Suppression of the Josephson current through a narrow, mesoscopic, semiconductor channel by a single impurity

Philip F. Bagwell
School of Electrical Engineering, Purdue University, West Lafayette, Indiana 47907
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We study the Josephson current through a ballistic, normal, one-dimensional quantum channel in contact with two superconducting electrodes. A single point impurity having reflection coefficient R is placed in the normal conductor. The impurity couples the Andreev energy levels of forward and reverse moving electrons inside the junction, opening energy gaps in the quasiparticle level spectrum versus superconducting phase difference ϕ. These "Andreev" energy gaps suppress the Josephson current in much the same way as disorder suppresses the magnetic flux driven currents in a normal mesoscopic ring. Finite temperature "energy averages" the contribution of Andreev levels above the Fermi energy with those below μ, further suppressing the Josephson current. The portion of the Josephson current carried by scattering states outside the superconducting gap is similarly suppressed by disorder and finite temperature.

I. INTRODUCTION

The Josephson effect in a clean superconductor-normal-superconductor (SNS) junction is very different from the Josephson effect in a tunnel junction. Coherent Andreev reflection,[1] whereby an electron incident on a superconductor from a normal conductor is reflected as a hole, is the mechanism responsible for supercurrent flow in clean SNS junctions.[2] Quasiparticles incident from the normal region of an SNS junction are Andreev reflected off the pair-potential discontinuity at each SN interface, and their wave interference produces a set of resonant levels which carry the Josephson current. Such a description for the Josephson current assumes quasiparticles in the normal region can move without losing phase information, that is, it assumes the SNS junction is a mesoscopic system.

When the normal region of the SNS junction forms a quantum point contact,[3,4] the natural Josephson current scale is $I_c = e\Delta/\hbar$ (Refs. 5 and 6) (2Δ is the energy gap of the superconductor) and the current phase relation is $I(\phi) = I_c \sin(\phi/2)$.[6,7] If a tunnel barrier is then introduced into the normal region of the point contact, electrons often reflect simply as electrons, rather than as a hole. The current through such a dirty SNS point contact then evolves gradually[8–13] with decreasing barrier transmissivity T into the usual Josephson form[14] $I(\phi) = I_c \sin(\phi)$, where $I_c = e\Delta T/2\hbar$. In long and clean SNS junctions, where the extent of the normal region L is longer than the Bardeen-Cooper-Schrieffer (BCS) healing length ξ_0, the natural Josephson current scale is $I_c = ev_F/L$ (Refs. 2 and 15–18) (v_F is the Fermi velocity) and the current phase relation assumes a triangular shape. These developments, and other calculations where different types of mesoscopic systems are embedded in the normal region of an SNS junction,[18–20] are reviewed in Refs. 21 and 22.

In this paper we apply the Bogoliubov–de Gennes equations[23,24] to describe the Josephson current in both clean SNS junctions and in SNS junctions containing a tunnel barrier. We obtain expressions for the Josephson current versus superconducting phase difference ϕ for both short ($L < \xi_0$) and long ($L > \xi_0$) SNS junctions, and which account for both the bound levels inside the superconducting energy gap and scattering states outside the gap. We show that introducing a tunnel barrier forces quasiparticles in the normal region to form standing waves at $\phi = \pm\pi$, so that the effect of disorder on the Josephson current can be interpreted in a manner very analogous to electrons confined to a normal mesoscopic ring[25,26] or to electrons in a spatially periodic potential. Many magnetic flux sensitive effects predicted for normal mesoscopic rings[27–29] therefore have analogs for the Josephson effect in SNS junctions.

II. SNS JUNCTION WITH AN IMPURITY

We model the SNS junction[2] by a step change in the pair potential shown in Fig. 1, namely

$$\Delta(x) = \begin{cases} \Delta e^{i\phi_1}, & x < 0 \\ 0, & 0 < x < L \\ \Delta e^{i\phi_2}, & x > L. \end{cases} \quad (1)$$

This model for the pair potential requires that the superconductor widen adiabatically to an infinite width.[6,17] The large superconducting banks can then be viewed as an "order parameter reservoir," where the supercurrent density is effectively zero. In that case, phase gradients of the superconducting order parameter in the contacts, required to support supercurrent flow, are negligible compared to phase gradients near the junction itself. The disorder we describe by a single point impurity placed in the junction

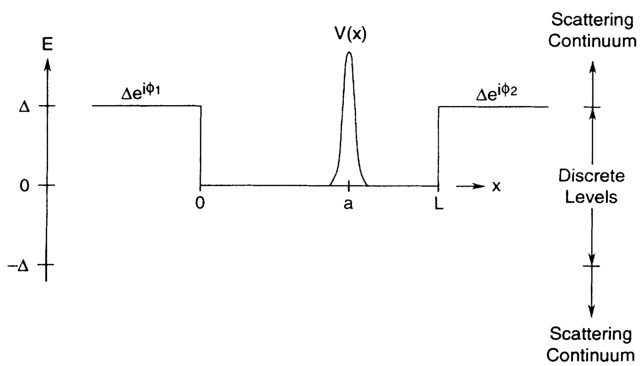

FIG. 1. Scattering potential which describes a mesoscopic Josephson junction. "Andreev" bound states form in the pair potential well. Both the bound levels and scattering states carry parts of the Josephson current.

$$V(x) = V_s \delta(x-a), \tag{2}$$

where $0 \leq a \leq L$.

The potentials $\Delta(x)$ and $V(x)$ enter the Bogoliubov–de Gennes equation of motion for the quasi-particles[23]

$$\begin{pmatrix} H(x)-\mu & \Delta(x) \\ \Delta^*(x) & -[H^*(x)-\mu] \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \tag{3}$$

where the one-electron Hamiltonian $H(x)$ is

$$H(x) = \frac{1}{2m}\left(-i\hbar\frac{d}{dx} - eA(x)\right)^2 + V(x). \tag{4}$$

Equation (3) embodies the quasiparticle dispersion relation for a uniform superconductor,

$$E^2 = \left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta^*\Delta, \tag{5}$$

and does not depend on the specific form of the microscopic Hamiltonian giving rise to the pairing potential. We neglect the vector potential A, the electron spin, and the self-consistency condition for $\Delta(x)$. Neglecting the self-consistency condition for $\Delta(x)$ is equivalent to neglecting the proximity effect.

The Andreev levels [the discrete eigenvalues E from Eq. (3) occurring for $|E| < \Delta$] are determined by matching the solutions of Eq. (3) for a spatially uniform pair potential at $x = 0$, $x = a$, and $x = L$. Details of this procedure are given in Appendix A. For a clean SNS junction this was first done by Kulik,[2] who found the Andreev levels are determined from

$$2\cos^{-1}\left(\frac{E}{\Delta}\right) + \left(\frac{L}{\xi_0}\right)\left(\frac{E}{\Delta}\right) \pm \phi = 2\pi n. \tag{6}$$

Here $\xi_0 = \hbar v_F/2\Delta$ is the BCS healing length, $\phi = \phi_2 - \phi_1$ the superconducting phase difference, and $n = 0, \pm 1, \pm 2, \ldots$.

Equation (6) can be interpreted in terms of Bohr-Sommerfeld quantization of the periodic electron-hole orbits in the normal region.[30] The term $2\cos^{-1}(E/\Delta)$ is the phase shift acquired from the evanescent quasiparticle waves penetrating into the superconducting regions. The term $(EL/\Delta\xi_0)$ is the phase shift acquired from free electron and hole propagation in the normal region. Finally the superconducting phase difference ϕ enters Eq. (6) because, in Andreev reflection, the reflected electron and holes acquire an additional phase shift equal to the phase of the superconducting order parameter. Changing ϕ therefore affects the energy levels in a manner similar to changing the size of a confining potential well in ordinary quantum mechanics.

