Vortex dynamics in a type-II superconducting film and complex linear-response functions

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The dynamics of interacting vortices in a type-II superconducting film responding to a distribution of currents parallel to the film is investigated. The nonlocality of vortex interactions is taken into account by use of an expansion in normal modes of the flux-line lattice and a wave-vector-dependent dynamical matrix. The theory is applied to obtain linear response functions including the complex mutual inductance and film impedance. For a particular geometry of driving current and pickup coil, the complex mutual inductance is calculated and it is shown that only longitudinal modes contribute to the measurable change in flux when pinning is absent or when it is represented as a linear restoring force acting the same way on each vortex. The configuration so described provides a sensitive means of studying vortex dynamics.

I. INTRODUCTION

Complex response functions such as impedance, susceptibility, self- and mutual inductance, and conductivity form a related family that can be used to describe the electrodynamic response of superconductors. For instance, in the absence of vortices, the imaginary part of the impedance (or conductivity) contains information on the intrinsic (or London) penetration depth, and the real part of the impedance (or conductivity) contains information on the quasiparticle excitations. Similarly, the real part of the susceptibility (or inductance) is connected with the diamagnetic response or intrinsic penetration depth, while the imaginary part of the susceptibility (or inductance) is associated with power dissipation.[1] Sufficiently detailed measurements of response functions then allow the possibility of deriving basic information concerning the superconducting pairing state and pair concentration. Moving vortices are known to influence the response functions, sometimes very strongly, when a type-II superconductor is in the mixed state. An effect of oscillating and flowing vortices is to alter the effective penetration depth for small-amplitude time-varying electromagnetic fields.

In this paper we consider the response to external driving currents of a vortex lattice in a type-II superconducting film. This theory is then applied to obtain various linear response functions, including the film impedance and the complex mutual inductance between the drive coil and a receive coil. Such a study is all the more pertinent since high-quality thin films of high-temperature superconductors can now be made. Of these, commercial applications in electronic devices may be possible.[2] The theory here complements that of Refs. 3 and 4, where an external microwave magnetic field applied parallel to a superconductor surface results in a driving current density also parallel to the surface. In the present geometry the external driving magnetic field has components both perpendicular and parallel to the film surface. Such an arrangement is realized in practice with two displaced coaxial coils[5–9] whose axes are normal to the film's surface.

In the following section we derive general results for the film's linear response when an external drive-coil current distribution flows parallel to the film. We calculate the response initially in the absence of vortices, giving the Meissner response of the film. We use Fourier transforms extensively throughout the paper, and Sec. II contains our conventions. In Sec. III, our general results are specialized for a drive coil producing an oscillating magnetic dipole field. In particular, we show that the dipole moment generated by the currents induced in an infinite superconducting film is exactly opposite that of the driving dipole. The resulting magnetic field can then be discussed in terms of the driving and induced dipoles. In Sec. IV, we calculate the response in the presence of a vortex lattice in the film. In this treatment we calculate the total current density induced in the film, and include all interactions between vortices by calculating the Lorentz force on vortices self-consistently. We also calculate the resulting magnetic field throughout all space. In Sec. V, we obtain the complex dynamic mutual inductance between a drive coil above and pickup coil above or below the film.

The structure of an isolated vortex in a superconducting film was investigated in Ref. 10. Results for the vortex vector potential and supercurrent and their Fourier transforms are used here. The nonlocality of vortex interactions is taken into account by use of an expansion in normal modes of the flux-line lattice and a wave-vector-dependent dynamical matrix. For a particular geometry of circular driving current distribution and circular pickup coil, it is shown that only longitudinal modes contribute to the measured mutual inductance. The flux relationships derived for the inductance make this calculation similar to, although more complicated than, that for finding the complex permeability of a type-II superconducting slab with a parallel driving field.[3] In the succeeding section, an additional response function is calculated, namely, the complex film impedance. The present theory for a two-coil mutual inductance method provides a sensitive technique for measuring superconducting sheet impedances and penetration depth.[7]

The frequency of the driving current is assumed to be well below the gap frequency so that pairs of supercurrent carriers are not broken. We also assume the effects of vortex inertia[11,12] to be negligible. Such effects are probably significant only for frequencies near the gap frequency.[11-14] Under these assumptions, a good approximation to the electrodynamic response of the superconducting film is given by the quasi-static London theory, where a local relation exists between the supercurrent density and vector potential[15] in the absence of vortices.

In the following we include the quasiparticle contribution to the current density by using a two-fluid model. This feature allows our results to be continuously valid through the transition temperature, and also in principle through the upper critical field for high-temperature superconductors. The two-fluid inclusion then allows the description of eddy currents in the normal state. In this sense, our treatment generalizes eddy current probes to the superconducting state. Since such probes have found valuable applications in nondestructive evaluation of metals[16] we may expect similar applications for type-II superconducting films.[17] In fact, measurements of high-T_c film transition temperature and critical current density have been made in a contactless way with these procedures.[17] Especially since high-T_c film preparation and processing have been constantly changing and the material properties are not always easily reproducible, nondestructive testing of the films is a desirable method.

An important early work on analytical solutions for eddy-current problems in axially symmetric geometry is that of Dodd and Deeds.[16] Our normal-state mutual inductance expression involving integrals of first-order Bessel functions corresponds to what would be derived from the solution for the magnetic field in planar geometry by Dodd and Deeds.

An analysis of sheet impedance based on an integral equation linking the total vector potential and sheet current density was used in Refs. 5–7. A numerical solution of this equation gives the complex mutual inductance and sheet impedance. The response of a vortex lattice was considered as well as the possibility of a Kosterlitz–Thouless-type transition.[5-7]. Another approach to the electromagnetic response of thin-film superconductors was used in Ref. 18. There, an analysis based on the BCS electrodynamical kernel was developed for calculating the surface impedance. The wave-vector dependence of the microwave-superconductor interaction was examined by variation of the film thickness and mean free path. The dominant wavelengths for the BCS kernel were of the order of the coherence length, whereas here they are either of the order of the London penetration depth or of macroscopic size. The vortex response was not included in the analysis of Ref. 18.

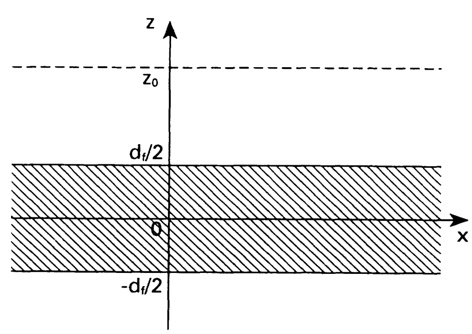

FIG. 1. Sketch of the film geometry studied in this paper. The superconducting film (crosshatched) is in the region $|z| < d_f/2$, and the external drive coil is in the space $z > z_0$.

II. MEISSNER-RESPONSE CURRENTS IN AND FIELDS ABOUT A SUPERCONDUCTING FILM DUE TO A DISTRIBUTION OF PARALLEL DRIVING CURRENTS

For specificity, we consider the following geometry (see Fig. 1). The superconducting film of thickness d_f is centered on the xy plane. The film, with surfaces at $z=\pm d_f/2$, is taken to extend infinitely in the x and y directions. The external drive currents are assumed to be confined to the semi-infinite region of space $z \geq z_0 > d_f/2$. The regions $d_f/2 \leq z < z_0$ above and $z \leq -d_f/2$ below the film are current-free.

We consider a single frequency component of the ac driving current, given by

$$\mathbf{J}_d(\mathbf{r},t) \equiv \mathbf{J}_0(\boldsymbol{\rho},z)e^{-i\omega t}, \tag{2.1}$$

where $\boldsymbol{\rho}=\hat{\mathbf{x}}x+\hat{\mathbf{y}}y$. The currents are assumed to flow parallel to the xy plane; $\hat{\mathbf{z}}\cdot\mathbf{J}_0(\boldsymbol{\rho},z)=0$. We are initially interested in the electromagnetic behavior of the superconductor in the Meissner state. This will be useful as a starting point when vortices are introduced. The perturbations in the fields and current densities due to moving vortices will be taken up in Sec. IV.

Here we derive general results for an external current distribution flowing parallel to the film. The external currents generate a magnetic field, which is screened by the superconducting film. It is desired to calculate the total (or net) current density in the film, from which the Lorentz force on vortices can be obtained, together with the magnetic field throughout all space. Since the film is assumed to be of infinite extent in two dimensions (2D), a Fourier transform approach to finding the current density and vector potential is advantageous. We employ the 2D Fourier transforms defined as

$$\mathbf{f}(\boldsymbol{\rho},z)=\int \frac{d^2q}{(2\pi)^2}\mathbf{f}(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}}, \tag{2.2a}$$

$$\mathbf{f}(\mathbf{q},z)=\int d^2\rho\, \mathbf{f}(\boldsymbol{\rho},z)e^{-i\mathbf{q}\cdot\boldsymbol{\rho}}. \tag{2.2b}$$

We use the usual convention in which the argument $\boldsymbol{\rho}$ or $\mathbf{q}$ distinguishes a function from its Fourier transform. Because of the parallel driving current distribution assumption, $J_{0z}(\mathbf{q},z)=0$. In addition, from

$$\nabla\cdot\mathbf{J}_0(\boldsymbol{\rho},z)=0 \tag{2.3}$$

we have, by taking the Fourier transform,

$$\mathbf{J}_0(\mathbf{q},z)=\hat{\mathbf{q}}_t J_{0t}(\mathbf{q},z)\ , \tag{2.4}$$

where $\hat{\mathbf{q}}_t=\hat{\mathbf{z}}\times\hat{\mathbf{q}}$ is the transverse wave vector. (The vectors $\hat{\mathbf{z}}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{q}}_t$ form an orthonormal triad used henceforth.) We will see that relation (2.3) implies that the magnetic field can be described by a transverse potential. In the following λ denotes the field-dependent penetration depth[19] and $\Lambda=2\lambda^2/d_f$ denotes the 2D screening length appropriate for thin films.[10,20,21]

A. Image fields

When the film is very thick, an excellent approximation to the fields is provided by the method of images. In the limit of $d_f \gg \lambda$ and $z_0 \gg \lambda$, the magnetic field distribution above the superconductor is given to good approximation by a vector sum $\mathbf{a}_0(\rho,z)=\mathbf{A}_0(\mathbf{p},z)+\mathbf{A}_0^i(\boldsymbol{\rho},z)$ of the vector potential $\mathbf{A}_0(\boldsymbol{\rho},z)$ generated by $\mathbf{J}_0(\boldsymbol{\rho},z)$ in the absence of the superconductor and the image potential $\mathbf{A}_0^i(\boldsymbol{\rho},z)$ generated by an image current density $\mathbf{J}_0^i(\boldsymbol{\rho},z)$. The image current density is given by $\mathbf{J}_0^i(\boldsymbol{\rho},z)=-\mathbf{J}_0(\boldsymbol{\rho},d_f-z)$, i.e., the reflection of $\mathbf{J}_0(\boldsymbol{\rho},z)$ in the top surface of the superconductor. By this image construction, sketched in Fig. 2, the field satisfies the boundary condition $b_z=0$ at $z=d_f/2$. When the conditions $d_f \gg \lambda$ and $z_0 \gg \lambda$ do not hold, the screening produced by the superconductor is less complete, and correction terms need to be added to $\mathbf{a}_0$. We solve for these terms below after finding $\mathbf{a}_0$.

