{ab} are defined, respectively, as $t_{ab}t_{ab}^\dagger$ and $r_{ab}r_{ab}^\dagger$. The explicit expressions for these quantities simplify after the ensemble average, yielding

$$\langle T_{ab}\rangle_s=\sum_{\alpha\alpha'}\langle u_{a\alpha}u_{a\alpha'}^\dagger\rangle_0\langle v_{\alpha b}v_{\alpha' b}^\dagger\rangle_0 \times\langle(1+\lambda_\alpha)^{-1/2}(1+\lambda_{\alpha'})^{-1/2}\rangle_s , \quad (3.16)$$

$$\langle R_{ab}\rangle_s=\sum_{\alpha\alpha'}\langle\bar{v}_{\alpha a}v_{\alpha b}v_{\alpha' b}^\dagger v_{\alpha' a}^*\rangle_0 \times\langle(\lambda_\alpha\lambda_{\alpha'})^{1/2}(1+\lambda_\alpha)^{-1/2}(1+\lambda_{\alpha'})^{-1/2}\rangle_s , \quad (3.17)$$

where $\langle\ \rangle_0$ denotes the average on the quaternion unitary group and $\langle\ \rangle_s$ is the expectation value obtained through the solution of (3.8).

The conductance of the model is given by the two-probe Landauer formula[26–28]

$$g=\sum_{a,b}T_{ab}=T . \quad (3.18)$$

IV. ANDERSON INSULATORS AND OSELEDEC'S THEOREM

The model is an Anderson insulator for sufficiently large L, at fixed N, simply because it is quasi-one-dimensional. The insulating regime is reached by solving (3.8) in the limit $L\gg\xi$, where ξ is the localization length. In this limit, one expects Oseledec's theorem[29] to apply. That is, if M is a transfer matrix for the whole sample, resulting from the successive multiplication of L statistically independent transfer matrices for individual slices

$$M(L)=\prod_{k=1}^{L}M_k , \quad (4.1)$$

then there exists a matrix $O(L)$ defined as

$$O(L)=(M^\dagger M)^{1/2L} \quad (4.2)$$

such that the logarithms of its eigenvalues self-average in the large L limit to a set of Lyapunov exponents $\{\alpha_a\}$. The N distinct, positive exponents define the inverse decay lengths of the system. The localization length is

$$\xi=\frac{1}{\alpha_{\min}} , \quad (4.3)$$

where $\alpha_{\min}$ is the smallest positive Lyapunov exponent. We shall now prove that these consequences of Oseledec's theorem also follow naturally from the parametrization (2.14) and the diffusion equation (3.8).

First, define a positive diagonal matrix ν satisfying

$$[\cosh(2\nu)-1]/2=\lambda . \quad (4.4)$$

Substituting (4.4) and (2.14), we get

$$M=\begin{bmatrix}u & 0\\ 0 & u^*\end{bmatrix}\begin{bmatrix}\cosh\nu & \sinh\nu\\ \sinh\nu & \cosh\nu\end{bmatrix}\begin{bmatrix}v & 0\\ 0 & v^*\end{bmatrix} . \quad (4.5)$$

Now from (4.5), (4.1), and (4.2) we find

$$\alpha(L)\equiv\ln O(L)=\frac{\nu(L)}{L} . \quad (4.6)$$

The theorem is verified if we can show that the matrix (4.6) self-averages in the large L limit. From Eq. (3.8) we can write an evolution equation for the expectation value

$$\langle F\rangle_s^{(\beta)}=\int F(\lambda)w_s^{(\beta)}(\lambda)d^N\lambda , \quad (4.7)$$

if we multiply both sides by $F(\lambda)$ and integrate over λ. The result is

$$\frac{\partial \langle F \rangle_s^{(\beta)}}{\partial s} = \frac{2}{\beta N + 2 - \beta} \times \left\langle \frac{1}{J_\beta} \sum_{a=1}^{N} \frac{\partial}{\partial \lambda_a} \left[\lambda_a (1+\lambda_a) J_\beta \frac{\partial F}{\partial \lambda_a} \right] \right\rangle_s^{(\beta)}, \tag{4.8}$$

