Long-range interactions and the quantum Hall effect

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(Received 28 October 1991; revised manuscript received 26 May 1992)

We have studied the consequences of long-range interactions for the properties of the quasiparticles in the quantum Hall effect. We find that the quasiparticle states have long-range tails in their current and density distributions. These tails are an effect of Landau-level mixing and are absent if the state space is restricted to the lowest Landau level. We also discuss a violation of the spin-statistics relation for the quasiparticles.

The theory of the quantum Hall effect[1] is constructed around the observation that a two-dimensional electron gas in a transverse magnetic field is incompressible at certain rational filling factors. The sharp commensuration between magnetic field and density that produces the incompressibility also determines the character of the elementary charged excitations—the quasiparticles are the defects that accommodate the deviations of the density from a commensurate value. The properties of these quasiparticles are of great interest—at noninteger filling factors they have fractional charge and statistics.[2-4]

In this paper we report results on the structure of the quasiparticles that arise from going beyond a common approximation—the neglect of Landau-level mixing—and are specific to long-range interactions. We show that for long-range interactions Landau-level mixing has important consequences for the structure of the quasiparticles, in that their density and current profiles become long ranged. For example, Coulomb ($1/r$) interactions produce tails in the density and current profiles of the quasiparticles that fall off as $1/r^3$ and $1/r^2$, respectively. As these profiles are short ranged in the absence of Landau-level mixing it follows that regardless of the ratio of the cyclotron gap $\hbar\omega_c$ to the typical interaction energy $e^2/\epsilon l$, one *cannot* study their long-distance behavior in a state space restricted to a single Landau level. The same is true of the asymptotic behavior of the connected ground-state correlation functions. This is our most important result.

We also address two other overlapping problems. First we show that two natural definitions of the spin of the quasiparticles are in conflict with the spin-statistics relation. Finally we comment on a tentative connection between our work on quasiparticle current profiles and the magnetization of Hall systems.

PRELIMINARIES

In this work we consider only the fully spin-polarized electron gas. Hence we consider a system of spinless particles of charge e that are confined to a plane perpendicular to a uniform magnetic field. The Hamiltonian of the system in second-quantized notation is

$$H = H_K + H_I ,$$
$$H_K = \frac{1}{2m^*} \int d^2r\, \psi^\dagger(\mathbf{r}) \left[\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(\mathbf{r}) \right]^2 \psi(\mathbf{r}) , \quad (1)$$
$$H_I = \tfrac{1}{2} \int d^2r\, d^2r'\, \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) - \bar{\rho} \int d^2r\, d^2r'\, V(\mathbf{r}-\mathbf{r}')[\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}') - \tfrac{1}{2}\bar{\rho}] ,$$

where m^* is the (effective) mass of the particles, $\mathbf{A}$ is the vector potential of the background magnetic field $\mathbf{B}$, $\psi^\dagger$ and ψ are, respectively, the particle creation and destruction operators, and $V(\mathbf{r})$ is the interparticle potential. [The second term in H_I is the interaction with the compensating background of density $\bar{\rho}$; this can be ignored for potentials for which $\int d^2r\, V(\mathbf{r}) < \infty$].

In subsequent calculations we use $\mathbf{A} = -(\frac{1}{2})\mathbf{r}\times\mathbf{B}$ (symmetric gauge) with $\mathbf{B} = B\hat{\mathbf{z}}$. Also we denote the cyclotron frequency by $\omega_c (\equiv eB/m^*c)$ and the Landau length by $l(\equiv\sqrt{\hbar c/eB}\,)$. The (familiar) eigenfunctions in this gauge are

$$u_{nm}(\mathbf{r}) = \left[\frac{n!}{2\pi l^2 (n+m)!} \right]^{1/2} \left[\frac{re^{i\theta}}{\sqrt{2}l} \right]^m \times L_n^m \left[\frac{r^2}{2l^2} \right] e^{-r^2/4^2} , \quad (2)$$

where the L_n^m are Laguerre polynomials. We expand the field operators in terms of the creation and destruction operators for the orbitals in the usual fashion:

$$\psi(\mathbf{r}) = \sum_{nm} u_{nm}(\mathbf{r}) c_{nm} ,$$
$$\psi^\dagger(\mathbf{r}) = \sum_{nm} u_{nm}^*(\mathbf{r}) c_{nm}^\dagger . \quad (3)$$

Here n is the Landau-level index and $\hbar m\,(m \ge -n)$ is the eigenvalue of the canonical angular momentum L_z.

We discuss the Laughlin fractions ($1/m$ with m odd and inclusive of 1) but it is clear that the analysis applies *mutatis mutandis* to other filling factors that exhibit quantum Hall ground states as well.