In the presence of a point impurity potential $V(x)$, Eq. (6) is modified to[31]

$$2\cos^{-1}\left(\frac{E}{\Delta}\right) + \left(\frac{L}{\xi_0}\right)\left(\frac{E}{\Delta}\right) \pm \alpha = 2\pi n, \tag{7}$$

where the "effective phase" α is determined from

$$\cos(\alpha) = T\cos(\phi) + R\cos\left[\left(\frac{L-2a}{\xi_0}\right)\left(\frac{E}{\Delta}\right)\right]. \tag{8}$$

This effective phase α depends on the the normal electron current transmission probability T, given in our point defect model by

$$T = 1 - R = \frac{1}{1 + (mV_s/\hbar^2 k_F)^2}, \tag{9}$$

where $\hbar k_F = \sqrt{2m\mu}$ and μ is the Fermi energy. When $T \to 1$ we recover the Andreev levels for the clean junction, since $\alpha \to \phi$. In the opposite limit of small transmission, where $T \to 0$, α also becomes small and is nearly independent of the actual phase difference ϕ. If the junction is long compared to the healing length ($L > \xi_0$), and if an impurity is present ($T \neq 1$), α also depends on the particle's energy and the impurity position $|L - 2a|$.

To compute the Josephson current induced by the superconducting phase difference, one must compute both the electrical current $I_n^{\pm}(\phi)$ carried by quasiparticles occupying each Andreev level $E_n^{\pm}(\phi)$ ("discrete" levels with $|E_n^{\pm}| < \Delta$) and the imbalance $I(E,\phi)$ in the electrical current per unit energy carried by quasiparticles flowing in the "continuum" levels (where $|E| > \Delta$). Once both $I_n^{\pm}(\phi)$ and $I(E,\phi)$ are known, the total current can be written down if one knows the occupation probabilty for these energy levels. Since the Josephson current flows near equilibrium, we assume that the single particle levels are thermally populated according to the Fermi distribution function $f(E)$. We therefore obtain the contribution $I_d(\phi)$ to the Josephson current from the discrete spectrum as

$$I_d(\phi) = \sum_n \left\{ I_n^+(\phi) f(E_n^+(\phi)) + I_n^-(\phi) f(E_n^-(\phi)) \right\}, \tag{10}$$

and the contribution $I_c(\phi)$ to the Josephson current from the continuous spectrum as

$$I_c(\phi) = \left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right) I(E,\phi) f(E)\, dE. \tag{11}$$

The total supercurrent $I(\phi)$ is then computed from

$$I(\phi) = I_d(\phi) + I_c(\phi). \tag{12}$$

In this viewpoint only energies below the Fermi energy ($E \leq 0$) contribute to the Josephson current flow at $T = 0$. An additional contribution to the Josephson current results if the pair potential itself changes with superconducting phase difference.[13,21,32]

For a clean junction ($R = 0$), we obtain the electrical current[24] $I_n^{\pm}(\phi)$ carried by each occupied Andreev level from the normalized quasiparticle wave functions $u(x)$ and $v(x)$ as

$$I_n^{\pm}(\phi) = \mp \frac{ev_F}{L + 2\xi(E_n^{\pm}(\phi))}, \tag{13}$$

where $\xi(E)$ is the energy dependent healing length[24]

$$\xi(E) = \left(\frac{\hbar v_F}{2\Delta}\right)\left(\frac{\Delta}{\sqrt{\Delta^2 - E^2}}\right) = \xi_0 \left(\frac{\Delta}{\sqrt{\Delta^2 - E^2}}\right), \tag{14}$$

and $E_n^{\pm}(\phi)$ is found from Eq. (6). One could, in principle, perform the same calculation with the impurity present. Unfortunately, the analytical calculation proves intractable. However, if such a calculation could be done, it must yield

$$I_n^{\pm}(\phi) = \mp \frac{ev_F}{L + 2\xi(E_n^{\pm}(\phi))} \frac{T\sin(\phi)}{\sin(\alpha)} \frac{1}{\gamma}, \tag{15}$$

with $E_n^{\pm}(\phi)$ now found from Eqs. (7) and (8). Here γ is a "feedback" factor

$$\gamma = 1 \pm \left(\frac{\hbar}{2e\Delta}\right)\left(\frac{ev_F}{L + 2\xi(E_n^{\pm}(\phi))}\right)\left(\frac{R}{\sin(\alpha)}\right) \times \left(\frac{L - 2a}{\xi_0}\right) \sin\left(\frac{(L - 2a)E_n^{\pm}(\phi)}{\Delta\xi_0}\right). \tag{16}$$

Equations (15) and (16) follow from free energy considerations given below.

The continuous spectrum also contributes to the Josephson current flow in a long SNS junction having $L > \xi_0$. This "continuous spectrum" can be viewed as being composed of "leaky" Andreev levels.[17] When $|E| > \Delta$, it is possible for either the electron or hole composing the Andreev level to escape from the junction by simply transmitting over the pair potential. Although this leakage away from the junction leads to a finite lifetime for the Andreev levels, reducing the total contribution to $I(E, \phi)$ from the continuum Andreev level and spreading it out over a finite-energy range,[18] these broadened Andreev levels nonetheless exist and contribute significantly to the supercurrent flow in long SNS junctions.

We find it most convenient to calculate the continuum current $I_c(\phi)$ using the quasiparticle transmission formalism of van Wees, Lenssen, and Harmans.[18] The method of van Wees, Lenssen, and Harmans is closely analogous to the Landauer-Büttiker transmission formalism[4] for normal currents, and is mathematically equivalent to the transmission formalism of Furusaki, Takayanagi, and Tsukada.[20] Applying this formalism, we find (in Appendix B) the continuum contribution $I_c(\phi)$ for a clean SNS junction to be

$$I_c(\phi) = \frac{2e}{h}\left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right) |u_0^2 - v_0^2| \times \left(\frac{1}{D(E,-\phi)} - \frac{1}{D(E,\phi)}\right) f(E)dE, \tag{17}$$

where the function $D(E, \phi)$ is

$$D(E,\phi) = u_0^4 + v_0^4 - 2u_0^2 v_0^2 \cos\left[\left(\frac{E}{\Delta}\right)\left(\frac{L}{\xi_0}\right) + \phi\right]. \tag{18}$$

Here u_0 and v_0 are the standard BCS "coherence" factors

$$2u_0^2 = 1 + \frac{\sqrt{E^2 - \Delta^2}}{E}, \tag{19}$$

and

$$2v_0^2 = 1 - \frac{\sqrt{E^2 - \Delta^2}}{E}. \tag{20}$$

The contribution $I_c(\phi)$ to the Josephson current in a clean SNS junction has been calculated previously.[2,15–18,20] We feel Eq. (17) is probably identical to the analytical results of Ishii,[15] Svidzinski, Antsygina, and Bratus,[16] and Furusaki, Takayanagi, and Tsukada.[20] In the presence of the point impurity, Eq. (17) is modified to

$$I_c(\phi) = \frac{2e}{h}T\left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right) |u_0^2 - v_0^2| \times \left(\frac{1}{D(E,-\alpha)} - \frac{1}{D(E,\alpha)}\right) \frac{\sin(\phi)}{\sin(\alpha)} f(E)dE, \tag{21}$$

with the effective phase α defined in Eq. (8). A computation given in Appendix B yields Eq. (21).