Setting

$$\mathbf{a}_0(\boldsymbol{\rho},z)=\mathbf{A}_0(\boldsymbol{\rho},z)+\mathbf{A}_0^i(\boldsymbol{\rho},z)\ , \tag{2.5a}$$

$$\mathbf{j}_0(\boldsymbol{\rho},z)=\mathbf{J}_0(\boldsymbol{\rho},z)+\mathbf{J}_0^i(\boldsymbol{\rho},z)\ , \tag{2.5b}$$

and recalling that[24]

$$\mathbf{A}_0(\boldsymbol{\rho},z)=\frac{\mu_0}{4\pi}\int d^3r'\frac{\mathbf{J}_0(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ , \tag{2.6}$$

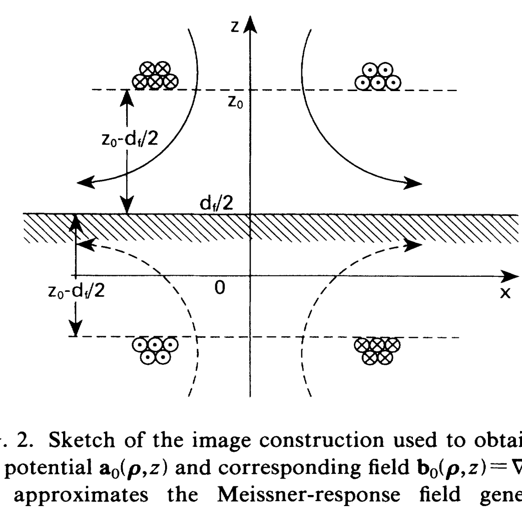

FIG. 2. Sketch of the image construction used to obtain the vector potential $\mathbf{a}_0(\boldsymbol{\rho},z)$ and corresponding field $\mathbf{b}_0(\boldsymbol{\rho},z)=\nabla\times\mathbf{a}_0$, which approximates the Meissner-response field generated above the superconductor ($z>d_f/2$) by current density distribution $\mathbf{J}_0(\boldsymbol{\rho},z)$ in the drive coil ($z>z_0$). Several magnetic field lines generated by $\mathbf{J}_0(\boldsymbol{\rho},z)$ and the image current density distribution $\mathbf{J}_0^i(\boldsymbol{\rho},z)=-\mathbf{J}_0(\boldsymbol{\rho},d_f-z)$ are shown.

we have

$$\mathbf{a}_0(\boldsymbol{\rho},z)=\frac{\mu_0}{4\pi}\int d^2\rho'\int dz'\frac{\mathbf{j}_0(\boldsymbol{\rho}',z')}{[(\boldsymbol{\rho}-\boldsymbol{\rho}')^2+(z-z')^2]^{1/2}}\ . \tag{2.7}$$

By Fourier transforming and using integration results for Bessel functions[22,23] we have

$$\mathbf{a}_0(\mathbf{q},z)=\hat{\mathbf{q}}_t a_{0t}(\mathbf{q},z)\ , \tag{2.8a}$$

where

$$a_{0t}(\mathbf{q},z)=\frac{\mu_0}{2q}\int_{-\infty}^{\infty}dz' j_{0t}(\mathbf{q},z')e^{-q|z-z'|}\ . \tag{2.8b}$$

In obtaining Eq. (2.8) we have made use of $\mathbf{j}_0(\mathbf{q},z)=\hat{\mathbf{q}}_t j_{0t}(\mathbf{q},z)$, which follows from Eq. (2.4) for the driving current density $\mathbf{J}_0$, and similar relations for the image current density $\mathbf{J}_0^i$ and the sum $\mathbf{j}_0=\mathbf{J}_0+\mathbf{J}_0^i$. Equation (2.8a) shows that the vector potential is transverse, with the defining Fourier transform property $\hat{\mathbf{q}}\cdot\mathbf{a}_0(\mathbf{q},z)=0$.[24] Noting that $J_{0t}(\mathbf{q},z)=0$ for $z<z_0$, we find that Eq. (2.8b) can be written more explicitly as

$$a_{0t}(\mathbf{q},z)=\frac{\mu_0}{2q}\int_{z_0}^{\infty}dz' J_{0t}(\mathbf{q},z')(e^{-q|z-z'|}-e^{-q|z-d_f+z'|})\ . \tag{2.8c}$$

where $z\geq d_f/2$.

B. Correction fields

As noted above, for arbitrary values of d_f and z_0, the vector potential (2.5a) needs to be modified to account for the incomplete screening of the superconductor. In the region $z\geq d_f/2$ we take the vector potential to be

$$a_>(\boldsymbol{\rho},z)=\mathbf{a}_0(\boldsymbol{\rho},z)+\mathbf{a}_1(\boldsymbol{\rho},z)\ , \tag{2.9}$$

where $\mathbf{a}_0(\boldsymbol{\rho},z)$ is the vector potential generated by the driving current $\mathbf{J}_0$ and its image $\mathbf{J}_0^i$, and the function $\mathbf{a}_1(\boldsymbol{\rho},z)$ is to be determined. Because $\mathbf{a}_0$ includes the effects of the driving current density and $\mathbf{a}_1$ is transverse, $\mathbf{a}_1(\boldsymbol{\rho},z)$ satisfies the Laplace equation in the region above the film. Using the 2D Fourier transform (2.2) we then obtain the differential equation

$$\left[-q^2+\frac{\partial^2}{\partial z^2}\right]\mathbf{a}_1(\mathbf{q},z)=0\ . \tag{2.10}$$

Choosing the solution which decays as $z\to\infty$, we can write

$$\mathbf{a}_1(\mathbf{q},z)=\hat{\mathbf{q}}_t a_{1t}(\mathbf{q})e^{-q(z-d_f/2)}\ . \tag{2.11}$$

For fields within the superconducting film, we use the subscript f. The vector potential in the film is $\mathbf{a}_f$, and the local magnetic induction in the film is $\mathbf{b}_f=\nabla\times\mathbf{a}_f$. At low frequencies and temperatures the response of the film is almost purely inductive, and the fields may be assumed to obey the London equation, where spatial variation of the fields is governed by a real penetration depth λ. At higher frequencies and temperatures, however, dissipation from excited quasiparticles (the normal fluid) gives rise to an imaginary part of the penetration depth. We approximate this effect by using the two-fluid model, in which the total electrical current density is expressed as $\mathbf{j}_f=\mathbf{j}_{fs}+\mathbf{j}_{fn}$, where the superfluid component obeys $\mathbf{j}_{fs}=-\mathbf{a}_f/\mu_0\lambda^2$ and the normal-fluid component obeys $\mathbf{j}_{fn}=\sigma_{nf}\mathbf{e}_f$. The conductivity of the normal fluid σ_{nf} can be modeled as $\sigma_{nf}=(T/T_c)^4\sigma_n$ at temperatures T below T_c, where σ_n is the normal-state conductivity. Combining these equations using Ampere's law ($\mu_0\mathbf{j}_f=\nabla\times\mathbf{b}_f$) and Faraday's law ($\mathbf{e}_f=-\dot{\mathbf{a}}_f=i\omega\mathbf{a}_f$), and taking the Fourier transform, we obtain

$$\left[-q^2+\frac{\partial^2}{\partial z^2}\right]\mathbf{a}_f(\mathbf{q},z)=\frac{1}{\lambda_\omega^2}\mathbf{a}_f(\mathbf{q},z)\ , \tag{2.12}$$

where $\lambda_\omega^{-2}=\lambda^{-2}-2i\delta_{nf}^{-2}$ and $\delta_{nf}^2\equiv 2/\mu_0\omega\sigma_{nf}$ is the normal-fluid skin depth. At very low temperature, the complex penetration depth λ_ω reduces to the London penetration depth λ, while for $T\to T_{c2}(H)$ or $B\to B_{c2}(T)$, the upper critical field, $\lambda(B,T)$ diverges, $\delta_{nf}\to\delta_n=(2/\mu_0\omega\sigma_n)^{1/2}$, and the normal-state value $\lambda_\omega\to(1+i)\delta_n/2$ is obtained. [Here $T_{c2}(H)$ is the field-dependent transition temperature and $\delta_n\equiv(2/\mu_0\omega\sigma_n)^{1/2}$ is the normal-state skin depth.] Once the normal state is attained, our results describe the response of induced eddy currents in the film. The solution of Eq. (2.12) may be written as

$$\mathbf{a}_f(\mathbf{q},z)=\hat{\mathbf{q}}_t[a_{fst}(\mathbf{q})\sinh Qz+a_{fct}(\mathbf{q})\cosh Qz]\ , \tag{2.13}$$

where $Q=(q^2+\lambda_\omega^{-2})^{1/2}$.