QUASIPARTICLE PROFILES

The theory of the Laughlin fractions simplifies in the high-field limit. Initially one restricts the state space to the lowest Landau level. Subsequently the effects of higher Landau levels are included perturbatively. Though this is an expansion around the infinite B limit it is technically more convenient to describe it as a small m^* expansion. The small m^* limit has the advantage that it allows the cyclotron gap to diverge without requiring that we change the number of particles to keep the filling factor constant. The wave functions and various physical quantities are then expressed as series in powers of m^* with the leading term arising from the lowest Landau level alone. At issue here is whether subleading terms in the mass expansion (i.e., the effects of Landau-level mixing) can invalidate conclusions about the structure of the quasiparticles reached on the basis of the leading term (i.e., the lowest Landau-level approximation). Previously it has been assumed that quasiparticle properties such as their density and current profiles are qualitatively similar to their lowest Landau-level forms. We show below that this expectation is incorrect for long-range interactions; the leading long-distance form of the quasiparticle profiles is altered completely by terms that are subleading in the m^* expansion.

Recall first that the properties of the quasiparticles in the lowest Landau level are well described by the wave functions introduced by Laughlin in his seminal work on the subject.[2] The plasma interpretation of the wave functions shows that they describe density profiles that relax exponentially on the scale of l to the density of the parent fluid. The current in these states is exponentially localized as well. This is a consequence of the current-density identity

$$\mathbf{j}(\mathbf{r})=\frac{\hbar}{2m^*}[\hat{\mathbf{z}}\times\nabla\rho(\mathbf{r})] , \tag{4}$$

due to Girvin, MacDonald, and Platzman[5] and valid in the sense of equality of matrix elements between arbitrary states in the lowest Landau level. In short, the quasiparticles in the lowest Landau level have a size of the order of l.

We now show that this cannot be the whole story by deriving the true asymptotic form of the quasiparticle current profile by means of a hydrodynamic argument. Consider a quasiparticle centered at the origin. Since the Hall liquid is incompressible and hence does not screen, at distances much greater than the size of the quasiparticle but far from the boundary of the system there exists a radial electric field in the system with a magnitude

$$E_r=\frac{e^*}{r^2} , \tag{5}$$

where e^* is the quasiparticle charge. Since the system has a nonvanishing Hall conductance σ_H, the presence of this field must cause a circular (particle) current to flow with a form

$$j_\theta(r)=\sigma_H\frac{e^*/e}{r^2} . \tag{6}$$

The presence of this power-law tail makes clear the inadequacy of the lowest Landau-level approximation when the interactions are long ranged. In fact, we can already guess that this tail arises in the next order of the mass expansion by noting that (6) is $O[(m^*)^0]$ and that the current operator has an explicit factor of $1/m^*$. In the next two sections we shall rederive this result first by perturbation theory and then using the Landau-Ginzburg theory of the Hall effect. Before we turn to the details of these derivations we would like to list some other examples of changed asymptotic behavior that we have derived similarly.

For the density profile of the quasiparticles we find the leading large distance term

$$\delta\rho(r)=\frac{\sigma_H}{\omega_c}\frac{e^*/e}{r^3} , \tag{7}$$

which has the effect of reducing the accuracy of quantization of the quasiparticle charge.[6]

Analogous terms arise in ground-state correlation functions. For example, the static structure factor $S(k)$ picks up a nonanalytic $|\mathbf{k}|^3$ contribution from the interactions which yields

$$\langle\rho(\mathbf{r})\rho(\mathbf{0})\rangle=\bar{\rho}^2\left[1-\nu\frac{9}{4}\frac{e^2/l}{\hbar\omega_c}\left[\frac{l}{r}\right]^5+O[(l/r)^7]\right] \tag{8}$$

for the density-density correlation function.[7]

Finally we would like to emphasize that these results are a consequence of the absence of long-wavelength screening due to the incompressibility of the electron gas *and* the presence of a long-ranged interaction. It is crucial that the incompressibility itself, unlike in the case of a charged system with a finite plasma frequency, does *not* arise from the long-range nature of the interactions.

PERTURBATIVE CALCULATIONS

The task of deriving (6) analytically is straightforward for $\nu=1$ and for the other integer Hall states. We treat H_K in (1) as the unperturbed problem whose eigenstates are Slater determinants constructed from the orbitals in (2). It follows from the earlier comments on the small m^* limit that we need only calculate the wave functions perturbatively to first order in H_I and take the matrix elements of $\mathbf{j}$ between the zeroth- and first-order pieces. We have done this and confirmed that (6) is correct to $O[(m^*)^0]$, and to leading order in l/r for large r. In fact, it arises entirely from mixing to the next Landau level. The relevant asymptotics are detailed in the appendix.