in which $J_\beta \equiv J_\beta(\lambda)$ and $F \equiv F(\lambda)$. The right-hand side of this equation simplifies in the large s limit because distinct λ_a's become exponentially separated. We order the eigenvalues of $\nu(L)$ by

$$1 < \nu_N(L) < \nu_{N-1}(L) < \cdots < \nu_1(L) , \tag{4.9}$$

and find

$$\langle \nu_k(L) \rangle_s^{(\beta)} = \left[\frac{\beta N + 1 - \beta k}{\beta N + 2 - \beta} \right] \left[\frac{L}{l} \right], \quad k = 1,2,\ldots,N , \tag{4.10}$$

$$\mathrm{Var}\{\nu_k(L)\} = \frac{1}{\beta N + 2 - \beta} \left[\frac{L}{l} \right] . \tag{4.11}$$

Diagonalizing (4.6) and using (4.11), we obtain

$$\mathrm{Var}\{\alpha_k(L)\} = \frac{1}{L^2} \mathrm{Var}\{\nu_k(L)\} = \frac{1}{\beta N + 2 - \beta} \left[\frac{1}{Ll} \right] , \tag{4.12}$$

and, thus,

$$\lim_{L \to \infty} \mathrm{Var}\{\alpha_k(L)\} = 0 ,$$

as wanted. The Lyapunov exponents α_k are given by

$$\alpha_k = \lim_{L \to \infty} \alpha_k(L) = \frac{1}{l} \left[\frac{\beta N + 1 - \beta k}{\beta N + 2 - \beta} \right], \quad k = 1,2,\ldots,N . \tag{4.13}$$

Using (4.3) and (4.13), we can obtain the localization length

$$\xi(\beta) = (\beta N + 2 - \beta) l \tag{4.14}$$

which, in the limit of large number of channels, i.e., $N \gg 1$, yields

$$\xi(\beta) \simeq \beta \xi(1) , \tag{4.15}$$

and gives the universal multiplication factors that accompany the transitions between the ensembles.[17,33–35] If, for instance, time-reversal symmetry is broken in the presence of strong spin-orbit scattering, the localization length makes a transition $\xi \to \xi/2$.

The density of positive Lyapunov exponents in the limit $N \to \infty$ is, from (4.13), uniform on (0,1) and independent of β. The same behavior has been obtained in numerical calculations.[16,17]

In summary, the simplicity of the local maximum-entropy approach has enabled us to obtain rather complete results. These agree broadly with those obtained from the global maximum-entropy approach, but there are two differences that are worth highlighting: (i) the denominator, $\beta N + 2 - \beta$, of the expression for α_k, Eq. (4.13), is replaced by βN in the global approach;[19] (ii) the variance of $\alpha_N(L)$, Eq. (4.12) (which is physically the variance of the logarithm of conductance) takes half the present value in the global approach.[19]

V. THE METALLIC REGIME AND UNIVERSAL CONDUCTANCE FLUCTUATIONS

The metallic or diffusive regime is defined as the one in which the length L of the system is much larger than the mean free path l, but much smaller than the localization length $\xi(\beta) \sim \beta N l$, i.e.,

$$l \ll L \ll Nl \quad \text{or} \quad 1 \ll s \ll N . \tag{5.1}$$

We can thus, using (5.1), (4.8), and (3.15), write $\langle T \rangle_s^{(\beta)}$ as a power series of N, whose coefficients are functions of s. This procedure was introduced by Mello,[11] and was shown[13] to produce the same result as microscopic calculations for systems with lengths much greater than their widths L_t,

$$L \gg L_t . \tag{5.2}$$

The resulting $\langle T \rangle_s^{(\beta)}$ is

$$\langle T \rangle_s^{(\beta)} \simeq \frac{N}{(1+s)} - \frac{1}{3} \left[\frac{2-\beta}{\beta} \right] \frac{s^3}{(1+s)^3} + \frac{1}{45N} \left[\frac{2-\beta}{\beta} \right]^2 \frac{s^8}{(1+s)^7} - \frac{2}{45N} \frac{(\beta-1)(4-\beta)s^6}{\beta^2(1+s)^5} . \tag{5.3}$$