Equations (17) and (21) show that the continuous spectrum makes no contribution to the Josephson current if $L \ll \xi_0$, since Eq. (18) gives $D(E, \alpha) = D(E, -\alpha)$ when $L \ll \xi_0$. Further, only energies within a few Δ of the Fermi level make any sizable contribution to $I_c(\phi)$, since the phase sensitive term in Eq. (18) for $D(E, \phi)$ is significant only within a few Δ of the Fermi energy. Thus, the Josephson current at $T = 0$ is not limited to flowing only at the Fermi energy (the case for normal currents at $T = 0$ and small bias voltages), or even limited to flow within an energy range Δ below the Fermi level (false only if $L \ll \xi_0$), but flows over an energy range extending several Δ below the Fermi energy when $L > \xi_0$. Equations (17) and (21) also show that the continuum contribution $I_c(\phi)$ is negligible near $\phi = \pm\pi$, and that $I_c(\phi = \pm\pi) = 0$.

An alternate method to compute Josephson currents requires the change in Helmholtz free energy F with superconducting phase difference[13,21,32,33]

$$I(\phi) = \frac{2e}{\hbar}\frac{dF(\phi)}{d\phi}. \tag{22}$$

Equation (22) provides additional insight into the supercurrent formulas presented in this section. Since Eqs. (7)

and (8) hold for the SNS junction with an impurity, our results for the dirty SNS junction follow from those of the clean SNS junction plus the chain rule of differentiation

$$\frac{dF}{d\phi} = \frac{dF}{d\alpha}\frac{d\alpha}{d\phi}, \tag{23}$$

where we have from Eq. (8)

$$\left.\frac{\partial\alpha}{\partial\phi}\right|_E = \frac{T\sin(\phi)}{\sin(\alpha)}. \tag{24}$$

The factor (24) appears both in our expression (15) for the current $I_n^{\pm}(\phi)$ carried by each Andreev level and in the expression (21) for the continuum supercurrent $I_c(\phi)$. As we argue more precisely in Appendix C, Eq. (23) for the derivative of the free energy, together with Eqs. (7) and (8), implies that Eq. (15) for $I_n^{\pm}(\phi)$ in the dirty junction follows from Eq. (13) for the clean junction. By similar arguments, Eq. (21) for $I_c(\phi)$ in the dirty junction follows from Eq. (17) for the clean junction.

III. JUNCTION SHORTER THAN THE HEALING LENGTH

In the limit of a "point contact" or "short" junction, for which $L \ll \xi_0$, we immediately find from Eq. (8) that the Andreev level spectrum is determined as

$$E^{\pm} = \pm\Delta\cos(\alpha/2) = \pm\Delta\sqrt{1 - T\sin^2(\phi/2)}, \tag{25}$$

where

$$\cos(\alpha) = T\cos(\phi) + R. \tag{26}$$

Equation (25) is a special case of the more general expression derived by Beenakker[13] when multiple quantum channels are open.

We graph Eq. (25) in Fig. 2.[34] The dotted lines show the position of the Andreev levels versus phase difference when no impurities are present.[6,35] The level $E^+(\phi)$ corresponds to an electron traveling in the negative-x direction, while the level $E^-(\phi)$ describes an electron moving along the positive-x direction. These two sets of Andreev levels are independent when no impurity is present in the junction. However, since the impurity will reflect electrons, it will couple these unperturbed Andreev levels. This effect is most pronounced when the levels are degenerate in energy, i.e., at $\phi = \pm\pi$, so that the impurity opens an energy gap in the Andreev level spectrum versus superconducting phase difference having size

$$E_{\rm gap} = 2\Delta\sqrt{R}. \tag{27}$$

Figure 2 is reminiscent of the effects of impurities on the energy level spectrum of a small normal metal ring subject to an Aharonov-Bohm flux,[25,26] and has also been previously suggested for a composite ring made of part normal and part superconducting material when the normal segment is disordered.[36] However, the motion of Andreev reflected electrons in the SNS junction is already periodic, so that a ring geometry (or other type of spatially periodic potential) is not required for $V(x)$ to open gaps in the quasiparticle energy spectrum. It is these "standing waves" forming in the energy level spectrum $E_n^{\pm}(\phi)$, due to the impurity, which suppress the Josephson current.

Using the properties $E^+ = -E^-$ and $f(E^+) = 1 - f(E^-)$, we readily obtain from Eqs. (25), (15), and (10)

$$I(\phi) = \frac{e\Delta}{\hbar}T\sin(\alpha/2)\left(\frac{\sin(\phi)}{\sin(\alpha)}\right)\tanh\left(\frac{\Delta}{2kT}\cos(\alpha/2)\right). \tag{28}$$

Writing this in terms of the phase difference ϕ, we find

$$I(\phi) = \frac{e\Delta}{2\hbar}T\left(\frac{\sin(\phi)}{\sqrt{1 - T\sin^2(\phi/2)}}\right) \times \tanh\left(\frac{\Delta}{2kT}\sqrt{1 - T\sin^2(\phi/2)}\right), \tag{29}$$

again a limiting case of the multichannel formula from Ref. 13. The single-channel formula, Eq. (29), has also been obtained previously.[8–13]

We consider the zero-temperature limit of Eqs. (28) and (29) in Fig. 3, where we can set $\tanh(E^+/2kT) = 1$. In the $kT = 0$ case, Eq. (28) becomes

$$I(\phi) = \frac{e\Delta}{\hbar}T\sin(\alpha/2)\left(\frac{\sin(\phi)}{\sin(\alpha)}\right). \tag{30}$$

Equation (30) interpolates nicely between the point contact and tunnel junction, as discussed in Refs. 13 and 21. When the transmission is nearly one, so that the effective phase difference $\alpha \simeq \phi$, we obtain the result[7] for a point contact, $I(\phi) = (\pi G\Delta/e)\sin(\phi/2)$, where $G = 2e^2/h$. In the other limit, where the transmission T is small, the effective phase α is also small and is nearly independent of ϕ. We can then approximate $\sin(\alpha) \simeq \alpha$

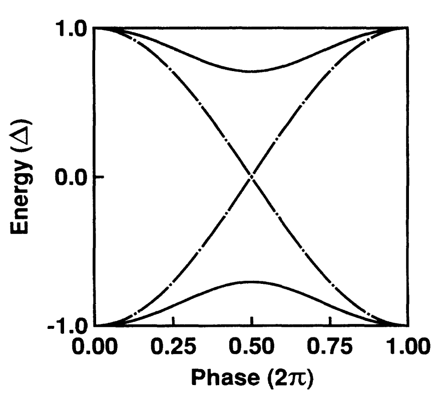

FIG. 2. Andreev levels $E^{\pm}(\phi)$ as a function of superconducting phase difference ϕ for a junction shorter than the healing length ($L \ll \xi_0$). The unperturbed levels in the clean junction (dotted), corresponding to a left-moving (E^+) and right-moving (E^-) electrons, become coupled by the impurity. As a result, the energy degeneracy at $\phi = \pm\pi$ is lifted and an energy gap of size $E_{\rm gap} = 2\Delta\sqrt{R}$ appears in the quasiparticle energy spectrum for the dirty SNS junction (solid).