For $\nu<1$ the noninteracting problem is degenerate and therefore the corresponding point of departure would be the exact eigenstates of the Hamiltonian restricted to the lowest Landau level. Unfortunately these are not available and we are forced to reformulate the problem in a way that can take advantage of known (nonrigorous) theorems about the physics of the lowest Landau level.[8] Consider a different decomposition of H, namely

$$\begin{aligned} H &= H_1 + H_2 , \\ H_1 &= H_K + PH_1P , \\ H_2 &= H_I - PH_IP , \end{aligned} \tag{9}$$

where P is the operator that projects onto the lowest Landau level. The strategy is to include the effects of higher Landau-level mixing (present solely in H_2) by constructing a unitarily equivalent Hamiltonian,

$$\begin{aligned} \tilde{H} &= e^T H e^{-T} \\ &= H + [T,H] + \frac{1}{2!}[T,[T,H]] + \cdots \end{aligned} \tag{10}$$

(T is anti-Hermitian) which has vanishing matrix elements between states that lie wholly in the lowest Landau level and those that do not, i.e., the lowest Landau level is an invariant subspace of $\tilde{H}$. To calculate the current in the eigenstates of H we calculate the expectation value of

$$\tilde{\mathbf{j}}(\mathbf{r}) = e^T \mathbf{j}(\mathbf{r}) e^{-T} \tag{11}$$

in the eigenstates of $\tilde{H}$. At this point the reader might despair of our sanity since we know even less about the eigenstates of $\tilde{H}$ than about those of H. The point, however, is this: We expect that the physics is relatively insensitive to the choice of Hamiltonian in the lowest Landau level. Consequently the nontrivial modifications to the quasiparticle profiles will arise from the modified *form* of the current and density operators. By making fairly general assumptions (see below) about the physics of the lowest Landau level we will be able to extract the *exact* leading long-distance behavior of the quasiparticle profiles. The particular (nonrigorous) theorem we shall use asserts that all reasonable Hamiltonians in the lowest Landau level have incompressible ground states at $\nu = 1/m$ which support excitations of charge $1/m$ and that their connected ground-state correlations as well as the profiles of their quasiparticles decay exponentially with l.[9] Since the theorem is believed to hold for H and the interactions in $\tilde{H}$ are not longer ranged than those in H it should hold for $\tilde{H}$ as well. Consequently the leading long-distance piece of the $O[(m^*)^0]$ correction to the current can be calculated by taking the expectation value of the $O[(m^*)^0]$ term in the current *operator* in the quasiparticle eigenstate of H. It is worth emphasizing that there is another correction to the current profile at $O[(m^*)^0]$, which is the change in the expectation value of the $O[(m^*)^{-1}]$ term in the current operator $[\mathbf{j}(\mathbf{r})]$ due to $O(m^*)$ changes in the eigenstates; however, by virtue of our theorem this is of finite range.

The operator T can be calculated perturbatively by using the expansion in (10) and it suffices to obtain it to lowest order in H_2. To this order the relevant nonzero matrix elements are

$$\begin{aligned} \langle i | T^1 | \alpha \rangle &= \langle i | H_2 | \alpha \rangle / (E_i - E_\alpha) \\ &= -\langle \alpha | T^1 | i \rangle^* , \end{aligned} \tag{12}$$

where i labels a state in the lowest Landau level, α labels a state with higher Landau-level content. To the same order (11) simplifies to

$$\tilde{\mathbf{j}}(\mathbf{r}) = \mathbf{j}(\mathbf{r}) + [T^1, \mathbf{j}(\mathbf{r})] ; \tag{13}$$

the second term on the right-hand side is the desired $O[(m^*)^0]$ term in the current operator. The demonstration that taking the expectation value of this term in the quasiparticle state reproduces (6) is contained in the Appendix. We turn next to an alternative derivation of these results within the framework of the Landau-Ginzburg theory of the Hall effect.

LANDAU-GINZBURG THEORY

In their construction of a Landau-Ginzburg theory for the Hall effect, Zhang, Hansson, and Kivelson[10] reformulated the problem of interacting fermions in a magnetic field as a problem of interacting bosons with an additional Cherm-Simons interaction. In their language the quantum dynamics of our system is governed by the Lagrangian density

$$\begin{aligned} \mathcal{L}(\mathbf{r}) = {} & \bar{\phi}(\mathbf{r})[i\hbar\partial_t - ea_0]\phi(\mathbf{r}) - \frac{1}{2m^*}\left| \left[\frac{\hbar}{i}\nabla - \frac{e}{c}[\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r})] \right] \phi(\mathbf{r}) \right|^2 \\ & - \tfrac{1}{2}\int d^2r' V(\mathbf{r}-\mathbf{r}')[|\phi(\mathbf{r})|^2 - n][|\phi(\mathbf{r}')|^2 - n] - \frac{e^2}{4\hbar c\theta}\epsilon^{\mu\nu\sigma} a_\mu(\mathbf{r})\partial_\nu a_\sigma(\mathbf{r}) , \end{aligned} \tag{14}$$

provided $\theta = (2k+1)\pi$. [$V(\mathbf{r})$ is, as in (1), the interparticle potential. For notational convenience we pick $\mathbf{A}$ so that $\mathbf{B} = -B\hat{\mathbf{z}}$, i.e., the field is reversed from that used in (1).]