In the region below the superconductor, $z\le -d_f/2$, the vector potential is $\mathbf{a}_<(\boldsymbol{\rho},z)$, which satisfies Laplace's equation. Then $\mathbf{a}_<$ satisfies an equation of the form (2.10), so that

$$\mathbf{a}_<(\mathbf{q},z)=\hat{\mathbf{q}}_t a_{<t}(\mathbf{q})e^{q(z+d_f/2)}\ . \tag{2.14}$$

By applying boundary conditions at the film surfaces, we can evaluate the q-dependent coefficients a_{1t}, a_{fst}, a_{fct}, and $a_{<t}$. We require the continuity of both the vector potential and magnetic field at the planes $z=\pm d_f/2$. Enforcing the continuity of the (2D) Fourier transforms of these two functions provides the four needed equations. For reference, we here give the form of the Fourier transform of the magnetic field in each of the three regions: above, in, and below the superconductor.

Above the superconductor, we have the field $\mathbf{b}_>(\boldsymbol{\rho},z)=\mathbf{b}_0(\boldsymbol{\rho},z)+\mathbf{b}_1(\boldsymbol{\rho},z)$, where $\mathbf{b}_0=\nabla\times\mathbf{a}_0(\boldsymbol{\rho},z)$ and $\mathbf{b}_1=\nabla\times\mathbf{a}_1(\boldsymbol{\rho},z)$. Using Eqs. (2.8c) and (2.11) and Fourier transforming, we have

$$\mathbf{b}_0(\mathbf{q},z)=\hat{\mathbf{q}}b_{0l}(\mathbf{q},z)+\hat{\mathbf{z}}b_{0z}(\mathbf{q},z)\ , \tag{2.15}$$

where

$$b_{0l}(\mathbf{q},z)=-\frac{\mu_0}{2}\int_{z_0}^{\infty}dz' J_{0t}(\mathbf{q},z')\times(e^{-q(z'-z)}+e^{-q(z'-d_f+z)})\ , \tag{2.16a}$$

$$b_{0z}(\mathbf{q},z)=\frac{\mu_0 i}{2}\int_{z_0}^{\infty}dz' J_{0t}(\mathbf{q},z')(e^{-q(z'-z)}-e^{-q(z'-d_f+z)}), \tag{2.16b}$$

[note that $b_{0z}(\mathbf{q},d_f/2)=0$] and

$$\mathbf{b}_1(\mathbf{q},z)=\hat{\mathbf{q}}b_{1l}(\mathbf{q},z)+\hat{\mathbf{z}}b_{1z}(\mathbf{q},z)\ , \tag{2.17}$$

where

$$b_{1l}(\mathbf{q},z)=qa_{1t}(\mathbf{q})e^{-q(z-d_f/2)}\ , \tag{2.18a}$$

$$b_{1z}(\mathbf{q},z)=iqa_{1t}(\mathbf{q})e^{-q(z-d_f/2)}\ . \tag{2.18b}$$

Inside the superconductor we have, using Eq. (2.13),

$$\mathbf{b}_f(\mathbf{q},z)=\hat{\mathbf{q}}b_{fl}(\mathbf{q},z)+\hat{\mathbf{z}}b_{fz}(\mathbf{q},z)\ , \tag{2.19}$$

where

$$b_{fz}(\mathbf{q},z)=-Q[a_{fst}(\mathbf{q})\cosh Qz+a_{fct}(\mathbf{q})\sinh Qz]\ , \tag{2.20a}$$

$$b_{fl}(\mathbf{q},z)=iq[a_{fst}(\mathbf{q})\sinh Qz+a_{fct}(\mathbf{q})\cosh Qz]\ . \tag{2.20b}$$

Below the superconductor we have, using Eq. (2.14),

$$\mathbf{b}_<(\mathbf{q},z)=\hat{\mathbf{q}}b_{<l}(\mathbf{q},z)+\hat{\mathbf{z}}b_{<z}(\mathbf{q},z)\ , \tag{2.21}$$

where

$$b_{<l}(\mathbf{q},z)=-qa_{<t}(\mathbf{q})e^{q(z+d_f/2)}\ , \tag{2.22a}$$

$$b_{<z}(\mathbf{q},z)=iqa_{<t}(\mathbf{q})e^{q(z+d_f/2)}\ . \tag{2.22b}$$

In order to enforce continuity, we apply equations (2.5a), (2.9), (2.11), (2.13), and (2.14) for the vector potential and Eqs. (2.15), (2.17), (2.19), and (2.21) for the magnetic field at $z=\pm d_f/2$. In writing the solution for the coefficients we define

$$\alpha(\mathbf{q})\equiv\frac{\mu_0}{q}\int_{z_0}^{\infty}dz' J_{0t}(\mathbf{q},z')e^{-q(z'-d_f/2)}\ . \tag{2.23}$$

We then have

$$a_{1t}=\frac{\alpha(\mathbf{q})}{2}\left[\frac{1+(q/Q)\tanh Qd_f}{1+[(Q^2+q^2)/2qQ]\tanh Qd_f}\right]\ , \tag{2.24a}$$

$$a_{fst}(\mathbf{q})=\frac{q\alpha(\mathbf{q})/2}{Q\cosh(Qd_f/2)+q\sinh(Qd_f/2)}\ , \tag{2.24b}$$

$$a_{fct}(\mathbf{q})=\frac{q\alpha(\mathbf{q})/2}{Q\sinh(Qd_f/2)+q\cosh(Qd_f/2)}\ , \tag{2.24c}$$

$$a_{<t}(\mathbf{q})=\frac{\alpha(\mathbf{q})}{2}\frac{\operatorname{sech}Qd_f}{\{1+[(Q^2+q^2)/2qQ]\tanh Qd_f\}}\ . \tag{2.24d}$$

In the limit of vanishing London penetration depth λ, where $Q\to\infty$, we have a_{1t}, a_{fst}, a_{fct}, $a_{<t}\to 0$. This provides a partial check on the solution for the correction fields. The expressions (2.24) completely specify the vector potential for all values of z, as well as for all values of d_f, z_0, and λ_ω. We have therefore solved for all the Meissner-response fields everywhere as generated by the

driving currents $\mathbf{J}_d(\mathbf{r},t)$.

Note that, in performing this calculation, we also have solved for the fields everywhere when the film is normal and the currents in the film are induced eddy currents. We have only to set $\lambda=\infty$ and $\delta_{nf}=\delta_n$ in the expression for λ_ω such that $Q=(q^2-2i\delta_n^{-2})^{1/2}$ in Eqs. (2.24), where $\delta_n^2=2/\mu_0\omega\sigma_n$ for nonmagnetic normal metals. (For strongly magnetic metals with magnetic permeability $\mu\gg1$, we must use the expression $\delta_n^2=2/\mu_0\mu\omega\sigma_n$.)

C. Current density and net Lorentz force

We next solve for the induced current density in the film $\mathbf{j}_f$ in preparation for a calculation of the force per unit length on a vortex arising from the interaction with $\mathbf{J}_d(\mathbf{r},t)$. Writing the London equation in Fourier space we have

$$\mathbf{j}_f(\mathbf{q},z)=-(1/\mu_0\lambda_\omega^2)\mathbf{a}_f(\mathbf{q},z)\ , \tag{2.25}$$

where $\mathbf{a}_f$ is given by Eqs. (2.13), (2.24b), and (2.24c). Letting the surface current be

$$\mathbf{K}_f(\boldsymbol{\rho})=\int_{-d_f/2}^{d_f/2}dz\ \mathbf{j}_f(\boldsymbol{\rho},z)\ , \tag{2.26}$$

we find that the electromagnetic force on a vortex at $\boldsymbol{\rho}$ arising from the Meissner currents induced by driving currents is

$$\mathbf{F}_d(\boldsymbol{\rho})=\mathbf{K}_f(\boldsymbol{\rho})\times\boldsymbol{\phi}_0\ , \tag{2.27}$$

where ϕ_0 is the flux quantum. By Fourier transformation we obtain

$$\mathbf{F}_d(\mathbf{q})=-(\phi_0/\mu_0)q\alpha(\mathbf{q})f_1(q)f_2(q)\hat{\mathbf{q}}\ , \tag{2.28}$$

where $\alpha(\mathbf{q})$ is given in Eq. (2.23) and the functions f_1 and f_2 are defined by

$$f_1(q)=Q^{-2}\lambda_\omega^{-2}\ ,\qquad f_2(q)=[1+(q/Q)\coth(Qd_f/2)]^{-1}\ . \tag{2.29}$$

We then have in real space

$$\mathbf{F}_d(\boldsymbol{\rho})=-\int\frac{d^2q}{(2\pi)^2}\int d^2\rho'\int_{d_f/2}^{\infty}dz'\mathbf{J}_0(\boldsymbol{\rho}',z')\times\boldsymbol{\phi}_0 f_1(q)f_2(q)e^{-q(z'-d_f/2)}e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}\ . \tag{2.30}$$

In the case of a thick film, where $d_f\gg|\lambda_\omega|$, we have $f_1(q)f_2(q)\to(Q-q)/Q$.

III. CURRENTS AND FIELDS NEAR A SUPERCONDUCTING FILM GENERATED BY AN OSCILLATING DIPOLE

To demonstrate how this method can be applied, we now specialize the results of the previous sections to a certain driving current density distribution $\mathbf{J}_0$. We consider the case of a single-turn coil of radius R_d parallel to the superconducting film. The coil is located in the plane $z=z_d$, such that $z_d=d_f/2+D_d$. We further assume a single-turn pickup coil of radius R_p is located a distance D_p from the bottom surface of the film, i.e., in the plane $z_p=-d_f/2-D_p$ (Fig. 3). We easily can obtain the vector potential, from which the magnetic field, current density in the film, and Lorentz force on a vortex can be derived. These results are to be used later for finding the mutual inductance between the driver coil and the pickup coil located below a superconducting film that contains vortices.

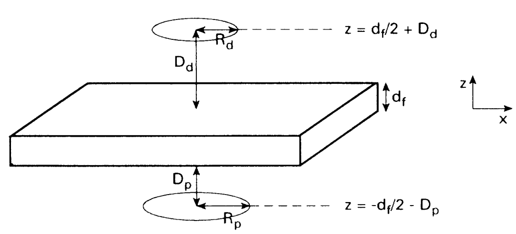

FIG. 3. Geometry of two-coil mutual inductance configuration considered in Secs. III and V. The drive coil in the plane $z=d_f/2+D_d$ has radius R_d and the pickup coil in the plane $z=-d_f/2-D_p$ has radius R_p.