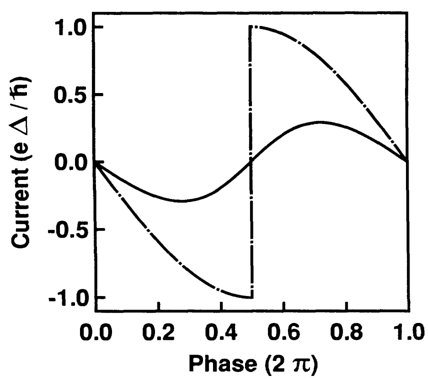

FIG. 3. The "discretized" Josephson current in a clean point contact (dotted) is suppressed by the presence of a tunnel barrier (solid). Regions analogous to "negative differential resistance" in a normal conductor, where the Josephson current decreases as the phase difference increases, are produced by the energy gap in Fig. 2.

so that we obtain the result for a tunnel junction[14] $I(\phi) = (\pi G\Delta/2e)\sin(\phi)$, where the normal conductance is given by a Landauer-type formula $G = 2e^2T/h$.[4]

The effect of finite temperature on the Josephson current is qualitatively similar to that of impurity scattering, since both introduce a mixture of Andreev levels associated with left- and right-moving electrons into the supercurrent. The temperature dependence of Eq. (28) arises from the combination of Fermi factors

$$f(E^-) - f(E^+) = 1 - 2f(E^+) = \tanh\left(\frac{E^+}{2kT}\right), \quad (31)$$

so that the temperature dependence of the Josephson current can be viewed as thermally "energy averaging"[37] the contribution of some left-moving electrons above the Fermi energy with right-moving ones below μ. The high-temperature limit of Eq. (28) as $kT \to kT_c$ is[13]

$$I(\phi) = \frac{e\Delta^2}{4\hbar kT_c}T\sin(\phi), \quad (32)$$

which has the same $\sin(\phi)$ dependence as a tunnel junction, again illustrating the similar effects of disorder and finite temperature on the Josephson current. Equation (32) resembles the well-known result of Aslamazov and Larkin[38] for the Josephson current through a point contact calculated from Ginzburg-Landau theory. But since the Ginzburg-Landau theory is valid only near T_c, it will never produce the correct low-temperature form $\sin(\phi/2)$ for the clean point contact.

The sharp discontinuity at $\phi = \pm\pi$ for the clean junction at zero temperature arises from the sharp discontinuity in the Fermi function at $T = 0$.[17] When $\phi = \pi$ the Andreev level E^- (carrying a current $I^- = e\Delta/\hbar$) moves above the Fermi level, while the level E^+ (carrying a current $I^+ = -e\Delta/\hbar$) passes below the Fermi level. Thus, the level E^- empties while E^+ becomes populated. This change in Andreev level occupation produces a discontinuity in the supercurrent of $2e\Delta/\hbar$. On the other hand, when the transmission T is small, the levels E^+ and E^- have combined to form a standing wave with $dE^{\pm}/d\phi = 0$ when $\phi = \pm\pi$, so that the Josephson current falls to zero near $\phi = \pm\pi$.

One would naively predict a critical "depairing current"[23,39,40] of order $e\Delta/\hbar$ for a one-dimensional (1D) superconductor. In the presence of a uniform superfluid flow having velocity v_s, the quasiparticle energies are shifted by an amount $\Delta E = p_F\cdot v_s$, where p_F is the Fermi momentum. Thus, the quasiparticle energy gap goes to zero when $\Delta = p_F v_c$, where v_c is the critical "depairing velocity," an argument originally advanced by Landau.[40] All the electrons below the Fermi energy then move at velocity v_c, giving a critical current I_c of

$$I_c = env_c = e\left(\frac{2k_F}{\pi}\right)\left(\frac{\Delta}{p_F}\right) = \left(\frac{e\Delta}{\hbar}\right)\left(\frac{2}{\pi}\right), \quad (33)$$

which is correct to within a numerical factor of order unity. A better treatment[41] would probably slightly increase the numerical value of this current. Thus, the "discretization of the Josephson current" in a superconducting quantum point contact,[6] which follows as a limiting case of Eq. (6), can be understood as simply the Landau depairing current of a 1D superconductor.

IV. JUNCTION LONGER THAN THE HEALING LENGTH

When the junction length L becomes comparable to the BCS healing length ξ_0, more Andreev levels become bound in the pair potential well and the continuum levels begin to carry a supercurrent. Despite these differences in detail, the basic picture given in the preceding section is still qualitatively correct. Quasiparticles trapped in the normal region of the SNS junction set up standing waves near $\phi = \pm\pi$ when an impurity is present, suppressing the Josephson current.

Figure 4 shows the Andreev levels [from Eqs. (7) and (8)] in a clean SNS junction (dashed line) and when an impurity is placed in the normal region (solid line).[42] The discrete energy level spectrum is quite similar to the normal mesoscopic ring[26] and to electrons in a periodic lattice. The energy gap near $E = 0$ for $\phi = \pi$, $a \simeq L/2$, and small R can be found (Appendix D) as

$$E_{\rm gap} \simeq 2\sqrt{R}\,(\hbar v_F/L_{\rm eff}), \quad (34)$$

where $L_{\rm eff}^2 = (L + 2\xi_0)^2 + R(L - 2a)^2$. For both long and short SNS junctions, the Andreev gap therefore obeys

$$\pi E_{\rm gap} \simeq \sqrt{R} I_c \Phi_0, \quad (35)$$

where I_c is the critical current of the clean junction and $\Phi_0 = h/e$ is the normal metal flux quantum. The statement (35) is exact for a short junction.

We plot the Josephson current corresponding to Fig. 4 in Fig. 5. In Fig. 5(a) we recover the known result for the Josephson current (solid) in a long and clean SNS junction.[15–18] The current phase relation is triangular with a critical current $I_c = ev_F/(L + 2\xi_0)$. The critical current I_c is a consequence solely of the current $I_d(\phi)$ carried by the Andreev levels (short dashes), since Eq. (21) shows the continuum current is zero at $\phi = \pi$. The crit-

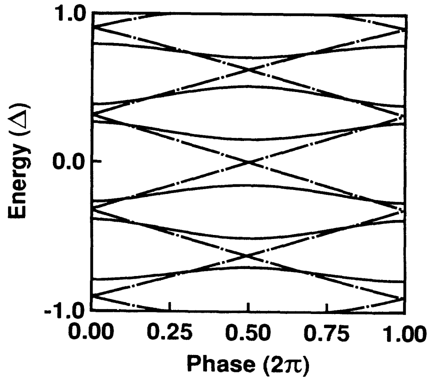

FIG. 4. Andreev levels in a long ($L > \xi_0$) junction with (solid) and without (dashed) an impurity present. As for the short junction, Andreev gaps open in the quasiparticle level spectrum. The analogy with "free electron" and "periodic potential" energy bands is evident in the figure.

ical current $I_c = ev_F/(L + 2\xi_0)$ follows from Eq. (13) and holds for all values of L, even for $L \ll \xi_0$ when the current phase relation is not triangular. The discretized Josephson current from Sec. III also follows by taking $L \to 0$ in this formula, namely $I_c = ev_F/2\xi_0 = e\Delta/\hbar$.

To obtain the correct dependence of the supercurrent on the phase (triangular as $L \to \infty$), it is necessary to account for the current $I_c(\phi)$ carried in the continuum levels (long dashes). The continuum currents can be understood naively and qualitatively by extrapolating the energy levels from Fig. 4 into the continuum and, ignoring their broadening, treating these continuum Andreev levels "as if" they carried the current from Eqs. (10) and (C4). Thus, the continuum currents $I_c(\phi)$ augment the discrete level currents $I_d(\phi)$ before the first Andreev level moves into the continuum near $\phi \simeq \pi/4$. $I_c(\phi)$ opposes $I_d(\phi)$ after the Andreev level closest to $E = -\Delta$ becomes unbound. (The precise value of ϕ where the lowest Andreev level unbinds depends on the junction length L.) We note the continuum supercurrent $I_c(\phi)$ is not sinusoidal.[15] We also note that $I(\phi)$ is a "small" difference between many larger currents[15–17] carried in each discrete level, along with the larger continuum currents.

The Josephson current through a long and dirty junction is shown in Fig. 5(b). The triangular current phase relation for the clean junction has evolved into an approximately sinusoidal dependence (solid). Neither $I_c(\phi)$ (long dashes) or $I_d(\phi)$ (short dashes) appears to be sinusoidal, although their sum $I(\phi)$ is approximately sinusoidal. For a long and dirty SNS junction, the continuum contribution $I_c(\phi)$ is not only essential to obtain a sinusoidal $I(\phi)$, but is also required to obtain the correct magnitude of the critical current. The continuum levels now contribute to the critical current magnitude, since I_c no longer occurs at $\phi = \pm\pi$. The parameters chosen[42] for this calculation give a transmission probability $T \simeq \frac{1}{2}$, so that comparing Figs. 5(a) and 5(b) reveals (for this calculation) that the critical current in the dirty junction is less than $T/2$ times the critical current of the clean junction. For a more transparent impurity, these ratios would be limited by the transmission coefficient T.