Following Ref. 10, the Landau-Ginzburg equations are the *classical* equations of motion derived from $\mathcal{L}$. For static field configurations they are a gauged nonlinear Schrödinger equation

$$\begin{aligned} & \frac{-\hbar^2}{2m^*}\left[\nabla - \frac{ie}{\hbar c}[\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r})] \right]^2 \phi(\mathbf{r}) + ea_0\phi(\mathbf{r}) \\ & \quad + \left[\int d^2r' V(\mathbf{r}-\mathbf{r}')[|\phi(\mathbf{r}')|^2 - n] \right] \phi(\mathbf{r}) = 0 , \end{aligned} \tag{15}$$

and the two components of the Chern-Simons field-current identity

$$|\phi(\mathbf{r})|^2=\left[\frac{e}{2\hbar c\theta}\right]\nabla\times\mathbf{a}(\mathbf{r}) , \tag{16}$$

$$\frac{1}{c}\mathbf{j}(\mathbf{r})=\left[\frac{e}{2\hbar c\theta}\right]\hat{\mathbf{z}}\times\nabla a_0(\mathbf{r}) . \tag{17}$$

Spatially uniform solutions correspond to the Laughlin states and exist only when

$$n=\frac{e}{2\hbar c\theta}B , \tag{18}$$

which is just $\nu=1/(2k+1)$ is disguise. Quasiparticles arise as charged vortices and they have charge $e/(2k+1)$. In the rest of this section we rederive our results for the tails in the current and density profiles of the quasiparticles within the framework of the Landau-Ginzburg theory.

For a vortex centered at the origin we write $\phi=f(r)e^{\pm i\theta}$, $\mathbf{A}+\mathbf{a}=\pm(\hbar c/e)\nabla\theta+g(r)\hat{\theta}$, $a_0(r)=h(r)/m^*$, and express the unknown functions f, g, and h as series in powers of m^*:

$$\begin{aligned} f(r)&=f^0(r)+m^*f^1(r)+\cdots , \\ g(r)&=h^0(r)+m^*h^1(r)+\cdots , \\ h(r)&=h^0(r)+m^*h^1(r)+\cdots . \end{aligned} \tag{19}$$

Upon inserting these series in the Landau-Ginzburg equations and collecting terms of each order in m^* we get a hierarchy of coupled equations. The nth order equations for f^n, g^n, and h^n involve the solutions to the lower order equations. The lowest-order ($O[(m^*)^{-1}]$) equations are independent of the potential interaction. The asymptotic (large r) behavior of their solutions is

$$\begin{aligned} f^0(r)&=\sqrt{n}+O(e^{-r/l}) , \\ g^0(r)&=O(e^{-r/l}) , \\ h^0(r)&=O(e^{-r/l}) , \end{aligned} \tag{20}$$

where l is the Landau length. At $O[(m^*)^0]$ the equations depend upon the detailed form of the $O[(m^*)^{-1}]$ solutions but their form at large r can be determined using (20) alone. For $1/r$ interactions [$V(\mathbf{r})=e^2/r$] we find the equations

$$\frac{-\hbar^2}{2}\nabla^2 f^1(r)+\sqrt{n}\,eh^1(r)+\sqrt{n}\,\frac{ee^*}{r}=O\left[\frac{1}{r^3}\right] , \tag{21}$$

$$2\sqrt{n}f^1(r)=\frac{e}{2\hbar\theta}\frac{1}{r}\partial_r[rg^1(r)]+O(e^{-r/l}) , \tag{22}$$

$$-\frac{e}{c}ng^1(r)=\frac{e}{2\hbar\theta}\partial_r h^1(r)+O(e^{-r/l}) . \tag{23}$$

Their solution is

$$\begin{aligned} f^1(r)&=\frac{e}{2n^{3/2}(2\hbar\theta)^2}\frac{e^*}{r^3}\left[1+O\left[\frac{1}{r^2}\right]\right] , \\ g^1(r)&=-\frac{c}{2n\hbar\theta}\frac{e^*}{r^2}\left[1+O\left[\frac{1}{r^2}\right]\right] , \\ h^1(r)&=-\frac{e^*}{r}\left[1+O\left[\frac{1}{r^2}\right]\right] . \end{aligned} \tag{24}$$

From these we recover (6) and (7) for the current and density. One feature of these results deserves comment. It is evident from (7) that a new length $\lambda=\nu\sigma_H/\omega_c$ enters the finite mass problem. It is straightforward to show[19] that the corresponding length for an arbitrary interaction is determined by the equation $\lambda^2=\tilde{V}(1/\lambda)\nu^2 m^*/\hbar^2 n$, where $\tilde{V}$ is the Fourier transform of the potential.