The current density in the coil is

$$\mathbf{J}_d(\mathbf{r},t)=\mathbf{J}_0(\rho,z)e^{-i\omega t}\ , \tag{3.1}$$

where

$$\mathbf{J}_0(\rho,z)=\hat{\boldsymbol{\phi}}I\delta(\rho-R_d)\delta(z-z_d)\ . \tag{3.2}$$

Its 2D Fourier transform is[22]

$$\mathbf{J}_0(\mathbf{q},z)=\hat{\mathbf{q}}_t J_{0t}(q,z)\ , \tag{3.3}$$

where

$$J_{0t}(q,z)=-2\pi iJ_1(qR_d)IR_d\delta(z-z_d)\ . \tag{3.4}$$

(Here J_1 is the first-order Bessel function.[22]) The quantity $\alpha(\mathbf{q})$ of Eq. (2.23) furnishes both the vector potential via Eq. (2.24) and the Fourier transform of the Lorentz force on a vortex in the film via Eq. (2.28). For the current distribution (3.2) we have

$$\alpha(q)=-i(2\pi\mu_0IR_d/q)J_1(qR_d)e^{-qD_d}\ . \tag{3.5}$$

We note that expressions such as (3.4) and (3.5) considerably simplify when the radius of the drive coil becomes very small. For instance, using $J_1(x)\to x/2$ as $x\to0$,[22] we have

$$\alpha(q)\simeq-i\mu_0me^{-qD_d}\ , \tag{3.6}$$

where $m=\pi R_d^2I$ is the magnetic dipole moment of the driver loop. Thus the important wave vectors introduced by the dipole obey $0\le q\le1/D_d$, where we assume that $D_d\gg|\lambda_\omega|$.

Of interest for finding the magnetic flux through the pickup coil is the nature of the field below the film. It is appropriate to think of the vector potential in this region, $\mathbf{a}_<(\rho,z)$, as the vector potential generated by two current distributions, that of the original driving dipole and that of a dipole generated by the induced current distribution in the film. The currents in the film generate a dipole moment[24]

$$\mathbf{m}_f=\tfrac{1}{2}\int_{\text{film}}\mathbf{r}\times\mathbf{j}_f d^3r \ . \tag{3.7}$$

From Eqs. (2.13) and (2.25) and the London equation we see that the dipole moment $\mathbf{m}_f$ has only a z component. By using Fourier representations, we calculate m_{fz} in Appendix A and show it to be $m_{fz}=-m$. Therefore the induced dipole moment $\mathbf{m}_f=-\mathbf{m}$ exactly. In Appendix B, we describe the resulting quadrupole character of the magnetic field at large distances from the film. The quadrupole moment is calculated explicitly in both the thin- and thick-film limits.

In summary, the magnetic field distribution far from the superconductor is to lowest approximation that of two opposed dipoles of magnitude m. At large distances, in both the regions above and below the film, due to cancellation from the image dipole and driving dipole, there remains only a quadrupole field. To lowest approximation the magnetic field is parallel to the surface.

IV. VORTEX LATTICE RESPONSE INCLUDING THE EFFECTS OF PINNING

A. Expansion of vortex positions in normal modes

We assume that vortices in the superconducting film form a 2D lattice. The elastic response of a flux-line lattice is nonlocal in the sense that the interaction force on a vortex in general depends on the position of all other vortices. Typically the range of the vortex interaction is λ, which we assume satisfies $\lambda \gg a_0$, the intervortex spacing. Since the vortex lattice response is nonlocal, the associated elastic constants are wave vector dependent. To mathematically treat the nonlocality, we employ an expansion in Fourier space. We first recall a few facts concerning the normal modes of a flux-line lattice; Ref. 25, e.g., may be consulted for more detail.

We take $\mathbf{u}(\mathbf{l},t)$ to be the average displacement of a vortex whose equilibrium position is at $\mathbf{l}$. Here $\mathbf{u}$ can be obtained from an average of the microscopic vortex displacement $\mathbf{s}$ over an area of size l_{av}^2 where $l_p \ll l_{\text{av}} \ll D_d$, l_p being a characteristic distance between pinning centers, i.e., $\mathbf{u}(\mathbf{l},t)=\langle \mathbf{s}(\mathbf{l},t)\rangle_{l_{\text{av}}}$. We assume a linear response for the lattice; the vortex displacements are taken to be small in comparison with a_0. We take the pinning force to be characterized by a constant κ (Labusch parameter[26]). In general, κ depends upon temperature, vanishing as $T\to T_{c2}(H)$, the field-dependent transition temperature.

In the harmonic approximation, the interaction energy between vortices can be written in quadratic form as

$$V(\mathbf{u})=\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'}\mathbf{u}(\mathbf{l},t)G(\mathbf{l}-\mathbf{l}')\mathbf{u}(\mathbf{l}',t) \ , \tag{4.1}$$

where $G(\mathbf{h})$ is the elastic matrix, with entries given by

$$G_{ij}(\mathbf{l}-\mathbf{l}')=\left.\frac{\partial^2 V}{\partial u_i(\mathbf{l})\partial u_j(\mathbf{l}')}\right|_{\mathbf{u}=0}, \quad i,j=1,2 \ . \tag{4.2}$$

The dynamical matrix $D(\mathbf{q})$ is the Fourier transform over real space lattice vectors of the elastic matrix:

$$D(\mathbf{q})=\sum_{\mathbf{h}}G(\mathbf{h})e^{i\mathbf{q}\cdot\mathbf{h}} \ . \tag{4.3}$$

Its eigenvectors are the polarization vectors $\hat{\epsilon}_p(\mathbf{q})$ which we employ, with eigenvalues D_{qp}:

$$D(\mathbf{q})\hat{\epsilon}_p(\mathbf{q})=D_{qp}\hat{\epsilon}(\mathbf{q}) \ . \tag{4.4}$$

In the long wavelength limit, the polarizations can be identified as either transverse (with index $p=t$) or longitudinal ($p=l$).[25] In terms of the basis $\hat{\epsilon}_p(\mathbf{q})$ we have

$$\mathbf{u}(\mathbf{l},t)=\sum_{\mathbf{q},p}e^{i\mathbf{q}\cdot\mathbf{l}}\hat{\epsilon}_p(\mathbf{q})Q_{qp}(t) \ , \tag{4.5}$$

where $Q_{qp}(t)$ are the normal-mode amplitudes and the sum on $\mathbf{q}$ is over the first Brillouin zone. The explicit inverse relation and accompanying orthogonality relations for the polarization vectors are given in Ref. 25.

B. Vortex dynamics

The phenomenological equation of motion for a vortex in the film is

$$\eta\frac{\partial\mathbf{u}(\mathbf{l},t)}{\partial t}=\mathbf{f}_{\text{ext}}(\mathbf{l},t)-\kappa\mathbf{u}(\mathbf{l},t)-\sum_{\mathbf{l}'}G(\mathbf{l}-\mathbf{l}')\mathbf{u}(\mathbf{l}',t) \ , \tag{4.6}$$

where η is the viscous drag coefficient (e.g., Refs. 27 and 28). In writing Eq. (4.6) we ignore the effects of a vortex inertia term.[11,12] The external force per unit length on the vortex, excluding the viscous drag, pinning, and interaction forces, due to the current in the drive coil, is

$$\mathbf{f}_{\text{ext}}(\mathbf{l},t)=(1/d_f)\mathbf{F}_d(\mathbf{l})e^{-i\omega t} \ . \tag{4.7}$$

By expanding Eq. (4.6) in normal modes we have

$$\eta\dot{Q}_{qp}(t)+(\kappa+D_{qp})Q_{qp}(t)=f_{qp}(t) \ , \tag{4.8}$$

where the 2D discrete Fourier transform of the external force per unit length is

$$f_{qp}(t)=\frac{1}{N}\sum_{\mathbf{l}}e^{-i\mathbf{q}\cdot\mathbf{l}}\hat{\epsilon}_p(\mathbf{q})\cdot\mathbf{f}_{\text{ext}}(\mathbf{l},t) \ . \tag{4.9}$$

Here N is the number of vortices, such that $NA_{\text{cell}}=A$, the area of the sample, where A_{cell} is the area of the unit cell of the flux-line lattice. Whether pinning plays an important role or not depends upon the q-dependent quantities D_{qp}/κ. Converting the sum in Eq. (4.9) to an integral gives

$$f_{qp}(t)=(1/Ad_f)\hat{\epsilon}_p(\mathbf{q})\cdot\mathbf{F}_d(\mathbf{q})e^{-i\omega t} \ . \tag{4.10}$$

Now $F_d(\mathbf{q})$ is large only for $q\le R_d^{-1}$, $D_d^{-1}\ll q_{\text{ZB}}$, where q_{ZB} is a Brillouin zone-boundary wave vector (of order $1/a_0$). For such small q's, $\hat{\epsilon}_l(\mathbf{q})\simeq\hat{\mathbf{q}}$ and

$\hat{\epsilon}_t(\mathbf{q}) \simeq \hat{\mathbf{q}}_t = \hat{\mathbf{z}} \times \hat{\mathbf{q}}$. Thus, using Eqs. (2.28) and (4.10), we have

$$f_{ql}(t) = -(\phi_0/\mu_0 A d_f) q\alpha(q) f_1(q) f_2(q) e^{-i\omega t} \quad (4.11)$$

and $f_{qt}(t)=0$. That is, the dipole excites only longitudinal modes of the vortex lattice. (The transverse mode amplitudes are zero.) The equation of motion for the longitudinal modes is Eq. (4.8) with the index $p=l$. This gives for the longitudinal mode amplitudes

$$Q_{ql}(t) = f_{ql}(t)/(D_{ql}+\kappa - i\omega\eta) \ . \quad (4.12)$$

From Eqs. (4.5), (4.12), and (4.11), the displacement field is given by

$$\mathbf{u}(\mathbf{l},t) = \int \frac{d^2q}{(2\pi)^2} \mathbf{u}(\mathbf{q},t) e^{i\mathbf{q}\cdot\mathbf{l}} \ , \quad (4.13)$$

where

$$\mathbf{u}(\mathbf{q},t) = -\frac{\phi_0}{\mu_0 d_f} \frac{\mathbf{q}\alpha(q) f_1(q) f_2(q)}{(D_{ql}+\kappa - i\omega\eta)} e^{-i\omega t} \ . \quad (4.14)$$