The continuum contribution $I_c(\phi)$ to the Josephson current is suppressed by finite temperature in the same way as the discrete contribution $I_d(\phi)$. Equation (21) can be written as an integral solely over the excitation spectrum as

$$I_c(\phi) = \frac{2e}{h}T\int_{\Delta}^{\infty} |u_0^2 - v_0^2| \left(\frac{1}{D(E,-\alpha)} - \frac{1}{D(E,\alpha)}\right) \times \frac{\sin(\phi)}{\sin(\alpha)}[2f(E) - 1]dE. \qquad (36)$$

The temperature-dependent factor in Eq. (36) is $1 - 2f(E) = \tanh(E/2kT)$, implying the same physics of thermal "energy averaging"[37] for currents carried in the continuum levels as that discussed for the discrete levels in Sec. III. Thus, in this viewpoint, all the Josephson current flowing in the continuum at $T = 0$ is carried below the superconducting gap at $E \leq -\Delta$.

One could plausibly argue that moving the single impurity inside the normal region would produce mesoscopic fluctuations[13,43] in the critical current I_c. Equa-

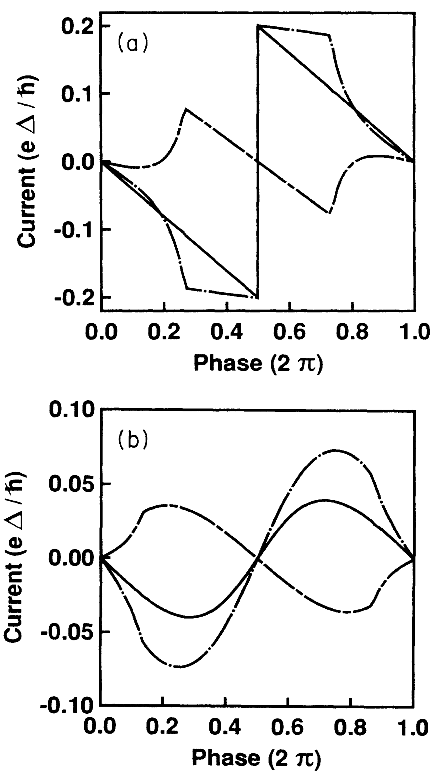

FIG. 5. Josephson current (solid) in a long junction that (a) is clean (b) has an impurity. The impurity rounds the triangular current versus phase relation into a sinusoidal shape, similar to its effect on the short SNS junction. Currents flowing in the scattering continuum (long dashes) and discrete levels (short dashes) combine to give the total Josephson current.

tions (7) and (8) show that moving the single impurity will change the Andreev energy levels $E_n^{\pm}$ in a long junction ($L \gg \xi_0$). However, it is not obvious that these fluctuating energy levels will cause the Josephson current to fluctuate. For example, it was originally believed from an analogous argument that the critical current would oscillate by varying L even in a clean SNS junction.[2] Later it was realized[15–17] that the critical current does not oscillate, but instead decreases monotonically with L as $I_c = ev_F/(L + 2\xi_0)$. This is because, as the various Andreev levels are forced into the continuum by reducing L, they do not immediately stop carrying a supercurrent. Instead, their contribution to I_c gradually decreases as they move farther outside the superconducting gap. A similar situation holds when moving the single impurity inside the SNS junction.

Moving the impurity away from the center of the junction ($a = L/2$) monotonically reduces the critical current I_c in Fig. 6,[44] showing that I_c does not fluctuate or oscillate with a. As the Andreev levels are forced farther away from the Fermi level by moving the impurity away from the center of the SNS junction, their contribution to I_c gradually decreases. The approximate analysis given in Appendix D agrees with this numerical result. If the impurity is moved a few healing lengths inside one of the superconducting contacts, the critical current should recover its full ballistic value ($I_c^{\text{clean}} = 0.2e\Delta/\hbar$ for the parameters used in Fig. 6), though we have not performed this calculation. In contrast, when the contacts are in their normal state, moving the impurity has no effect on the conductance. The normal state conductance[4] $G = 2e^2T/h$ does not fluctuate as a function of impurity position a, since the transmission coefficient T from Eq. (9) is independent of the impurity position. Therefore, the changing I_c as a function of impurity position a in Fig. 6 is a mesoscopic effect due to the interference of electron waves with the pair potential discontinuities at each SN interface. Mesoscopic fluctuations of the critical current studied in Refs. 13 and 43 have a different origin, arising from the wave interference between multiple impurity scattering events inside the normal region.

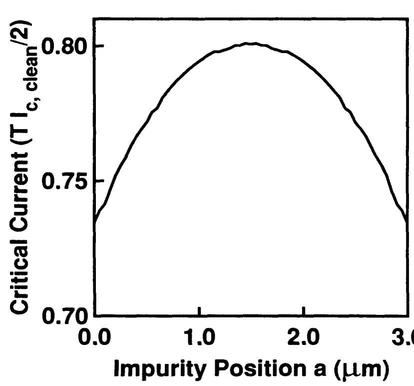

FIG. 6. Moving the lone impurity away from the center of the normal region suppresses the Josephson current in a long SNS junction, even though the normal state conductance of the junction remains unchanged. This mesoscopic effect is due to wave interference of the quasiparticles with the discontinuity in the pairing potential.

V. CONCLUSIONS

Bulk supercurrents flow in response to gradients in the phase $\phi(x)$ of the order parameter $\Delta(x) = |\Delta(x)|\exp[i\phi(x)]$, and are maximum when the phase gradient $\nabla\phi(x)$ is large. From this perspective, the Josephson relation for a tunnel junction, $I = I_c\sin(\phi)$, is quite counterintuitive. The Josephson tunneling current $I = I_c\sin(\phi)$ decreases as the driving phase difference increases, and is smallest when the superconducting phase difference is a maximum. The current phase relation $I = I_c\sin(\phi)$ is thus analogous to negative differential resistance in a normal conductor, where the normal current decreases as the driving voltage difference increases.

A more intuitive current phase relation exists for clean SNS junctions. For either long or short junctions, the Josephson current always increases as the superconducting phase difference increases, and is maximum when the superconducting phase difference is a maximum. Only when an impurity, tunnel barrier, or geometrical irregularity is present in the junction is the supercurrent zero when the superconducting phase difference is maximum at $\phi = \pm\pi$. The quasiparticle energy gap opening at $\phi = \pm\pi$ in the Andreev level spectrum versus superconducting phase difference when these imperfections are present, and the associated formation of "standing waves" with $dE_n^{\pm}/d\phi = 0$, explain the resulting suppression of the Josephson current. Finite temperature also mixes in a contribution to the Josephson current from the Andreev levels associated with reverse-moving electrons to those of the forward-moving ones, via thermal "energy averaging" instead of directly modifying the energy level spectrum, and therefore has a qualitatively similar effect as disorder on suppressing the Josephson current.