EFFECT ON STATISTICS

The long-ranged profiles of the quasiparticles have implications for their statistics as well. It has been shown by Lee and Hanna[11] that the phase obtained by braiding two quasiparticles at a finite distance differs from its asymptotic value (the statistics) by terms that fall off at most inversely with the distance. This is to be contrasted with the exponential approach found by Arovas, Schrieffer, and Wilczek.[4] Heuristically, this follows from our result for the density profile as the quasiparticles view the density of the fluid as flux.

SPIN AND STATISTICS

We turn now to the question of whether the quasiparticles obey a spin-statistics relation. The inspiration to look for such a connection is evidently the spin-statistics theorem in relativistic quantum field theory which holds for fractional statistics particles as well.[12] The relevance of that result to our system is not evident; nevertheless it is interesting to pursue this possibility.

To go any further we need a definition of quasiparticle spin. We define the spin of a quasiparticle as the (orbital) angular momentum of the state which contains exactly one quasiparticle at rest. This is analogous to the definition of a spin for fundamental particles that arise as quasiparticles in a relativistic field theory; the Pauli-Lubanski construction implements this definition covariantly.[13] In defining the spin we have ignored the intrinsic spin of the bare particles, i.e., the electrons. This is justified on the grounds that the (interesting case of the) fractional quantum Hall effect is essentially a property of spinless electrons. If the behavior of spinless electrons is sufficient to produce fractional statistics it should suffice to produce a commensurate spin as well.

In our problem there are at least two "natural" choices for the angular momentum operator. The first,

$$\begin{aligned} L^{(1)}&=\int d^2r\,\mathbf{r}\times\mathrm{Re}\left\{\psi^\dagger(\mathbf{r})\left[\frac{\hbar}{i}\nabla-\frac{e}{c}\mathbf{A}(\mathbf{r})\right]\psi(\mathbf{r})\right\} \\ &=m^*\int d^2r\,\mathbf{r}\times\mathbf{j}(\mathbf{r}) , \end{aligned} \tag{25}$$

is the mechanical angular momentum (j is the particle current operator) and is related to the magnetization by $M=(e/2m^*c)L^{(1)}$. $L^{(1)}$ is gauge invariant but it does not commute with the Hamiltonian, i.e., it is not a constant of the motion. The relevant commutator is

$$[H,L^{(1)}]=\hbar^2\omega_c\sum_{nm}\sqrt{(n+m+1)(n+1)}c^\dagger_{nm}c_{n+1m+1} - \sqrt{(n+m)n}\,c^\dagger_{nm}c_{n-1m-1}\ . \quad (26)$$

In this form it is clear that the matrix elements of (26) between arbitrary states in the lowest Landau level vanish and hence *only* if we restrict the state space of the system to the lowest Landau level does the commutator vanish.

The second choice,

$$L^{(2)}=L^{(1)}+\frac{e}{2c}(\mathbf{B}\cdot\hat{\mathbf{z}})\int d^2r\, r^2\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\ , \quad (27)$$

which is also gauge invariant, generates rotations and is a constant of the motion.[14] In symmetric gauge it reduces to the canonical angular momentum and takes the familiar form

$$L^{(2)}=\hbar\sum_{nm} mc^\dagger_{mn}c_{mn}\ . \quad (28)$$

To isolate the spin of the quasiparticle, which reflects the effect of a local rotation, one needs to exclude the effects of the boundary. We consider the definitions

$$S^{(i)}=\lim_{a\to\infty}\lim_{V\to\infty}\left\langle\int d^2r\, f\left[\frac{r}{a}\right]L_i(r)\right\rangle_{\rm QP}\ ,\quad i=1,2 \quad (29)$$

where V is the area of the system, a is a sampling length, $f(x)$ is a smooth monotonically deceasing function which takes the value unity at the origin and falls to zero at infinity, $L_i(\mathbf{r})$ are the densities corresponding to $L^{(i)}$, and the label QP denotes that the expectation value is taken in the one-quasiparticle state.

EVALUATION OF SPIN

First we evaluate (25) in the lowest Landau-level approximation. The current-density identity (4) enables us to obtain a general relation between $S^{(1)}$ and charge. For states that asymptotically reach a uniform density and are characterized by an integrated density excess Q we find

$$S^{(1)}=\hbar Q\ . \quad (30)$$

Consequently, quasiholes (electrons) in the Laughlin state at filling $\nu=1/m$ have spin

$$S^{(1)}=\pm\frac{\hbar}{m}\ , \quad (31)$$

while the ground state itself has spin zero.[15] Since the charge of the quasiparticles is expected (in the presence of a gap in the spectrum) to depend only upon the filling factor ν, independent of the details of the interaction, the same should be true of $S^{(1)}$. Though the quasiparticle spin is fractional for $m>1$ it nevertheless violates the spin-statistics connection since it changes sign between quasihole and quasielectron.[16] In fact it is off by a factor of 2 in magnitude as well; e.g., for $m=1$ $S^{(1)}$ should equal $\hbar/2$ to satisfy the spin-statistics relation.