From conditions (5.1), (5.2), and the fact that $N \equiv (k_F L_t)^{d-1}$ we find, for $d \geq 3$, the condition

$$\sqrt{N} \ll s \ll N . \tag{5.4}$$

Now we can use (5.4) to asymptotically expand (5.3), yielding

$$\langle T \rangle_s^{(\beta)} \simeq \frac{Nl}{L} + \frac{\beta - 2}{3\beta} . \tag{5.5}$$

So, the same conditions that yield weak localization in the orthogonal ensemble [the correction, $(\beta-2)/3\beta$, is *negative* for $\beta = 1$], give weak antilocalization in the presence of sufficiently strong spin-orbit scattering, i.e., the symplectic case (the correction is *positive* for $\beta = 4$). We recall that this result is predicted by the weak-localization theory and arises from interference between time-reversed paths. In the presence of spin-orbit scattering, this interference is destructive in the backscattering direction and, therefore, enhances the conductance, as seen in (5.5).

In the same way used to obtain (5.3), we can calculate the second moment

$$\langle T^2\rangle_s^{(\beta)} \simeq \frac{N^2}{(1+s)^2} - \frac{2}{3}\left[\frac{2-\beta}{\beta}\right]\frac{Ns^3}{(1+s)^4} + \frac{13}{15}\left[\frac{2-\beta}{\beta}\right]^2\frac{s^8}{(1+s)^8} + \frac{2}{45}\frac{(\beta-1)(4-\beta)s^6}{\beta^2(1+s)^6} . \quad (5.6)$$

After squaring (5.3) we find

$$(\langle T\rangle_s^{(\beta)})^2 \simeq \frac{N^2}{(1+s)^2} - \frac{2}{3}\left[\frac{2-\beta}{\beta}\right]\frac{Ns^3}{(1+s)^4} + \frac{2}{45}\left[\frac{2-\beta}{\beta}\right]^2\frac{s^8}{(1+s)^8} - \frac{4}{45}\frac{(\beta-1)(4-\beta)s^6}{\beta^2(1+s)^6} + \frac{1}{9}\left[\frac{2-\beta}{\beta}\right]^2\frac{s^6}{(1+s)^6} . \quad (5.7)$$

We use condition (5.4) to asymptotically expand (5.6) and (5.7) and obtain

$$\mathrm{Var}\{T\} \simeq \frac{6}{45\beta^2}[(2-\beta)^2+(\beta-1)(4-\beta)] = \frac{2}{15\beta} . \quad (5.8)$$

For the conductance, $g=2T$, (restoring the factor 2 for Kramers degeneracy) we get

$$\mathrm{Var}\{g\} \simeq \frac{8}{15\beta} , \quad (5.9)$$

which gives precisely the factor $\frac{1}{4}$ (for the symplectic case) obtained by perturbative microscopic calculations.[8] This result, which motivates the term universal-conductance fluctuations, shows that the leading contribution to $\mathrm{Var}\{g\}$ depends only on the particular universality class of the system and, as a consequence of the macroscopic approach used here, does not depend on any information of a microscopic nature.

From Eqs. (3.16) and (3.17), and using (D5) and (D6) from Appendix D, we find for the first moment of the transmission and reflection coefficients, respectively,

$$\langle T_{ab}\rangle_s^{(\beta)} = \frac{\langle T\rangle_s^{(\beta)}}{N^2} , \quad (5.10)$$

$$\langle R_{ab}\rangle_s^{(\beta)} = \frac{\beta+\delta_{ab}(2-\beta)}{N(\beta N+2-\beta)}\langle R\rangle_s^{(\beta)} , \quad (5.11)$$

where

$$\langle R\rangle_s^{(\beta)} = N - \langle T\rangle_s^{(\beta)} . \quad (5.12)$$