From the vortex continuity equation,[21] the density of vortices is perturbed from n_0 by an amount

$$n_{\text{ind}}(\boldsymbol{\rho},t) = -n_0 \nabla\cdot\mathbf{u}(\boldsymbol{\rho},t)$$
$$= \int \frac{d^2q}{(2\pi)^2} \delta n(\mathbf{q},t) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} \ , \quad (4.15)$$

where

$$n_{\text{ind}}(\mathbf{q},t) = -i n_0 \mathbf{q}\cdot\mathbf{u}(\mathbf{q},t) \ . \quad (4.16)$$

The longitudinal eigenvalue of the dynamical matrix, accounting only for the electromagnetic interaction, is[10,21]

$$D_{ql} = \frac{B_0\phi_0 q^2}{\mu_0} f_1(q) \left[1 + \frac{2}{q d_f} f_1(q) f_2(q) \right] . \quad (4.17)$$

Equation (4.17) is the form of D_{ql} which results from the interaction energy per unit length U_0 between two vortices in the film whose cores are treated in the London limit.[10] This limit for the vortex core is suitable for the present purposes and can be replaced with the variational-core model[10] without difficulty. The interaction force per unit length on a vortex at $\boldsymbol{\rho}$ due to a vortex at the origin is[21]

$$\mathbf{f}_0(\rho) = -\nabla U_0(\boldsymbol{\rho}) \ , \quad (4.18a)$$

giving

$$\mathbf{f}_0(\mathbf{q}) = -i\mathbf{q} U_0(\mathbf{q}) \ . \quad (4.18b)$$

The Fourier transform of the interaction force per unit length for a vortex lattice is $\mathbf{f}_{\text{ind}}(\mathbf{q},t) = n_{\text{ind}}(\mathbf{q},t) f_0(\mathbf{q})$.[21] Then expression (4.17) follows from Eqs. (4.18b) and (4.16) and the relations

$$\mathbf{f}_{\text{ind}}(\mathbf{q},t) = -n_0 q^2 U_0(\mathbf{q}) \mathbf{u}(\mathbf{q},t) = -D_{ql} \mathbf{u}(\mathbf{q},t)$$

so that $D_{ql} = n_0 q^2 U_0(\mathbf{q})$. A useful approximation for the longitudinal eigenvalue is obtained for $q \ll |\lambda_\omega|^{-1}$, where

$$D_{ql} = \frac{B_0\phi_0 q^2}{\mu_0} \left[1 + \frac{2}{q d_f} \frac{1}{[1+q\lambda_\omega \coth(d_f/2\lambda_\omega)]} \right] , \quad (4.19a)$$

which in the thin-film limit becomes

$$D_{ql} = \frac{B_0\phi_0 q^2}{\mu_0} \left[1 + \frac{2}{q d_f} \frac{1}{(1+q\Lambda_\omega)} \right] , \quad (4.19b)$$

where $\Lambda_\omega = 2\lambda_\omega^2/d_f$ is the frequency-dependent 2D screening length and $B_0 = n_0\phi_0$ is the equilibrium flux density of the vortex lattice.

V. FLUX IN PICKUP COIL AND COMPLEX MUTUAL INDUCTANCE

We now wish to calculate the flux, due to the change in vortex positions, up through a pickup coil, which may be located either below or above the film. This will provide the contribution of the moving vortices to the mutual inductance between the driving and pickup coils. Consider a vortex at the origin. Let $\mathbf{a}_v(\boldsymbol{\rho},z) = \hat{\boldsymbol{\phi}} a_{v\phi}(\rho,z)$ denote the vector potential generated by this vortex at $\boldsymbol{\rho}$ and z, where $a_{v\phi}(\rho,z)$ was derived in Ref. 10. By Fourier transforming, we have for $z < -d_f/2$

$$\mathbf{a}_v(\hat{\mathbf{q}},z) = \hat{\mathbf{q}}_t a_{vt}(q,z) \ , \quad (5.1)$$

where [Ref. 10, Eq. (12)]

$$a_{vt}(q,z) = -i(\phi_0/q) f_1(q) f_2(q) e^{q(z+d_f/2)} \ . \quad (5.2)$$

The magnetic flux density at ρ,z from a vortex at the origin is $\mathbf{b}_v(\boldsymbol{\rho},z) = \nabla\times\mathbf{a}_v(\boldsymbol{\rho},z)$, so that the z component of its Fourier transform is given by $b_{vz}(q,z) = iqa_{vt}(q,z)$. As a consequence of the altered vortex positions in the film, the z component of the flux density at $\boldsymbol{\rho},z$ is

$$\delta b_z(\boldsymbol{\rho},z,t) = \int d^2\rho' n_{\text{ind}}(\boldsymbol{\rho}',t) b_{vz}(\boldsymbol{\rho}-\boldsymbol{\rho}',z)$$
$$= \int \frac{d^2q}{(2\pi)^2} n_{\text{ind}}(\mathbf{q},t) b_{vz}(q,z) e^{i\mathbf{q}\cdot\boldsymbol{\rho}} \ . \quad (5.3)$$

The corresponding flux up through a pickup coil below the film at $z=z_p=-d_f/2-D_p$ generated by the moving vortices is

$$\Phi_{pv}(t) = \int_{\rho<R_p} d^2\rho \, \delta b_z(\boldsymbol{\rho},z_p,t) \ . \quad (5.4a)$$

By employing the Fourier transforms $n_{\text{ind}}(\mathbf{q},t)$ and $b_{vz}(q,z_p)$, Eqs. (4.14), (4.16), and (5.2), and an integration rule for Bessel functions,[22,23] we find

$$\Phi_{pv}(t) = \frac{2\pi\phi_0 B_0 I R_d R_p}{d_f} \times \int_0^\infty dq \frac{q f_1^2(q) f_2^2(q) J_1(qR_d) J_1(qR_p)}{D_{ql}+\kappa-i\omega\eta} \times e^{-qD} e^{-i\omega t} \ , \quad (5.4b)$$

where $D \equiv D_d + D_p$. The net z flux through the pickup coil is

$$\Phi_p(t) = \Phi_{pd}(t) + \Phi_{pv}(t) , \tag{5.5}$$

where Φ_{pv} is the contribution from the moving vortices and Φ_{pd} is the straight-through flux generated by the driver coil but reduced because of the screening effect of the film. From Eq. (2.22b) we have for the z-flux density below the film

$$b_{<z}(\rho,z) = \int \frac{d^2q}{(2\pi)^2} b_{<z}(\mathbf{q},z) e^{i\mathbf{q}\cdot\rho} = \int \frac{d^2q}{(2\pi)^2} iqa_{<t}(q,z) e^{i\mathbf{q}\cdot\rho} . \tag{5.6}$$

From Eqs. (2.14), (2.23), and (2.24d), we thus have

$$\Phi_{pd}(t) = \pi\mu_0 I R_d R_p \times \int_0^\infty dq \frac{\operatorname{sech}(Qd_f) J_1(qR_d) J_1(qR_p)}{1 + [(Q^2+q^2)/2qQ]\tanh(Qd_f)} \times e^{-qD} e^{-i\omega t} . \tag{5.7}$$

For the sake of simplicity, in the remainder of this section we assume that the macroscopic lengths R_d, R_p, and $D = D_d + D_p$ dominate the q dependence of the integrals of Eqs. (5.4b) and (5.7). We also consider only the thin-film limit $d_f \ll \lambda \ll L$: we assume that for the important wave vectors, $0 \le q \le 1/L$, where L is one of the macroscopic lengths. Thus, $0 \le qd_f \ll q\lambda \le \lambda/L \ll 1$ and $Q \approx \lambda_\omega^{-1}$, $f_1(q) \approx 1$, and $f_2(q) \approx (1+q\Lambda_\omega)^{-1}$, where Λ_ω is possibly of the size of L. In this approximation, the complex mutual inductance $M(\omega)$, defined via

$$\Phi_p(t) = M(\omega) I e^{-i\omega t} , \tag{5.8}$$

becomes

$$M(\omega) = \pi\mu_0 R_d R_p \left[\int_0^\infty dq \frac{q\Lambda_\omega}{(1+q\Lambda_\omega)} J_1(qR_d) J_1(qR_p) e^{-qD} + \frac{B_0\phi_0}{2\pi d_f} \int_0^\infty dq \frac{q}{(1+q\Lambda_\omega)^2} \frac{J_1(qR_d) J_1(qR_p)}{(D_{ql}+\kappa - i\omega\eta)} e^{-qD} \right] . \tag{5.9}$$

In the absence of the film, the mutual inductance can be obtained by ignoring the vortex contribution and taking the thin-film screening length $\Lambda_\omega \to \infty$. The resulting integral can be evaluated[22] and we obtain[24]

$$M_0 = \mu_0 \left[\frac{D^2 + R_d^2 + R_p^2}{[D^2 + (R_d+R_p)^2]^{1/2}} K(k) - [D^2 + (R_d+R_p)^2]^{1/2} E(k) \right] , \tag{5.10a}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, and

$$k^2 \equiv \frac{4R_d R_p}{[D^2 + (R_d+R_p)^2]} . \tag{5.10b}$$

When $R_d \ll D$ and $R_p \ll D$, we have

$$M_0 \approx \frac{2A_d A_p}{D^3} , \tag{5.11}$$

where the coil areas are $A_d = \pi R_d^2$ and $A_p = \pi R_p^2$. This is a result of the driving dipole producing a field of magnitude $2m/D^3$ at the site of the pickup coil.