In the lowest Landau level the expression (28) for $L^{(2)}$ reduces to

$$L^{(2)}=\hbar\sum_m mc^\dagger_{m0}c_{m0}\ . \quad (32)$$

Given the structure of the orbitals $u_{m0}(\mathbf{r})$ and the reduced Hilbert space[17] we can implement (29) by considering

$$S^{(2)}=\lim_{M\to\infty}\lim_{V\to\infty}\hbar\sum_{m=0}^{M} m\langle c^\dagger_{m0}c_{m0}\rangle_{\rm QP}\ . \quad (33)$$

Unfortunately this diverges in the one-quasiparticle state and indeed in any state of interest in the infinite volume limit (including the ground state) since the occupancy of the orbitals does not vanish with m for an infinite system. We consider instead the difference between the one-quasiparticle state and the ground state

$$S^{(2)}=\lim_{M\to\infty}\lim_{V\to\infty}\hbar\sum_{m=0}^{M} m\{\langle c^\dagger_{m0}c_{m0}\rangle_{\rm QP}-\langle c^\dagger_{m0}c_{m0}\rangle_G\}\ , \quad (34)$$

which is finite. In contrast to our result for $S^{(1)}$ we were unable to find a general relation between $S^{(2)}$ and the charge of the quasiparticles. In fact one might suspect from (27) that $S^{(2)}$ is sensitive to the details of the interaction since it differs from $S^{(1)}$ by the second moment of the density in the quasiparticle state. However, for the Laughlin wave functions the second moment is fixed by the constant screening sum rule for plasmas,[2] and $S^{(2)}$ vanishes for all filling factors. It is conceivable that a similar sum rule can be derived for more general quasiparticle states and that as a result $S^{(2)}$ is always zero;[18] however, it would still be in conflict with the result for statistics.

As the reader might expect, the situation deteriorates considerably when we remove the restriction to the lowest Landau level and consider the case of $1/r$ interactions. The presence of the long-range tail (6) leads to a divergent correction to $S^{(1)}$,

$$\delta S^{(1)}=m^*\sigma_H\left[\frac{e^*}{e}\right]R\ , \quad (35)$$

where R is a length that diverges with the size of the system. From (6) and (7) we discover that $S^{(2)}$ has a divergent correction as well.

To summarize: Both $L^{(1)}$ and $L^{(2)}$ fail to yield definitions of spin that are in agreement with the spin-statistics relation. Indeed, in the physically relevant problem they fail to yield definitions in any useful sense. Of course we cannot exclude the possibility that a different, satisfactory definition can be constructed.

MAGNETIZATION

Finally we would like to note a different estimate of (35) that may have implications for the magnetization calculations for Hall systems. Consider a droplet of Hall liquid of size R that contains a quasiparticle at the center.

If we increase the magnetic field through the system by an amount, δB, the flux through the system changes by $\delta\phi=\delta B\pi R^2$. For $\delta\phi<\phi_0$ the system responds by shrinking uniformly so that a charge $\delta q=\nu(\delta\phi/\phi_0)e$ appears at the boundary. Up to factors of order unity the change in energy due to the interactions (ignoring the change in $\hbar\omega_c$ which is interaction independent) is

$$\delta E=\frac{e^*\delta q}{R}=\frac{\nu e^2 e^*}{hc}R\,\delta B\ . \tag{36}$$

Hence we can obtain the contribution of the interactions to $L^{(1)}$ by computing the correction to the magnetization as

$$\delta M=\frac{\partial(\delta E)}{\partial(\delta B)}=\frac{\nu e^2 e^*}{hc}R\ . \tag{37}$$

Since $L^{(1)}=(2m^*c/e)M$ we recover (35).

It is possible that this contribution to the magnetization in the presence of quasiparticles may lead to calculable experimental consequences.[20] We should note though that the quantity of experimental interest is the thermodynamic magnetization which is calculated by taking the limit $\delta B\to\infty$ *after* the infinite volume limit is taken. Our calculation of the (microscopic) magnetization for a *single* quasiparticle essentially reverses this order of limits; hence, its connection to the thermodynamic quantity requires clarification.

ACKNOWLEDGMENTS

We would like to thank J. Chayes, T. Einarsson, S. Girvin, H. Hansson, A. Karlhede, D.-H. Lee, and B. Spivak for useful discussions. This work was supported in part by NSF Grant No. DMR-90-11803 at UCLA.