One can easily see from (5.10) and (5.11) that

$$\langle T\rangle_s^{(\beta)} = \sum_{a,b}\langle T_{ab}\rangle_s^{(\beta)}$$

and

$$\langle R\rangle_s^{(\beta)} = \sum_{a,b}\langle R_{ab}\rangle_s^{(\beta)} ,$$

as expected. Equation (5.11) confirms the prediction of weak-localization theory and the comment just below (5.5), that the presence of sufficiently strong spin-orbit coupling ($\beta=4$) yields a backscattering depletion due to destructive quantum-interference effects. Observe that if $\beta=1$, the spinless orthogonal ensemble yields a backscattering enhancement, and that both effects are absent if time-reversal symmetry is broken by an applied magnetic field, i.e., for the unitary ensemble ($\beta=2$). Note also that from (5.12) we find for $\mathrm{Var}\{R\}$ the same result as in (5.8), i.e.,

$$\mathrm{Var}\{R\} \simeq \frac{2}{15\beta} . \quad (5.13)$$

We mention in passing that a lengthy but straightforward calculation as done in Ref. 13 with the results of Ref. 36 can be extended to the symplectic case to calculate the covariance of transmission and reflection coefficients. We do not undertake this.

VI. SUMMARY AND CONCLUSIONS

In this work we have defined the symplectic ensemble of random-transfer matrices in a quaternion representation, in order to take into account the effects of sufficiently strong spin-orbit scattering in the presence of time-reversal symmetry in quantum-transport theory for disordered conductors. Kramers degeneracy and the invariant measure characteristic of symplectic ensembles are natural consequences of the quaternion algebra. Some quasi-one-dimensional physical restrictions are imposed on the probability density in order to derive a Fokker-Planck equation, which describes a diffusion in the hyperbolic manifold associated with the symplectic ensemble. The quaternion representation greatly facilitates this derivation.

In the insulating regime, which is defined by the condition that the length of the sample is much greater than the localization length, we have shown that the system is characterized by quantities that self-average to a well-defined set of Lyapunov exponents in agreement with Oseledec's theorem.[29] In the limit of a large number of channels, the Lyapunov exponents were shown to have a uniform density independent of the ensemble. The universal multiplicative factors relating localization lengths in different ensembles, predicted by different macroscopic approaches,[17,33,34] were shown to be a direct consequence of the diffusion equation.

In the metallic regime, which is defined as the one in which the length of the system is much smaller than the localization length, but much larger than the mean free path, we obtained within the quasi-one-dimensional restrictions the expected weak antilocalization correction to the conductance, backscattering depletion, and universal-conductance fluctuations in agreement with diagrammatic calculations[8] and experimental observations.[1] The factor $\frac{1}{4}$ in the variance of the conductance is very striking and arises uniquely from the symmetries of the symplectic ensemble. We remark that our results in this so-called local maximum-entropy approach[9-13] agree with those of the global maximum-entropy approach,[14–19] which gives an ansatz for the probability density of the transfer matrix for the whole sample, instead of using a diffusion equation.

We conclude by observing that the conditions imposed to derive the diffusion equation limit the model to quasi-one-dimensional conductors, and that additional or different restrictions must be set up in order to give more realistic geometrical structure to the channels and, consequently, extend the model to higher dimensions.

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APPENDIX A: THE QUATERNION PARAMETRIZATION

An alternative representation for the symplectic case was recently derived in Ref. 30 and we shall follow very closely their method, adding the adaptions required by the quaternion algebra. It turns out that the quaternion representation provides significant simplifications.

Consider a transfer matrix M written in the form

$$M=\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}, \tag{A1}$$

where α, β, γ, and δ are $N\times N$ quaternion matrices. Conditions (2.4) and (2.5) imply that $\gamma=\beta^*$, $\delta=\alpha^*$, and

$$\alpha^\dagger\alpha-\bar{\beta}\beta^*=1 , \tag{A2a}$$

$$\alpha^\dagger\beta-\bar{\beta}\alpha^*=0 , \tag{A2b}$$

$$\alpha\alpha^\dagger-\beta\beta^\dagger=1 , \tag{A2c}$$

$$\alpha\bar{\beta}-\beta\bar{\alpha}=0 , \tag{A2d}$$

in which the matrix operations are defined in (2.11), (2.12), and (2.16). We now write α and β in the polar representation[37]

$$\alpha=ulv , \tag{A3a}$$

$$\beta=u'l'v' , \tag{A3b}$$

where u, u', v, and v' are $N\times N$ quaternion unitary matrices and l,l' are real, positive and diagonal $N\times N$ quaternion matrices. This decomposition is not unique, as can be easily seen by the transformation $u\to ud$ and $v\to d^{-1}v$ with $[d,l]=0$. Substituting (A3a) and (A3b) into (A2a)–(A2d), we get