ACKNOWLEDGMENTS

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APPENDIX A: SCATTERING FROM THE SNS POTENTIAL

In this appendix we obtain the electrical current[24] transmission amplitude for the scattering of an electronlike excitation incident from the left contact on the potentials $\Delta(x)$ and $V(x)$, shown schematically in Fig. 7. To

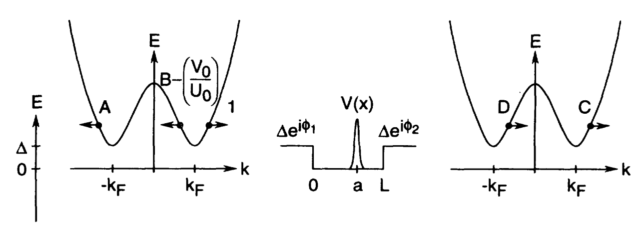

FIG. 7. An electronlike excitation incident from the left contact generates both reflected and transmitted quasiparticles. The transmission amplitudes C and D as a function of the superconducting phase difference $\phi_2 - \phi_1$ determine the Josephson current.

do this, we match the solutions of Eq. (3) in the regions of uniform pair potential shown in Fig. 1. The poles of this transmission amplitude[45] yield Eqs. (7) and (8) for the discrete Andreev levels, and therefore the electrical current $I_n^{\pm}(\phi)$ carried by the discrete levels. Appendix B shows how the transmission coefficient found in this appendix can be used to obtain Eq. (21) for the continuum current $I_c(\phi)$.

The energy dispersion curve from Eq. (5) can be described by the wave vectors k_e and k_h, where

$$\frac{\hbar^2 k_e^2}{2m} - \mu = \sqrt{E^2 - \Delta^2}, \tag{A1}$$

and

$$\frac{\hbar^2 k_h^2}{2m} - \mu = -\sqrt{E^2 - \Delta^2}, \tag{A2}$$

with $|k_e| > k_F$ and $|k_h| < k_F$. The "excitation spectrum" of this dispersion law, which includes only the states having $E > 0$, is plotted in Fig. 7.

For a homogeneous superconductor having a pair potential $\Delta(x) = \Delta e^{i\phi_1}$, Eq. (3) has both electronlike solutions for $E > 0$,

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} u_0 e^{i\phi_1} \\ v_0 \end{pmatrix} e^{\pm i k_e x}, \tag{A3}$$

and holelike solutions for $E > 0$,

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} v_0 e^{i\phi_1} \\ u_0 \end{pmatrix} e^{\pm i k_h x}. \tag{A4}$$

The "coherence" factors u_0 and v_0 are given by Eqs. (19) and (20). For $E^2 \leq \Delta^2$, $u(x)$ and $v(x)$ decay inside the superconductor[24] with an energy dependent healing length $\xi(E)$ given in Eq. (14). The current carried by normal electrons in the N region (for $E^2 < \Delta^2$) is then converted to a supercurrent within a length $\xi(E)$ inside the superconductor.[24]

We consider now an electronlike excitation incident from the left contact above the superconducting gap, as shown in Fig. 7. The waves generated by this incident electron can be grouped into four sets (denoted by coefficients A, B, C, and D) which properly include the Andreev reflections. For the electronlike excitations transmitted into right contact we have

$$C \begin{pmatrix} u_0 e^{i\phi_2} \\ v_0 \end{pmatrix} e^{i k_e (x - L)} \quad (x > L), \tag{A5}$$

supported by electrons in the normal region

$$C \begin{pmatrix} u_0 e^{i\phi_2} \\ 0 \end{pmatrix} e^{i \tilde{k}_e (x - L)} \quad (a < x < L), \tag{A6}$$

and holes in the normal region

$$C \begin{pmatrix} 0 \\ v_0 \end{pmatrix} e^{i \tilde{k}_h (x - L)} \quad (a < x < L). \tag{A7}$$

Similarly, for holelike excitations transmitted into the right contact, the waves

$$D \begin{pmatrix} v_0 e^{i\phi_2} \\ u_0 \end{pmatrix} e^{-i k_h (x - L)} \quad (x > L), \tag{A8}$$

$$D \begin{pmatrix} v_0 e^{i\phi_2} \\ 0 \end{pmatrix} e^{-i \tilde{k}_e (x - L)} \quad (a < x < L), \tag{A9}$$

and

$$D \begin{pmatrix} 0 \\ u_0 \end{pmatrix} e^{-i \tilde{k}_h (x - L)} \quad (a < x < L) \tag{A10}$$

are required. I use the notation $\tilde{k}_e$ and $\tilde{k}_h$ to denote the wave vectors in Eqs. (A1) and (A2) when $\Delta = 0$.

A similar procedure applied to the left SN interface is only slightly complicated by the presence of an incident electronlike wave (source term). This source term, which carries an electrical current[24] of $e\hbar k_e/m$, is

$$\begin{pmatrix} u_0 e^{i\phi_1} \\ v_0 \end{pmatrix} e^{i k_e x} \quad (x < 0). \tag{A11}$$

The Andreev reflected holelike excitation is

$$\left(B - \frac{v_0}{u_0}\right) \begin{pmatrix} v_0 e^{i\phi_1} \\ u_0 \end{pmatrix} e^{i k_h x} \quad (x < 0). \tag{A12}$$

The resulting electrons in the normal region are

$$\left(B - \frac{v_0}{u_0} + \frac{u_0}{v_0}\right) \begin{pmatrix} v_0 e^{i\phi_1} \\ 0 \end{pmatrix} e^{i \tilde{k}_e x} \quad (0 < x < a), \tag{A13}$$

and holes

$$B \begin{pmatrix} 0 \\ u_0 \end{pmatrix} e^{i \tilde{k}_h x} \quad (0 < x < a). \tag{A14}$$

The electronlike excitations reflected into the left contact are

$$A \begin{pmatrix} u_0 e^{i\phi_1} \\ v_0 \end{pmatrix} e^{-i k_e x} \quad (x < 0), \tag{A15}$$

supporting electrons in the normal region

$$A \begin{pmatrix} u_0 e^{i\phi_1} \\ 0 \end{pmatrix} e^{-i \tilde{k}_e x} \quad (0 < x < a), \tag{A16}$$

and holes

$$A \begin{pmatrix} 0 \\ v_0 \end{pmatrix} e^{-i \tilde{k}_h x} \quad (0 < x < a). \tag{A17}$$

The scattered waves written above assume $\mu \gg \Delta$, so that we can neglect the difference $(k_e - k_h)$ except when it appears in an exponent. This "Andreev approximation" is usually made in most works on the Josephson effect in SNS junctions.

The scattering matrix[46] (for both electrons and holes) connects the current amplitudes of incoming and outgoing waves at the impurity ($x = a$). The scattering matrix for electrons at $x = a$ gives

$$\begin{pmatrix} C u_0 e^{i\phi_2} e^{i \tilde{k}_e (a - L)} \\ A u_0 e^{i\phi_1} e^{-i \tilde{k}_e a} \end{pmatrix} = \begin{pmatrix} t(\tilde{k}_e) & r(\tilde{k}_e) \\ r(\tilde{k}_e) & t(\tilde{k}_e) \end{pmatrix} \begin{pmatrix} \left(B - \frac{v_0}{u_0} + \frac{u_0}{v_0}\right) v_0 e^{i\phi_1} e^{i \tilde{k}_e a} \\ D v_0 e^{i\phi_2} e^{-i \tilde{k}_e (a - L)} \end{pmatrix}. \tag{A18}$$

Equation (A18) contains the source term. A similar scattering matrix connects the the hole current amplitudes

$$\begin{pmatrix} Du_0 e^{-i\tilde{k}_h(a-L)} \\ Bu_0 e^{i\tilde{k}_h a} \end{pmatrix} = \begin{pmatrix} t^*(\tilde{k}_h)\, r^*(\tilde{k}_h) \\ r^*(\tilde{k}_h)\, t^*(\tilde{k}_h) \end{pmatrix} \begin{pmatrix} Av_0 e^{-i\tilde{k}_h a} \\ Cv_0 e^{i\tilde{k}_h(a-L)} \end{pmatrix}, \quad \text{(A19)}$$

where, for our point scatterer model, the current transmission and reflection amplitudes are

$$t(k) = \frac{1}{1 + i(mV_s/\hbar^2 k)}, \quad \text{(A20)}$$

and

$$r(k) = \frac{-i(mV_s/\hbar^2 k)}{1 + i(mV_s/\hbar^2 k)}. \quad \text{(A21)}$$

The electrical current[24] flowing into the right contact, due to the electrical current imposed from the left contact, is

$$J_Q \simeq \frac{e\hbar}{m}\tilde{k}_e \left(|C|^2 - |D|^2\right). \quad \text{(A22)}$$