APPENDIX

First we will derive the asymptotic current distribution for a quasihole at $\nu=1$. In the notation of (3) the unperturbed quasihole state is $|qh\rangle\equiv c_{00}|g\rangle$ where $|g\rangle$ is the filled lowest Landau level. It is straightforward to write down an expression for the first perturbative correction to the current in the quasihole state; the only excited states that enter contain a single particle-hole excitation. We define

$$V(a,b,c,d)=\int d^2r\,d^2r'\,V(\mathbf{r}-\mathbf{r}')u_a^*(\mathbf{r})\times u_b^*(\mathbf{r}')u_c(\mathbf{r}')u_d(\mathbf{r})\ ,$$
$$j_\theta(a,b,\mathbf{r})=u_a^*(\mathbf{r})\left[\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{e}{c}A_\theta\right]u_b(\mathbf{r}')\ . \tag{A1}$$

(Here the indices a,b label both the Landau level n and the angular momentum m.) The correction to the current is

$$\delta j_\theta(\mathbf{r})=\sum_{n\geq 1}\frac{1}{n\hbar\omega_c}\sum_{m\geq 1}f_{nm}[j_\theta(0m,nm,\mathbf{r})+\text{c.c.}]\ , \tag{A2}$$

where c.c. denotes the complex conjugate and

$$f_{nm}=V(00,nm,0m,00)-V(00,nm,00,0m)\ . \tag{A3}$$

For Coulomb interactions the needed matrix elements can be evaluated in terms of standard (if somewhat unhelpful) functions:

$$V(00,nm,0m,00)=\frac{e^2}{l}\frac{1}{2^{n+m+1}}\frac{\Gamma(\frac{1}{2}+n+m)}{\sqrt{(n+m)!n!m!}}\times{}_2F_1(\tfrac{1}{2},-m,\tfrac{1}{2}-m-n;2)\ ,$$
$$V(00,nm,00,0m)=\frac{e^2}{l}\frac{1}{2^{n+m+1}}\frac{\Gamma(\frac{1}{2}+n+m)}{\sqrt{(n+m)!n!m!}}\ . \tag{A4}$$

${}_2F_1$ is the Gauss hypergeometric function in the notation of Ref. 21.

At large m (i.e., large r) the first of these matrix elements dominates the second. Further, the contribution from the first Landau level dominates that from the higher Landau levels as $V(00,nm,0m,00)$ falls off as $1/m^{(n+1)/2}$ at fixed n. Replacing $V(00,nm,0m,00)$ by its leading large m piece we find that the asymptotic large r behavior is determined by the series

$$\delta j_\theta(\mathbf{r})=\sum_{m\geq 1}\frac{e^2/l}{\hbar\omega_c}\frac{1}{2\sqrt{2m}}[j_\theta(0m,1m,\mathbf{r})+\text{c.c.}]\ . \tag{A5}$$

The task of showing that *this* series reduces to (6) (with $\nu=1$ and $e^*=e$) when analyzed for its large r behavior is left to the dedicated reader as an exercise.

Next we supply the details for the fractional case. From (13) we obtain

$$\delta j_\theta(\mathbf{r})=\langle qh|[T^1,j_\theta(\mathbf{r})]|qh\rangle=2\,\text{Re}\{\langle qh|T^1 j_\theta(\mathbf{r})|qh\rangle\} \tag{A6}$$

As the current in the ground state must vanish we can consider instead

$$\delta j_\theta(\mathbf{r})=2\,\text{Re}\{\langle qh|T^1 j_\theta(\mathbf{r})|qh\rangle-\langle g|T^1 j_\theta(\mathbf{r})|g\rangle\}$$
$$=\frac{1}{2}\sum_{n\geq 1}\frac{1}{n\hbar\omega_c}\sum_{j,k,l,m\geq 0}[\langle qh|c_{0j}^\dagger c_{0k}^\dagger c_{0l}c_{0m}|qh\rangle-\langle g|c_{0j}^\dagger c_{0k}^\dagger c_{0l}c_{0m}|g\rangle]\times V(0j,0k,0l,0m)[j_\theta(0m,nm,\mathbf{r})+\text{c.c.}]\delta_{j+k,l+m}\ , \tag{A7}$$

where the Kronecker δ enforces L_z conservation and we have omitted the background term as it manifestly does not contribute at large r.