$$v^\dagger l^2 v-\bar{v}'l'^2v'^*=1 , \tag{A4a}$$

$$v^\dagger l u^\dagger u'l'v'=\bar{v}'l'\bar{u}'u^*lv^* , \tag{A4b}$$

$$ul^2u^\dagger-u'l'^2u'^\dagger=1 , \tag{A4c}$$

$$ulv\bar{v}'l'\bar{u}'=u'l'v'\bar{v}l\bar{u} . \tag{A4d}$$

Equation (A4c) can be written as

$$w\mu w^\dagger=\mu' , \tag{A5}$$

where $w=u'^\dagger u$, $\mu=l^2-1$, and $\mu'=l'^2$. It is easy to verify that a w satisfying (A5) can only be of the form

$$w=Pd_1 , \tag{A6}$$

where P is a quaternion real permutation matrix and d_1 is a unitary matrix such that $[d_1,\mu]=[d_1,\mu']=0$. We then get

$$u'=ud_1^\dagger\bar{P} \tag{A7}$$

and

$$l'=P\sqrt{l^2-1}\bar{P} . \tag{A8}$$

Similarly from (A4a) we obtain

$$v'=Pd_2^*v^* , \tag{A9}$$

where d_2 is another unitary matrix, such that $[d_2,l]=[d_2,l']=0$. If we now substitute (A7)–(A9) into (A4b) and (A4d), we find the condition

$$s\equiv d_1^\dagger d_2^*=\bar{s} , \tag{A10}$$

which means that s is a unitary self-dual matrix and that $[s,l]=0$. So, starting from (A3a) we find that the most general β satisfying conditions (A2a)–(A2d) is

$$\beta=us\sqrt{l^2-1}v^* , \tag{A11}$$

but according to Ref. 30, any unitary self-dual matrix can be written as

$$s=k\bar{k} , \tag{A12}$$

where k is a unitary matrix. The decomposition (A12) is not unique and we can choose k such that $[k,l]=0$. Substituting (A12) into (A11), we find

$$\beta=(uk)\sqrt{l^2-1}(k^\dagger v)^* . \tag{A13}$$

Now we use the nonuniqueness of (A3a) and (A3b) to transform

$$uk\to u,\quad k^\dagger v\to v , \tag{A14}$$

and after defining $\lambda\equiv l^2-1$, we get

$$\alpha=u\sqrt{1+\lambda}v , \tag{A15a}$$

$$\beta=u\sqrt{\lambda}v^* . \tag{A15b}$$

Substituting (A15a) and (A15b) into (A1), we find (2.14), which proves that any transfer matrix satisfying flux conservation and time-reversal symmetry can be written in this form in the symplectic case.

APPENDIX B: THE INVARIANT MEASURE

In a similar way as was done for the orthogonal[9] and unitary[13] cases, the invariant measure for the symplectic case can be worked out from the observation that if the arc element

$$ds^2=\mathrm{Tr}(\Sigma_z dM^\dagger\Sigma_z dM)=\sum_{i,j=1}^{\nu} g_{ij}(x)dx_i dx_j \tag{B1}$$

is invariant under the transformation $M\to M_0MM_1$, where M_0 and M_1 are fixed transfer matrices, then the volume element

$$d\mu(M)=|\det g(x)|^{1/2}\prod_{i=1}^{\nu} dx_i \tag{B2}$$

is also left unchanged.

Using (2.14) and (B1) we obtain

$$ds^2=\mathrm{tr}\left[2(\delta u)^\dagger(\delta u)+2(\delta v)^\dagger(\delta v) -\frac{1}{2}\frac{d\lambda\, d\lambda}{\lambda(1+\lambda)}+4\,\mathrm{Re}\sqrt{1+\lambda}(\delta u)^\dagger\sqrt{1+\lambda}(\delta v) -4\,\mathrm{Re}\sqrt{\lambda}(\overline{\delta u})\sqrt{\lambda}(\delta v)\right], \tag{B3}$$

where $\delta u \equiv u^\dagger du$ and $\delta v \equiv dv v^\dagger$. As δu and δv are anti-Hermitian matrices, they can be decomposed into a sum of two quaternion real matrices,[31]