In Eq. (5.9), the quantity $\tau_{ql} = \eta/D_{ql}$ can be identified as the exponential decay time for a longitudinal mode of wave vector q and $\tau = \eta/\kappa$ is the exponential decay time of a plucked vortex. In either of the limits $\kappa \gg D_{ql}$ or $\kappa \ll D_{ql}$, the second integrand of Eq. (5.9) can be simplified. In the latter case, pinning is negligible, and by employing the approximation

$$D_{ql} \approx \frac{2B_0\phi_0 q}{\mu_0 d_f} \frac{1}{(1+q\Lambda_\omega)} \tag{5.12}$$

from Eq. (4.16) for the thin-film limit, it can be seen that the mutual inductance depends on the frequency in the combination ω/B_0.

If the pickup coil should be located above the film, in the plane $z = z' = d_f/2 + D_p'$, the flux and mutual inductance can be found similarly. If only the contribution of the moving vortices is detected, then we require only the term $\Phi_{pv}(t)$. In this case we use the form of the vortex vector potential $\mathbf{a}_v$ suitable for the space above the film.[10] The result is simply to modify the distance D in Eq. (5.5) to $D_d + D_p'$.

VI. COMPLEX EFFECTIVE RESISTIVITY AND FILM IMPEDANCE

By knowing the dynamic response of the vortex lattice in the film, local total response functions can now be calculated. From the total time-dependent electric field, the complex effective resistivity and film impedance can be calculated. We show that a local connection exists between the total electric field and total surface current in the film, which yields these functions.

The total electric field is given as

$$\mathbf{e}_{\rm tot}(\rho,z,t) = \mathbf{e}_{\rm ext}(\rho,z,t) + \mathbf{e}_{\rm ind}(\rho,z,t) , \tag{6.1}$$

where $\mathbf{e}_{\rm ext}$ is the Meissner response to the external field and $\mathbf{e}_{\rm ind}$ is the response from the vortex motion. The corresponding total magnetic flux density is $\mathbf{b}_{\rm tot} = \mathbf{b}_{\rm ext} + \mathbf{b}_{\rm ind}$, where $\mathbf{b}_{\rm ext}$ is the flux density generated by the driver current density but modified by the Meissner response of the film and $\mathbf{b}_{\rm ind}$ is the flux density caused by the motion of the vortices and the departure $n_{\rm ind}$ of the vortex density in the film from the constant value $n_0 = B_0/\phi_0$. Averaging over the thickness of the film gives

$$\mathbf{e}_{\text{tot}}(\boldsymbol{\rho},t)=\frac{1}{d_f}\int_{d_f/2}^{d_f/2}dz\,\mathbf{e}_{\text{tot}}(\boldsymbol{\rho},z,t)$$
$$=\mathbf{e}_{\text{ext}}(\boldsymbol{\rho},t)+\mathbf{e}_{\text{ind}}(\boldsymbol{\rho},t)\ . \tag{6.2}$$

We obtain the electric field in the film from Faraday's law

$$\mathbf{e}(\boldsymbol{\rho},z,t)=-\frac{\partial}{\partial t}\mathbf{a}(\boldsymbol{\rho},z,t)\ . \tag{6.3}$$

We introduce the Fourier transform in time for quantities averaged over the thickness of the film by

$$\mathbf{f}(\boldsymbol{\rho},t)=\int\frac{d\omega}{2\pi}\mathbf{f}(\boldsymbol{\rho},\omega)e^{-i\omega t}\ , \tag{6.4a}$$

$$\mathbf{f}(\boldsymbol{\rho},\omega)=\int dt\ \mathbf{f}(\boldsymbol{\rho},t)e^{i\omega t}\ . \tag{6.4b}$$

Then by Fourier transforming the Meissner response in both space (2D) and time, Eq. (6.3) becomes

$$\mathbf{e}_{\text{ext}}(\mathbf{q},\omega)=i\omega\mathbf{a}_{\text{ext}}(\mathbf{q},\omega) \tag{6.5}$$

and the London equation Eq. (2.25) becomes

$$\mathbf{j}_{\text{ext}}(\mathbf{q},\omega)=-(1/\mu_0\lambda_\omega^2)\mathbf{a}_{\text{ext}}(\mathbf{q},\omega)\ , \tag{6.6}$$

where $\mathbf{j}_{\text{tot}}=\mathbf{j}_{\text{ext}}+\mathbf{j}_{\text{ind}}$ is the total current density in the film and $\mathbf{a}_{\text{tot}}=\mathbf{a}_{\text{ext}}+\mathbf{a}_{\text{ind}}$ is the total vector potential. The portion of the vector potential generated by the motion of the vortices, $\mathbf{a}_{\text{ind}}$, can be found from linear superposition by using the vector potential for a single vortex in the film, $\mathbf{a}_0$, and the variation in vortex density n_{ind}. From the convolution

$$\mathbf{a}_{\text{ind}}(\boldsymbol{\rho},t)=\int d^2\rho' n_{\text{ind}}(\boldsymbol{\rho}',t)\mathbf{a}_0(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \tag{6.7}$$

we have

$$\mathbf{a}_{\text{ind}}(\mathbf{q},t)=n_{\text{ind}}(\mathbf{q},t)\mathbf{a}_0(\mathbf{q})\ , \tag{6.8}$$

where

$$n_{\text{ind}}(\mathbf{q},\omega)=in_0\mathbf{q}\cdot\mathbf{u}(\mathbf{q},\omega)=-in_0qu_l(\mathbf{q},\omega)$$

as given by Eq. (4.16).

The vector potential for a vortex in the film located at the origin is[10]

$$\mathbf{a}_0(\boldsymbol{\rho},z)=\mathbf{a}_{0b}(\boldsymbol{\rho})+\mathbf{a}_{0s}(\boldsymbol{\rho},z)\ . \tag{6.9}$$

The "bulk" and "surface" contributions in Eq. (6.9) are given by

$$\mathbf{a}_{0b}(\boldsymbol{\rho})=\hat{\boldsymbol{\phi}}a_{0b\phi}(\rho)=\hat{\boldsymbol{\phi}}2\phi_0\int_0^\infty dq\,J_1(q\rho)f_1(q)\ , \tag{6.10a}$$

$$\mathbf{a}_{0s}(\boldsymbol{\rho},z)=\hat{\boldsymbol{\phi}}a_{0s\phi}(\rho,z)$$
$$=-\hat{\boldsymbol{\phi}}2\phi_0\int_0^\infty dq\,J_1(q\rho)f_1(q)$$
$$\times f_2(q)\frac{q\cosh(Qz)}{Q\sinh(Qd_f/2)}\ , \tag{6.10b}$$

where we do not include a variational core-radius parameter for the vortex[10] in this presentation. The same procedure for the complex effective resistivity may be followed including such a vortex-core-radius parameter. The respective 2D Fourier transforms of the "bulk" and "surface" terms are

$$\mathbf{a}_{0b}(\mathbf{q})=\hat{\mathbf{q}}_t a_{0bt}(\mathbf{q})=-\hat{\mathbf{q}}_t(4\pi i\phi_0/q)f_1(q) \tag{6.11a}$$

and

$$\mathbf{a}_{0s}(\mathbf{q},z)=\hat{\mathbf{q}}_t a_{0st}(\mathbf{q},z)$$
$$=\hat{\mathbf{q}}_t 4\pi i\phi_0 f_1(q)f_2(q)\frac{\cosh Qz}{Q\sinh(Qd_f/2)}\ . \tag{6.11b}$$

Averaging over the film thickness gives

$$\mathbf{a}_0(\mathbf{q})=\mathbf{a}_{0b}(\mathbf{q})+\mathbf{a}_{0s}(\mathbf{q})\ , \tag{6.12}$$

where

$$\mathbf{a}_{0s}(\mathbf{q})=\hat{\mathbf{q}}_t a_{0st}(\mathbf{q})=-\hat{\mathbf{q}}_t i(2\phi_0/d_f)f_1^2(q)f_2(q)\ . \tag{6.13}$$

From Eqs. (6.11) and (6.13) we obtain

$$\mathbf{a}(\mathbf{q})=[-i\phi_0 f_1(q)/q][1-q\Lambda_\omega f_1(q)f_2(q)]\hat{\mathbf{q}}\ . \tag{6.14}$$

Thus, from Faraday's law $\mathbf{e}_{\text{ind}}(\mathbf{q},\omega)=i\omega\mathbf{a}_{\text{ind}}(\mathbf{q},\omega)$ and Eqs. (6.8), (4.18), and (4.12), we have

$$\mathbf{e}_{\text{ind}}(\mathbf{q},\omega)=-i\omega B_0\phi_0 f_1(q)[1-q\Lambda_\omega f_1(q)f_2(q)]$$
$$\times[\mathbf{j}_{\text{ext}}(\mathbf{q},\omega)/(D_{ql}+\kappa-i\omega\eta)]\ , \tag{6.15}$$

where we used $f_{\text{ext},l}(\mathbf{q},\omega)=j_{\text{ext},t}(\mathbf{q},\omega)\phi_0$.

The current density in the film generated by a vortex at the origin is[10]

$$\mathbf{j}_0(\boldsymbol{\rho},z)=\mathbf{j}_{0b}(\boldsymbol{\rho})+\mathbf{j}_{0s}(\boldsymbol{\rho},z) \tag{6.16}$$

where

$$\mathbf{j}_{0b}(\boldsymbol{\rho})=\hat{\boldsymbol{\phi}}j_{0b\phi}(\rho)=\hat{\boldsymbol{\phi}}(\phi_0/2\pi\mu_0\lambda_\omega^3)K_1(\rho/\lambda_\omega) \tag{6.17}$$

whose 2D Fourier transform is

$$\mathbf{j}_{0b}(\mathbf{q})=\hat{\mathbf{q}}_t j_{0bt}(\mathbf{q})=-\hat{\mathbf{q}}_t i(\phi_0/\mu_0\lambda_\omega^2)qf_1(q) \tag{6.18}$$

and $\mathbf{j}_{0s}(\boldsymbol{\rho},z)=-(1/\mu_0\lambda_\omega^2)\mathbf{a}_{0s}(\boldsymbol{\rho},z)$. [In Eq. (6.17), K_1 is the modified Bessel function of the second kind of order one.[22]] Equation (6.13) can be used to find the Fourier transform of $\mathbf{j}_{0s}$ averaged over the film thickness.