Thus far our analysis of (A6) has been exact. As before, we are interested in the behavior of the series when m is large. We now invoke the hypothesis of short-ranged clustering to conclude: (a) that the only contribution to the difference of the expectation values arises when at least one of the remaining indices (quite generally we take this to be l) is near the origin (the location of the quasihole); and (b) that the expectation values are exponentially small unless k is near l and j is near m or vice versa. More precisely we find that, for $m \gg 1$ and l small,

$$\langle qh|c_{0j}^{\dagger}c_{0k}^{\dagger}c_{0l}c_{0m}|qh\rangle-\langle g|c_{0j}^{\dagger}c_{0k}^{\dagger}c_{0l}c_{0m}|g\rangle\sim\langle qh|c_{0j}^{\dagger}c_{0m}|qh\rangle\langle qh|c_{0k}^{\dagger}c_{0l}|qh\rangle-\langle qh|c_{0k}^{\dagger}c_{0m}|qh\rangle\langle qh|c_{0j}^{\dagger}c_{0l}|qh\rangle$$
$$-\langle g|c_{0j}^{\dagger}c_{0m}|g\rangle\langle g|c_{0k}^{\dagger}c_{0l}|g\rangle+\langle g|c_{0k}^{\dagger}c_{0m}|g\rangle\langle g|c_{0j}^{\dagger}c_{0l}|g\rangle\ . \quad \text{(A8)}$$

For $m \gg 1$, $\langle qh|c_{0j}^{\dagger}c_{0m}|qh\rangle\sim\langle g|c_{0j}^{\dagger}c_{0m}|g\rangle$ which equals $\nu\delta_{jm}$ by L_z conservation. Invoking L_z conservation once more forces the remaining indices to be equal. Making these replacements in (7) we find

$$\delta j_{\theta}(\mathbf{r})=\nu\sum_{n\ge 1}\frac{1}{n\hbar\omega_c}\sum_{l,m,\ge 0}[\langle qh|c_{0l}^{\dagger}c_{0l}|qh\rangle-\langle g|c_{0l}^{\dagger}c_{0l}|g\rangle]$$
$$\times[V(0l,0m,0m,0l)-V(0l,0m,0l,0m)][j_{\theta}(0m,nm,\mathbf{r})+\text{c.c.}]\ . \quad \text{(A9)}$$

As the sum over l will be cut off at the size of the quasihole ignoring the l dependence of the matrix elements will not affect the leading large m behavior This allows us to recognize the sum over l as the charge of the quasihole. Hence the leading behavior of the current is given by the series

$$\delta j_{\theta}(\mathbf{r})=\nu\frac{e^*}{e}\sum_{n\ge 1}\frac{1}{n\hbar\omega_c}\sum_{m\ge 0}f_{nm}[j_{\theta}(0m,nm,\mathbf{r})+\text{c.c.}]\ . \quad \text{(A10)}$$

Since this differs from (A5) only by the presence of the factor $\nu e^*/e$, (6) follows.

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[1] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990).

[2] R. B. Laughlin, in Ref. 1.

[3] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[4] D. P. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[5] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B **33**, 2481 (1986).

[6] For a field of 10 T and a sample of linear dimension 1 mm (7) yields $\delta e^*/e^*\sim 10^{-6}$.

[7] There are clearly other examples; these include the charge-current correlation functions which allow the intriguing possibility of extracting the Hall conductance from their asymptotic data.

[8] One can treat $\nu=1$ by the following analysis as well; the (nonrigorous) theorems are exact in that case. The separate treatment is intended to emphasize the simplicity of that case.

[9] Or possibly with a length connected with the magnetoroton minimum. There are arguments that suggest that this might in fact not be the case—for $1/r$ interactions there might be power laws in the lowest-Landau-level problem as well. However these would decay faster than the ones we are attempting to calculate and hence our (nonrigorous) theorem would be *essentially* correct.

[10] S.-C. Zhang, T. H. Hansson, and S. A. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[11] D. H. Lee and C. B. Hanna (unpublished).

[12] J. Fröhlich and P. A. Marchetti, Nucl. Phys. B **356**, 533 (1991).

[13] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[14] To construct $L^{(2)}$ we rotate a state in an arbitrary gauge by first gauge transforming to the symmetric gauge, rotating using the canonical angular momentum, and finally gauge transforming back to the original gauge. A similar procedure can be used to construct the conserved magnetic translation operators.

[15] Equation (11) has also been derived by S. M. Girvin (unpublished) and by T. Einarsson (unpublished).

[16] The quasihole and the quasielectron have the *same* statistics. This follows in the Chern-Simons description from the fact that quasiholes and quasielectrons carry opposite charge and flux. In Halperin's original formulation (Ref. 3) the same conclusion is reached by noticing that the choice of analyticity or antianalyticity of the pseudo-wave-functions for the quasiparticles depends upon their charge. For completeness we note that the "relative statistics" or the quasiholes and quasielectrons are opposite to their individual statistics.

[17] In the presence of Landau-level mixing (8) has to be implemented directly.

[18] S. M. Girvin (private communication).

[19] D.-H. Lee (private communication).

[20] S. L. Sondhi and S. A. Kivelson (unpublished).

[21] *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).