$$\delta u=\delta a+i\delta s\ , \tag{B4}$$

$$\delta v=\delta a'+i\delta s'\ , \tag{B5}$$

such that δa and $\delta a'$ are anti-self-dual and δs and $\delta s'$ are self-dual matrices, i.e., $\overline{\delta a}=-\delta a$, $\overline{\delta s}=\delta s$, and similarly for $\delta a'$ and $\delta s'$. Recalling that a self-dual (anti-self-dual) matrix can be expanded in the quaternion basis as a symmetric (antisymmetric) and three antisymmetric (symmetric) matrices, we can rewrite (B3) as

$$\begin{aligned} ds^2={}&4\sum_{a=1}^{N}[(\delta s_{aa}^{(0)})^2+(\delta s_{aa}'^{(0)})^2+2(1+2\lambda_a)\delta s_{aa}^{(0)}\delta s_{aa}'^{(0)}]+4\sum_{a=1}^{N}\sum_{i=1}^{3}[(\delta a_{aa}^{(i)})^2+(\delta a_{aa}'^{(i)})^2]\\ &+8\sum_{a<b}\sum_{i=0}^{3}[(\delta s_{ab}^{(i)})^2+(\delta s_{ab}'^{(i)})^2+2\Lambda_{ab}^{+}\delta s_{ab}^{(i)}\delta s_{ab}'^{(i)}]\\ &+8\sum_{a<b}\sum_{i=0}^{3}[(\delta a_{ab}^{(i)})^2+(\delta a_{ab}'^{(i)})^2+2\Lambda_{ab}^{-}\delta a_{ab}^{(i)}\delta a_{ab}'^{(i)}]-\sum_{a=1}^{N}\frac{(d\lambda_a)^2}{\lambda_a(1+\lambda_a)}\ , \end{aligned} \tag{B6a}$$

in which

$$\Lambda_{ab}^{\sigma}\equiv\sqrt{(1+\lambda_a)(1+\lambda_b)}+\sigma\sqrt{\lambda_a\lambda_b}\quad(\sigma=+,-)\ , \tag{B6b}$$

$\delta s^{(0)}$, $\delta a^{(i)}$, $\delta s'^{(0)}$ and $\delta a'^{(i)}$ $(i=1,2,3)$ are symmetric matrices and $\delta a^{(0)}$, $\delta s^{(i)}$, $\delta a'^{(0)}$, and $\delta s'^{(i)}$ $(i=1,2,3)$ are antisymmetric matrices. We mention that as M itself is invariant under the transformation (2.15), so is ds^2 and, therefore, $d\mu(M)$, and that we will preserve this symmetry for the reasons explained following (2.16).

Now, using (B1) and (B6a), we find

$$\det g \propto \prod_{a<b}|\lambda_a-\lambda_b|^8\ , \tag{B7}$$

and from (B2)

$$d\mu(M)=\prod_{a<b}|\lambda_a-\lambda_b|^4\prod_{a=1}^{N}d\lambda_a d\mu(u)d\mu(v)\ , \tag{B8}$$

where

$$d\mu(u)=\prod_{a=1}^{N}\delta s_{aa}^{(0)}\prod_{i=1}^{3}\delta a_{aa}^{(i)}\prod_{a<b}\prod_{i=0}^{3}\delta s_{ab}^{(i)}\delta a_{ab}^{(i)}\ , \tag{B9}$$

$$d\mu(v)=\prod_{a=1}^{N}\delta s_{aa}'^{(0)}\prod_{i=1}^{3}\delta a_{aa}'^{(i)}\prod_{a<b}\prod_{i=0}^{3}\delta s_{ab}'^{(i)}\delta a_{ab}'^{(i)}\ . \tag{B10}$$

We have thus proved (2.17).

APPENDIX C: THE DIFFUSION EQUATION

For a thorough derivation in the orthogonal and unitary cases the reader is referred to Refs. 9 and 13. We shall give a more concise derivation, just stressing the most important conceptual differences from previous work.