We obtain the current density generated by the motion of the vortices from

$$\mathbf{j}_{\text{ind}}(\boldsymbol{\rho},t)=\int d^2\rho' n_{\text{ind}}(\boldsymbol{\rho}',t)\mathbf{j}_0(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \tag{6.19}$$

giving

$$\mathbf{j}_{\text{ind}}(\mathbf{q},\omega)=n_{\text{ind}}(\mathbf{q},\omega)\mathbf{j}_0(\mathbf{q})\ , \tag{6.20}$$

where $\mathbf{j}_0(\mathbf{q})=\hat{\mathbf{q}}_t j_{0t}(\mathbf{q})$ and from Eqs. (6.13) and (6.16), $j_{0t}(\mathbf{q})=-iD_{ql}/B_0q$. Using Eqs. (4.12), (4.16), and (6.20) we obtain

$$\mathbf{j}_{\text{ind}}(\mathbf{q},\omega)=\frac{D_{ql}}{D_{ql}+\kappa-i\omega\eta}\mathbf{j}_{\text{ext}}(\mathbf{q},\omega) \tag{6.21}$$

so that $\mathbf{j}_{\text{tot}}=\mathbf{j}_{\text{ext}}+\mathbf{j}_{\text{ind}}$ is

$$\mathbf{j}_{\text{tot}}(\mathbf{q},\omega)=\frac{\kappa-i\omega\eta}{D_{ql}+\kappa-i\omega\eta_\eta}\mathbf{j}_{\text{ext}}(\mathbf{q},\omega)\ . \tag{6.22}$$

To find the complex effective resistivity, we write both $\mathbf{e}_{\rm tot}$ and $\mathbf{j}_{\rm tot}$ in terms of $\mathbf{j}_{\rm ext}$. Equation (6.15) relates $\mathbf{e}_{\rm ind}$ to $\mathbf{j}_{\rm ext}$ and Eqs. (6.5) and (6.6) give $\mathbf{e}_{\rm ext}(\mathbf{q},\omega)=-i\mu_0\omega\lambda_\omega^2\mathbf{j}_{\rm ext}(\mathbf{q},\omega)$. The total electric field is given by Eq. (6.1), which we approximate in the long wavelength limit $q \ll |\lambda_\omega|^{-1}$. In this limit, using Eq. (5.12) for D_{ql}, we have

$$\mathbf{e}_{\rm tot}(\mathbf{q},\omega)=-i\omega\left[\frac{k_l+\kappa-i\omega\eta}{D_{ql}+\kappa-i\omega\eta}\right]\mathbf{j}_{\rm ext}(\mathbf{q},\omega)\ . \tag{6.23}$$

where $k_l=B_0\phi_0/\mu_0\lambda_\omega^2$. Comparing Eqs. (6.22) and (6.23) we see that we may write

$$\mathbf{e}_{\rm tot}(\mathbf{q},\omega)=\rho_{\rm tot}(\omega)\mathbf{j}_{\rm tot}(\mathbf{q},\omega)\ , \tag{6.24}$$

where

$$\rho_{\rm tot}(\omega)=-i\mu_0\omega\lambda_\omega^2\frac{\omega+i/\tau+i/\tau_\perp}{\omega+i/\tau} \tag{6.25}$$

and $\tau_\perp=\eta/k_l=2\lambda_\omega^2/\omega\delta_f^2,\tau=\eta/\kappa$. We have introduced the flux-flow penetration depth, $\delta_f^2\equiv 2B_0\phi_0/\mu_0\eta\omega$.[29] One can rewrite Eq. (6.25) with the pinning resistivity, $\rho_p=-i\omega B_0\phi_0/\kappa$,[3] and the flux-flow resistivity $\rho_f=B_0\phi_0/\eta$. Since $\rho_{\rm tot}$ is independent of $\mathbf{q}$, Eq. (6.24) gives a local relation between $\mathbf{e}_{\rm tot}$ and $\mathbf{j}_{\rm tot}$. At high frequencies we have $\rho_{\rm tot}(\omega)\approx -i\mu_0\omega\lambda_\omega^2$, which agrees with the second London equation, $\mathbf{e}=\mu_0\lambda_\omega^2(\partial/\partial t)\mathbf{j}$. At low frequencies, where $\omega \ll \tau_\perp^{-1},\tau^{-1}$, we have $\rho_{\rm tot}(\omega)\approx -i\mu_0\omega\lambda_\omega^2(1+\lambda_C^2/\lambda^2)$ where $\lambda_C^2=B_0\phi_0/\mu_0\kappa$ is the square of the Campbell penetration depth.[30] In terms of characteristic lengths we have

$$\rho_{\rm tot}(\omega)=-i\mu_0\omega\lambda_\omega^2\left[\frac{\delta_f^2-2i\lambda_C^2+\lambda_C^2\delta_f^2/\lambda^2}{\delta_f^2-2i\lambda_C^2}\right]. \tag{6.26}$$

The relation between $\mathbf{e}_{\rm tot}$ and the total surface current

$$\mathbf{K}_{\rm tot}(\boldsymbol{\rho},t)=\int_{-d_f/2}^{d_f/2}dz\ \mathbf{j}_{\rm tot}(\boldsymbol{\rho},t)\ , \tag{6.27}$$

gives the film impedance $Z(\omega)$. By Fourier transforming in time, we have

$$\mathbf{e}_{\rm tot}(\rho,\omega)=Z(\omega)\mathbf{K}_{\rm tot}(\boldsymbol{\rho},\omega)\ . \tag{6.28}$$

From Eq. (6.24) we have $Z(\omega)=\rho_{\rm tot}(\omega)/d_f$.

VII. SUMMARY

In this paper the dynamics of interacting vortices in a type-II superconducting film responding to a distribution of currents parallel to the film was studied. The induced current and field distribution in the film were described. In particular, we described in detail the film dipole moment induced by a dipole drive coil (see Appendix A).

The present theory provides relevant analytical results for a typical geometry used in film measurements.[5–9] Here the applied rf magnetic field is not entirely parallel to the film surface. The applied magnetic field was found and its interaction with vortices, via the Lorentz force, was determined. The theory for the resulting vortex response used the vortex structure appropriate to a superconducting film. The inclusion of two-fluid effects allows the results to hold continuously through the transition temperature or upper critical field. In this way, earlier work on eddy-current probes was generalized to the superconducting state.

By means of Fourier transform techniques, we were able to solve for all of the fields in linear response. The coupled fields considered include the total current density, magnetic and electric fields, vortex displacement, and vortex density. The nonlocality of vortex interactions was explicitly taken into account by using an expansion of the flux-line lattice in normal modes. The theory was applied to find total local response functions, including the mutual inductance of two coaxial circular coils and the film impedance. The determined linear response functions are complex valued, with, e.g., the imaginary part of the mutual inductance representing the effect of dissipation. The real part of the mutual inductance is connected with the film's flux exclusion and is highly dependent on the two-dimensional screening length. We discussed the situation when pinning is neglected or else is simply represented as a linear restoring force acting the same way on each vortex, where only the longitudinal modes of the vortex lattice contribute to the mutual inductance.

All the above treatment including pinning holds, of course, only if the vortices do not move beyond the point where the linear restoring force applies. At large amplitudes the pinning force is no longer linear in the displacement, hysteresis comes into play, and the situation becomes more complicated. It is possible that our theory can be extended to take into account such effects by using the Bean critical-state model. With this model the magnetic field profiles could be determined for a given field sweep. An important dimensionless parameter in such a description is then expected to be the ratio of the microwave field, derived from the driving current density, to the field that drives the flux front to the middle of the film.[1]

We have ignored the thermal activation of vortices which considerably complicates the vortex dynamics. In the presence of this effect, both the transverse and longitudinal modes of the vortex lattice will contribute to the complex response functions.

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APPENDIX A: DIPOLE MOMENT OF THE INDUCED FILM CURRENT

Here we describe the calculation of the magnetic dipole moment $\mathbf{m}_f$ generated by the induced current density $\mathbf{j}_f$ flowing in the superconducting film. The driving current density, Eqs. (3.1) and (3.2), produces a field which is screened by the current density

$$\mathbf{j}_f(\boldsymbol{\rho},z) = -(1/\mu_0\lambda_\omega^2)\mathbf{a}_f(\boldsymbol{\rho},z) , \quad (A1)$$

where $\mathbf{a}_f$ is given by Eqs. (2.13), (2.24b), and (2.24c). Writing Eq. (3.7) explicitly for the film geometry we have

$$\mathbf{m}_f = \tfrac{1}{2}\int_{-d_f/2}^{d_f/2} dz \int d^2\rho(\boldsymbol{\rho}+z\hat{\mathbf{z}})\times\mathbf{j}_f(\boldsymbol{\rho},z) . \quad (A2)$$

By using the Fourier representation of the film current density,

$$\mathbf{j}_f(\boldsymbol{\rho},z) = \int \frac{d^2q}{(2\pi)^2}\mathbf{j}_f(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}} , \quad (A3)$$

by representing the 2D position vector $\boldsymbol{\rho}$ as a Fourier transform,

$$\boldsymbol{\rho} = \int \frac{d^2q}{(2\pi)^2}\boldsymbol{\rho}_q e^{i\mathbf{q}\cdot\boldsymbol{\rho}} , \quad (A4a)$$

$$\boldsymbol{\rho}_q = \int d^2\rho\, \boldsymbol{\rho} e^{-i\mathbf{q}\cdot\boldsymbol{\rho}} , \quad (A4b)$$

and then by integrating over $\boldsymbol{\rho}$ in Eq. (A2), we have

$$m_{fz} = \tfrac{1}{2}\hat{\mathbf{z}}\int_{-d_f/2}^{d_f/2}\int \frac{d^2q}{(2\pi)^2}\boldsymbol{\rho}_{-q}\times\mathbf{j}_f(\mathbf{q},z) . \quad (A5)$$