Hence, we must compute the coefficients C and D to obtain the electrical current. The matrix equation for C is

$$\left(\frac{v_0}{u_0}\right)^2 \begin{pmatrix} t(\tilde{k}_e)\, r(\tilde{k}_e) \\ r(\tilde{k}_e)\, t(\tilde{k}_e) \end{pmatrix} \begin{pmatrix} e^{i(\tilde{k}_e - \tilde{k}_h)a} & 0 \\ 0 & e^{i\phi} e^{-i(\tilde{k}_e - \tilde{k}_h)(a-L)} \end{pmatrix}$$
$$\times \begin{pmatrix} t^*(\tilde{k}_h)\, r^*(\tilde{k}_h) \\ r^*(\tilde{k}_h)\, t^*(\tilde{k}_h) \end{pmatrix} \begin{pmatrix} e^{-i(\tilde{k}_e - \tilde{k}_h)(a-L)} e^{-i\phi} & 0 \\ 0 & e^{i(\tilde{k}_e - \tilde{k}_h)a} \end{pmatrix} \begin{pmatrix} C' \\ A' \end{pmatrix} + \begin{pmatrix} t(\tilde{k}_e)\, r(\tilde{k}_e) \\ r(\tilde{k}_e)\, t(\tilde{k}_e) \end{pmatrix} \begin{pmatrix} \left[1 - \left(\frac{v_0}{u_0}\right)^2\right] e^{i\tilde{k}_e a} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \begin{pmatrix} C' \\ A' \end{pmatrix}. \quad \text{(A23)}$$

Here C' and A' are the same as C and A up to a phase factor, so that $C' = C\exp(i\phi)\exp[i\tilde{k}_e(a-L)]$ and $A' = A\exp(-i\tilde{k}_e a)$. The analogous matrix equation for D is

$$\left(\frac{v_0}{u_0}\right)^2 \begin{pmatrix} t^*(\tilde{k}_h)\, r^*(\tilde{k}_h) \\ r^*(\tilde{k}_h)\, t^*(\tilde{k}_h) \end{pmatrix} \begin{pmatrix} e^{-i(\tilde{k}_e - \tilde{k}_h)(a-L)} e^{-i\phi} & 0 \\ 0 & e^{i(\tilde{k}_e - \tilde{k}_h)a} \end{pmatrix}$$
$$\times \begin{pmatrix} t(\tilde{k}_e)\, r(\tilde{k}_e) \\ r(\tilde{k}_e)\, t(\tilde{k}_e) \end{pmatrix} \begin{pmatrix} e^{i(\tilde{k}_e - \tilde{k}_h)a} & 0 \\ 0 & e^{i\phi} e^{-i(\tilde{k}_e - \tilde{k}_h)(a-L)} \end{pmatrix} \begin{pmatrix} B' \\ D' \end{pmatrix}$$
$$+ \left(\frac{v_0}{u_0}\right) \begin{pmatrix} t^*(\tilde{k}_h)\, r^*(\tilde{k}_h) \\ r^*(\tilde{k}_h)\, t^*(\tilde{k}_h) \end{pmatrix} \begin{pmatrix} e^{-i(\tilde{k}_e - \tilde{k}_h)(a-L)} e^{-i\phi} & 0 \\ 0 & e^{i(\tilde{k}_e - \tilde{k}_h)a} \end{pmatrix} \begin{pmatrix} t(\tilde{k}_e)\, r(\tilde{k}_e) \\ r(\tilde{k}_e)\, t(\tilde{k}_e) \end{pmatrix} \begin{pmatrix} \left[1 - \left(\frac{v_0}{u_0}\right)^2\right] e^{i\tilde{k}_e a} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \begin{pmatrix} B' \\ D' \end{pmatrix}. \quad \text{(A24)}$$

Here B' and D' are the same as B and D up to a phase factor, where $B' = B\exp(i\tilde{k}_h a)$ and $D' = D\exp[-i\tilde{k}_h(a-L)]$.

Equations (A23) and (A24) are in a form which can be easily generalized to arbitrary elastic scattering potentials and multiple-moded electron waveguides. Further, Eqs. (A23) and (A24) can be interpreted directly in terms of the phase shifts acquired by the electron and hole Feynman paths multiply reflected from the scatterer and multiply Andreev reflected from the SN boundaries. The remaining considerations here and in Appendix B now specialize to the case where the transmission amplitude is independent of energy.

We directly invert Eqs. (A23) and (A24) to find

$$C' = t\frac{\left[1 - \left(\frac{v_0}{u_0}\right)^2 e^{i\phi} e^{i(\tilde{k}_e - \tilde{k}_h)L}\right]\left[1 - \left(\frac{v_0}{u_0}\right)^2\right] e^{i\tilde{k}_e a}}{\left[1 - \left(\frac{v_0}{u_0}\right)^2 e^{i\alpha} e^{i(\tilde{k}_e - \tilde{k}_h)L}\right]\left[1 - \left(\frac{v_0}{u_0}\right)^2 e^{-i\alpha} e^{i(\tilde{k}_e - \tilde{k}_h)L}\right]}, \quad \text{(A25)}$$

and

$$D' = \left(\frac{v_0}{u_0}\right) \frac{\left(tr^* e^{-i\phi} e^{i(\tilde{k}_e - \tilde{k}_h)(L-a)} + rt^* e^{i(\tilde{k}_e - \tilde{k}_h)a}\right)\left[1 - \left(\frac{v_0}{u_0}\right)^2\right] e^{i\tilde{k}_e a}}{\left[1 - \left(\frac{v_0}{u_0}\right)^2 e^{i\alpha} e^{i(\tilde{k}_e - \tilde{k}_h)L}\right]\left[1 - \left(\frac{v_0}{u_0}\right)^2 e^{-i\alpha} e^{i(\tilde{k}_e - \tilde{k}_h)L}\right]}. \tag{A26}$$

The angle α in Eqs. (A25) and (A26) is determined from

$$\cos(\alpha) = T\cos(\phi) + R\cos[(\tilde{k}_h - \tilde{k}_e)(L - 2a)]. \tag{A27}$$

If we remove the tunnel barrier ($\alpha \to \phi$ and $t \to 1$) and set the phase difference ϕ to zero, Eq. (A25) gives the result of Demers and Griffin[47] for the transmission amplitude of an electronlike excitation between two superconductors.

The discrete levels in the SNS potential with a tunnel barrier are found[45] by computing the poles of the transmission amplitudes C' or D'. Therefore, the Andreev levels are determined from

$$1 = \left(\frac{v_0}{u_0}\right)^2 e^{\pm i\alpha} e^{i(\tilde{k}_e - \tilde{k}_h)L}. \tag{A28}$$

Since $(v_0/u_0)^2 = \exp[2i\cos^{-1}(E/\Delta)]$ for $|E| < \Delta$, Eq. (A28) reduces to

$$2\cos^{-1}(E/\Delta) + (\tilde{k}_e - \tilde{k}_h)L \pm \alpha = 2\pi n. \tag{A29}$$

Using $\tilde{k}_e - \tilde{k}_h \simeq k_F E/\mu = E/\Delta\xi_0$, Eqs. (A27) and (A29) reduce to Eqs. (7) and (8).