The isotropic assumption implies that $\rho_L(MM'^{-1})$ in (3.4) depends only on the eigenvalues of the matrix

$$\tilde{\lambda}=\lambda+w\ , \tag{C1}$$

in which

$$\begin{aligned} w={}&\sqrt{1+\lambda}(v'^\dagger\lambda' v')\sqrt{1+\lambda}+\sqrt{\lambda}(\bar{v}'\lambda' v'^*)\sqrt{\lambda}\\ &-\sqrt{1+\lambda}[v'^\dagger\sqrt{\lambda'(1+\lambda')}v'^*]\sqrt{\lambda}-\sqrt{\lambda}\\ &\times[\bar{v}'\sqrt{\lambda'(1+\lambda')}v']\sqrt{1+\lambda}\ . \end{aligned} \tag{C2}$$

Since δL is very small, the matrix w can be seen as a small perturbation $\delta\lambda$ to the diagonal matrix λ. Using perturbation theory, we find

$$\delta\lambda_a=\delta\lambda_a^{(1)}+\delta\lambda_a^{(2)}+\cdots\ , \tag{C3}$$

where

$$\delta\lambda_a^{(1)}=w_{aa}\ , \tag{C4a}$$

$$\delta\lambda_a^{(2)}=\sum_{b(\neq a)}\frac{w_{ab}w_{ba}}{\lambda_a-\lambda_b}\ . \tag{C4b}$$

We will be interested only in corrections up to first order in δL and, therefore, using (3.6), (C4a), and (C2), we get

$$\begin{aligned} \langle\delta\lambda_a^{(1)}\rangle_{\delta L}&=(1+2\lambda_a)\langle v_{ca}'^\dagger v_{ca}'\rangle_0\sum_{c=1}^{N}\langle\lambda_c'\rangle_{\delta L}\ ,\\ \langle\delta\lambda_a^{(1)}\rangle_{\delta L}&=(1+2\lambda_a)\frac{\delta L}{l}\ , \end{aligned} \tag{C5}$$

where we have used Eq. (D4) from Appendix D. Now, from (C2) we find

$$\langle w_{ab}w_{ba}\rangle_{\delta L}\simeq\sum_{c,d}(\langle z^c_{ab}k^d_{ba}\rangle_{\delta L}+\langle k^c_{ab}z^d_{ba}\rangle_{\delta L})\times\langle v'^{\dagger}_{ca}v'^{*}_{cb}\bar{v}'_{db}v'_{da}\rangle_0\,, \tag{C6}$$

in which we have defined

$$z^c_{ab}\equiv\sqrt{1+\lambda_a}\sqrt{\lambda'_c(1+\lambda'_c)}\sqrt{\lambda_b}\,, \tag{C7}$$

$$k^c_{ab}\equiv\sqrt{\lambda_a}\sqrt{\lambda'_c(1+\lambda'_c)}\sqrt{1+\lambda_b}\,, \tag{C8}$$

and the average over the quaternion unitary group is given by

$$\langle v'^{\dagger}_{ca}v'^{*}_{cb}\bar{v}'_{db}v'_{da}\rangle_0=\frac{2\delta_{c,d}}{N(2N-1)}\,, \tag{C9}$$

if $a\neq b$, as shown in Appendix D.

Substitute (C9) into (C6), and the result into (C4b), to get

$$\langle\delta\lambda^{(2)}_a\rangle_{\delta L}=\frac{2}{2N-1}\left[\frac{\delta L}{l}\right]\sum_{b\,(\neq a)}\frac{\lambda_a+\lambda_b+2\lambda_a\lambda_b}{\lambda_a-\lambda_b}\,. \tag{C10}$$

We will also need to calculate

$$\langle\delta\lambda_a\delta\lambda_b\rangle_{\delta L}\simeq\langle w_{aa}w_{bb}\rangle_{\delta L}\simeq\sum_{c,d}(\langle z^c_{aa}k^d_{bb}\rangle_{\delta L}+\langle k^c_{aa}z^d_{bb}\rangle_{\delta L})\times\langle v'^{\dagger}_{ca}v'^{*}_{ca}\bar{v}'_{db}v'_{db}\rangle_0\,,$$

for which, using formula (D6), we easily find

$$\langle\delta\lambda_a\delta\lambda_b\rangle_{\delta L}\simeq\frac{2}{2N-1}\delta_{ab}\lambda_a(1+\lambda_a)\left[\frac{\delta L}{l}\right]. \tag{C11}$$