The singular expressions (A4) are to be interpreted in the sense of distributions. One means of evaluating them is to use a convergence factor. Instead, we use

$$\boldsymbol{\rho}_{-q} = \boldsymbol{\rho}_q^* = -i(2\pi)^2\nabla_q\delta_2(\mathbf{q}) , \quad (A6)$$

to write

$$m_{fz} = -\frac{i}{2}\hat{\mathbf{z}}\int_{-d_f/2}^{d_f/2} dz \int d^2q[\nabla_q\delta_2(\mathbf{q})]\times\mathbf{j}_f(\mathbf{q},z) . \quad (A7)$$

Using the vector identity

$$\nabla_q\delta_2(\mathbf{q})\times\mathbf{j}_f = \nabla_q\times[\mathbf{j}_f\delta_2(\mathbf{q})] - \delta_2(\mathbf{q})\nabla_q\times\mathbf{j}_f \quad (A8)$$

in Eq. (A7) and noting that the first term integrates to zero by Stokes' theorem, we have

$$m_{fz} = \frac{i}{2}\int_{-d_f/2}^{d_f/2} dz \lim_{\mathbf{q}\to 0}\hat{\mathbf{z}}\cdot\nabla_q\times\mathbf{j}_f(\mathbf{q},z) . \quad (A9)$$

Since $\mathbf{j}_f(\mathbf{q},z) = \hat{\mathbf{q}}_t j_{ft}(q,z)$, m_{fz} may be written as

$$m_{fz} = \frac{i}{2}\lim_{q\to 0}\int_{-d_f/2}^{d_f/2} dz \frac{1}{q}\frac{\partial}{\partial q}[q j_{ft}(q,z)] . \quad (A10)$$

We employ Eqs. (2.25), (2.13), (2.24b), and (2.24c) for $j_{ft}(q,z)$ so that m_{fz} may be written as

$$m_{fz} = -\frac{i}{2\mu_0\lambda_\omega^2}\lim_{q\to 0}\frac{1}{q}\frac{\partial}{\partial q}\left[q a_{fct}(q)\frac{2}{Q}\sinh\left[\frac{Qd_f}{2}\right]\right] \quad (A11)$$

which may be further rewritten in terms of the functions f_1 and f_2 of Eq. (2.29):

$$m_{fz} = -\frac{i}{2\mu_0}\lim_{q\to 0}\frac{1}{q}\frac{\partial}{\partial q}[q^2\alpha(q)f_1(q)f_2(q)] . \quad (A12)$$

In evaluating the limit $q\to 0$ we use the asymptotic form (3.6) of $\alpha(q)$, valid for small q, which contains the magnetic dipole moment m of the driver loop. Thus, we find that $m_{fz} = -m$.

The above derivation has employed distributions proportional to the gradient of the Dirac delta function. By well-known results in functional analysis,[31] the two-dimensional delta function lies in the Sobolev space $H^{-1-\epsilon}$ for $\epsilon > 0$. Therefore the quantity $\boldsymbol{\rho}_q$ lies in the Sobolev space $H^{-2-\epsilon}$.

An alternative derivation of the result $\mathbf{m}_f = -\mathbf{m}$ can be given by employing the delta function alone. This derivation also points up the importance of the long wavelength behavior of the quantities appearing in Eq. (A12). The following does not assume a thin film, although it readily reduces in that limit. It may therefore be of benefit to outline this approach.

By using the Fourier representation (A3) in Eqs. (A2) and (A1), integrating over the angular variables and the thickness of the film, we have

$$m_{fz} = -\frac{i}{2\mu_0}\int_0^\infty d\rho\,\rho^2\int_0^\infty dq\, q^2 f_1(q)f_2(q)J_1(q\rho)\alpha(q) . \quad (A13)$$

Upon the change of scale $v = q\Lambda$, $u = \rho/\Lambda$, we have

$$m_{fz} = -\frac{i}{2\mu_0}\int_0^\infty du\, u^2\int_0^\infty dv\, v^2 f_1(v/\Lambda)f_2(v/\Lambda) \times J_1(uv)\alpha(v/\Lambda) . \quad (A14)$$

Notice that Eqs. (A13) and (A14) involve a divergent integral for the ρ (or u) integration. Rather than performing the u integral with a convergence factor, we use the technique of introducing a delta function, by way of the limit

$$u/2 = \lim_{w\to 0}(1/w)J_1(uw) . \quad (A15)$$

Then m_{fz} can be evaluated with the aid of the representation[22]

$$\frac{\delta(v-w)}{v} = \int_0^\infty du\, uJ_p(uv)J_p(uw) , \quad (A16)$$

with $p=1$ to give

$$m_{fz} = -\frac{i}{\mu_0}\lim_{w\to 0}f_1\left[\frac{w}{\Lambda}\right]f_2\left[\frac{w}{\Lambda}\right]\alpha\left[\frac{w}{\Lambda}\right] . \quad (A17)$$

The similarity of Eq. (A17) to Eq. (A12) can be noted. Taking the indicated limit, using $\alpha(0) = -i\mu_0 m$, we again arrive at the result $m_{fz} = -m$.

APPENDIX B: QUADRUPOLAR MAGNETIC FIELD PRODUCED BY A SUPERCONDUCTING FILM

In this Appendix the magnetic field below the superconducting film is first considered. The geometry of the text is assumed, with the single-turn drive coil located above the film, as described in Sec. III. Specific results are obtained in both the thin- and thick-film limits. The vector potential below the film is

$$\mathbf{a}_<(\rho,z) = \int \frac{d^2q}{(2\pi)^2}\mathbf{a}_<(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}} , \quad (B1)$$

where $\mathbf{a}_<(\mathbf{q},z)$ is given in Eqs. (2.14) and (2.24d). In the dipole approximation, we use Eq. (3.6) for $\alpha(q)$, so that $a_{<t}(\mathbf{q})$ becomes

$$a_<(\mathbf{q})=-(i\mu_0/2)me^{-qD_d}[q\Lambda_\omega/(1+q\Lambda_\omega)] \quad \text{(B2)}$$

in the thin-film limit $d_f \ll |\lambda_\omega|$. Since $a_{<t}(q)$ depends only on the magnitude of q, the angular integration in (B1) can be performed, with the result

$$\mathbf{a}_<(\rho,z)=\hat{\boldsymbol{\phi}}\frac{\mu_0 m}{4\pi}\Lambda_\omega\int_0^\infty dq\frac{q^2J_1(q\rho)}{1+q\Lambda_\omega}e^{q(z+d_f/2-D_d)} . \quad \text{(B3)}$$

In the dipole approximation, the important wave numbers are those for which $0\le q\le 1/D_d$ or $0<q\Lambda_\omega\le\Lambda_\omega/D_d$. Thus, as long as we consider wavelengths larger than the 2D screening length $(q\Lambda_\omega\ll 1)$ we may replace $q\Lambda_\omega$ by zero in the integrand of Eq. (B3). The resulting integral may be evaluated (Ref. 23, p. 712) as

$$\mathbf{a}_<(\rho,z)\simeq\hat{\boldsymbol{\phi}}\frac{3\mu_0 m}{4\pi}\Lambda_\omega\frac{\beta(z)\rho}{[\beta^2(z)+\rho^2]^{5/2}} , \quad \text{(B4)}$$

where $\beta(z)=D_d-d_f/2-z$. The magnetic field described by Eq. (B4), $\mathbf{b}_<=\nabla\times\mathbf{a}_<$, has components given by

$$\mathbf{b}_{<z}(\rho,z)\simeq\frac{3\mu_0 m}{4\pi}\Lambda_\omega\frac{\beta(z)[2\beta^2(z)-3\rho^2]}{[\beta^2(z)+\rho^2]^{7/2}} , \quad \text{(B5)}$$

$$\mathbf{b}_{<\rho}(\rho,z)\simeq\frac{3\mu_0 m}{4\pi}\Lambda_\omega\frac{\rho[\rho^2-4\beta^2(z)]}{[\beta^2(z)+\rho^2]^{7/2}} . \quad \text{(B6)}$$

When $|z|\gg D_d$, we have, well below the film, far from the screening region, $\beta\approx -z$, and then Eqs. (B5) and (B6) are the same as the magnetic field components derived from the magnetic potential of a quadrupole moment $Q_<=-4m\Lambda_\omega$,

$$\Phi_<(\rho,z)=\tfrac{1}{2}Q_<[(z^2-\rho^2/2)/r^5] , \quad \text{(B7)}$$

where[24] the nonzero components of the quadrupole moment tensor are $Q_{11}=Q_{22}=-Q_{33}/2=-Q_</2$. Then the behavior of $\mathbf{b}_<(\rho,z)$ for $r=(\rho^2+z^2)^{1/2}\gg|\Lambda_\omega|$ is given by

$$\mathbf{b}_<(\rho,z)=-\nabla\Phi_z(\rho,z) . \quad \text{(B8)}$$

In the thick-film limit $d_f \ll |\lambda_\omega|$, approximating $\mathbf{a}_<(\mathbf{q},z)$ in Eq. (2.24d), we find that Eq. (B2) is replaced by

$$a_<(\mathbf{q})\simeq -2i\mu_0 mq\lambda_\omega e^{-d_f/\lambda_\omega}e^{-qD_d} . \quad \text{(B9)}$$

Then the results for the vector potential and field, Eqs. (B4)–(B6), are modified by a factor of $4\lambda_\omega e^{-d_f/\lambda_\omega}/\Lambda_\omega$. In this case the quadrupole moment for the field below the film, $Q_<=-16m\lambda_\omega e^{-d_f/\lambda_\omega}$, is exponentially small as expected due to the very effective screening of the film.

Similarly, the magnetic field at large distances in the region above the film can be examined and shown to be quadrupole in nature. By expanding the leading part of the vector potential $\mathbf{a}_0$, Eq. (2.5a) written in terms of the drive-coil magnetic moment, in powers of $r\equiv(\rho^2+z^2)^{1/2}\gg D_d$, the dominant part of the quadrupole moment above the film is found to be $Q_{0>}=8mD_d$. The correction field $\mathbf{a}_1$, Eq. (2.11), contributes an additional quadrupole moment of $Q_{1>}=4m\Lambda_\omega$.

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