Expanding both sides of (3.4) in powers of δL, we get

$$\frac{\partial\rho_L(\lambda)}{\partial L}\delta L\simeq\sum_{a=1}^{N}\frac{\partial\rho_L(\lambda)}{\partial\lambda_a}\langle\delta\lambda_a\rangle_{\delta L}+\frac{1}{2}\sum_{a,b}\frac{\partial^2\rho_L(\lambda)}{\partial\lambda_a\partial\lambda_b}\langle\delta\lambda_a\delta\lambda_b\rangle_{\delta L}+\cdots\,. \tag{C12}$$

Substituting (C11) and (C3) into (C12), and using the identities

$$\sum_{b(\neq a)}\frac{\lambda_a+\lambda_b+2\lambda_a\lambda_b}{\lambda_a-\lambda_b}=(1-N)(1+2\lambda_a)+2\lambda_a(1+\lambda_a)\sum_{b(\neq a)}\frac{1}{\lambda_a-\lambda_b}\,, \tag{C13}$$

$$\frac{\partial\ln J(\lambda)}{\partial\lambda_a}=4\sum_{b(\neq a)}\frac{1}{\lambda_a-\lambda_b}\,, \tag{C14}$$

where $J(\lambda)$ is given by (2.17b), we obtain

$$l\frac{\partial\rho_L(\lambda)}{\partial L}=\frac{1}{2N-1}\frac{1}{J(\lambda)}\times\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left[\lambda_a(1+\lambda_a)J(\lambda)\frac{\partial\rho_L(\lambda)}{\partial\lambda_a}\right]. \tag{C15}$$

Defining $w_s(\lambda)\equiv\rho_s(\lambda)J(\lambda)$ and $s\equiv L/l$, we finally get Eq. (3.7).

APPENDIX D: AVERAGES ON THE QUATERNION UNITARY GROUP

The averages needed in this paper can be easily obtained by observing that any element of an $N\times N$ quaternion unitary matrix can be written as

$$q_{ij}=\begin{bmatrix}u_{2i-1,2j-1} & u_{2i-1,2j}\\ u_{2i,2j-1} & u_{2i,2j}\end{bmatrix}, \tag{D1}$$

where u_{ab} are elements of a $2N\times 2N$ unitary matrix, for which the following relations hold:[36]

$$\langle u_{ab}u^*_{cd}\rangle_0=\frac{\delta_{ac}\delta_{bd}}{2N}\,, \tag{D2}$$

$$\langle u_{a'\alpha'}u_{b'\beta'}u^*_{a\alpha}u^*_{b\beta}\rangle_0=\frac{1}{4N^2-1}(\delta^a_{a'}\delta^b_{b'}\delta^\alpha_{\alpha'}\delta^\beta_{\beta'}+\delta^b_{a'}\delta^a_{b'}\delta^\beta_{\alpha'}\delta^\alpha_{\beta'})-\frac{1}{2N(4N^2-1)}(\delta^a_{a'}\delta^b_{b'}\delta^\beta_{\alpha'}\delta^\alpha_{\beta'}+\delta^b_{a'}\delta^a_{b'}\delta^\alpha_{\alpha'}\delta^\beta_{\beta'})\,. \tag{D3}$$

Now, using definition (2.11), together with (D2) and (D1), one finds

$$\langle q^\dagger_{ab}q_{cd}\rangle_0=\langle q_{ab}q^\dagger_{cd}\rangle_0=\frac{\delta_{ac}\delta_{bd}}{N}\,, \tag{D4}$$

and from definitions (2.11), (2.12), and (2.16), as well as with (D3) and (D1), we get

$$\langle q^\dagger_{ca}q^*_{cb}\bar{q}_{db}q_{da}\rangle_0=\frac{2\delta_{cd}}{N(2N-1)}\quad\text{if } a\neq b\,, \tag{D5}$$

and

$$\langle q^\dagger_{ca}q^*_{ca}\bar{q}_{db}q_{db}\rangle_0=\frac{\delta_{cd}\delta_{ab}}{N(2N-1)}\,. \tag{D6}$$

Higher-order averages can be obtained in the same way.

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