Equations (7) and (8) also determine the location of the "leaky" Andreev levels in the continuum. In the continuum, the solution of Eqs. (7) and (8) will occur at a complex energy $E = E_R + iE_I$. The imaginary part of this pole can then be interpreted as the lifetime $(\hbar/|E_I|)$ of the leaky Andreev level. The locations of both the discrete poles and the continuum poles of the transmission amplitude are shown in Fig. 8.

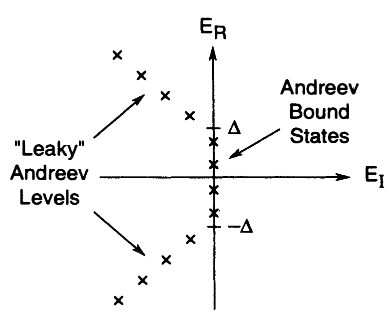

FIG. 8. Poles of the quasiparticle transmission amplitude determine the natural eigenergies of the SNS potential. Well-defined quantum levels exist inside the superconducting energy gap, while scattering resonances outside the gap have a finite lifetime $\hbar/|E_I|$.

APPENDIX B: CONTINUUM SUPERCURRENTS

Continuum supercurrents can only flow if there is an imbalance in the electrical currents carried by the quasiparticles incident from the left and right superconducting contacts. The imbalance $I(E,\phi)$ will be proportional to the difference in the transmission coefficients $T^e_{L\to R}(E,\phi) - T^e_{L\leftarrow R}(E,\phi)$ (transmission from left to right minus transmission from right to left) for electrical currents across the SNS junction. This conclusion follows because, near equilibrium, quantum states in the left and right contacts are equally thermally populated according to the Fermi function $f(E)$ at each energy, and because there are no inelastic scattering processes in the junction.

To obtain the continuum currents we invoke the insights of van Wees, Lenssen, and Harmans.[18] Reference 18 notes that that the electrical current $I(E)$ carried by a single filled quantum state per unit energy is $I(E) = ev_p(E)N_s^+(E)$, where v_p is the phase velocity and $N_s^+(E)$ the superconducting density of states for quasiparticles moving in one direction. We can write this current as $I(E) = e(v_p(E)/v_g(E))v_g(E)N_s^+(E)$, where v_g is the group velocity. (Reference 24 defines these phase and group velocities.) Just as in the normal metal,[4] the group velocity times the density of states cancels to $v_g(E)N_s^+(E) = 1/h$. The other factor,

$$v_p(E)/v_g(E) = 1/|u_0^2 - v_0^2| = N_s^+(E)/N_n^+(E),$$

is simply the ratio of the density of states in the superconductor to the density of states in the normal metal. [This factor $N_s^+(E)/N_n^+(E)$ appears prominently in the study of superconducting tunnel junctions.] The electrical current carried by electronlike excitations in the continuum is therefore

$$I_c^e(\phi) = \frac{2e}{h}\left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right)\frac{1}{|u_0^2 - v_0^2|} \times [T^e_{L\to R}(E,\phi) - T^e_{L\leftarrow R}(E,\phi)]\, f(E)dE. \tag{B1}$$

The additional factor of 2 in Eq. (B1) comes from the electron spin.

We now evaluate the transmission coefficients appearing in Eq. (B1) for our model problem. The transmission coefficient for the electrical currents carried by electronlike excitations incident from the left contact is

$$T^e_{L\to R}(E,\phi) = |C'|^2 - |D'|^2, \tag{B2}$$

with C' and D' determined from Eqs. (A25) and (A26) in Appendix A. Computing these quantities, we find

$$|C'|^2 = T|u_0^2 - v_0^2|^2 \frac{D(E,\phi)}{D(E,\alpha)D(E,-\alpha)}, \tag{B3}$$

and

$$|D'|^2 = |u_0^2 - v_0^2|^2 \frac{2TRu_0^2v_0^2}{D(E,\alpha)D(E,-\alpha)} \times\{1+\cos[(\tilde{k}_e - \tilde{k}_h)(L-2a)-\phi]\}, \tag{B4}$$

where $D(E,\alpha)$ is given by Eq. (18). $T^e_{L\leftarrow R}(E,\phi)$ is easily found from $T^e_{L\rightarrow R}(E,\phi)$ by the transformations $\phi \rightarrow -\phi$ and $a \rightarrow (L-a)$. Therefore,

$$T^e_{L\rightarrow R}(E,\phi) - T^e_{L\leftarrow R}(E,\phi) = T|u_0^2 - v_0^2|^2 \frac{D(E,\phi) - D(E,-\phi)}{D(E,\alpha)D(E,-\alpha)}. \tag{B5}$$

Equation (B5) can be recast as

$$T^e_{L\rightarrow R}(E,\phi) - T^e_{L\leftarrow R}(E,\phi) = T|u_0^2 - v_0^2|^2 \frac{\sin(\phi)}{\sin(\alpha)} \frac{D(E,\alpha) - D(E,-\alpha)}{D(E,\alpha)D(E,-\alpha)}. \tag{B6}$$

Combining Eqs. (B6) and (B1), we find

$$I^e_c(\phi)|_{E>\Delta} = \frac{2e}{h}T\int_\Delta^\infty |u_0^2 - v_0^2| \left(\frac{1}{D(E,-\alpha)} - \frac{1}{D(E,\alpha)}\right) \times \frac{\sin(\phi)}{\sin(\alpha)} f(E)dE. \tag{B7}$$

One can now choose several alternate paths to Eq. (21). We have chosen what seems (to us) the most straightforward. Consider the full dispersion relation shown in Fig. 9. It can be separated into an electron branch (standard parabola) and hole branch (inverted parabola). These branches should all be populated according to a Fermi function $f(E)$ at finite temperatures, including the hole branches. To compute the Josephson current, one must therefore work also with the negative energy ($E < 0$) solutions of the Bogoliubov–de Gennes equations. We consider quasiparticles incident in all possible branches of the dispersion curve, as shown pictorially in Fig. 10.

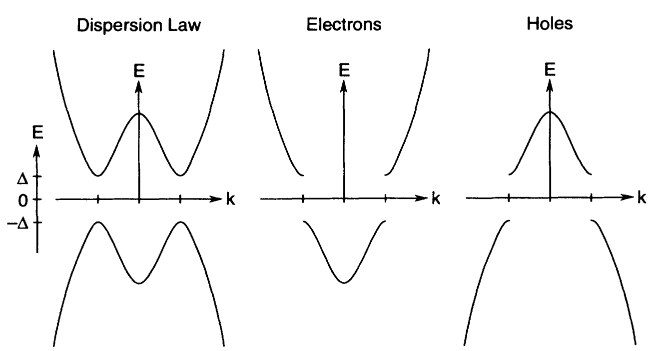

FIG. 9. The total dispersion law for a uniform superconductor can be decomposed into electronlike and holelike states.

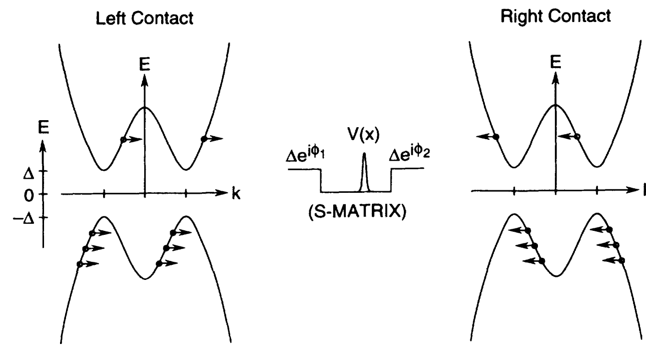

FIG. 10. Both electronlike and holelike states are incident on the SNS potential from all possible branches of the dispersion curve. These quantum levels are populated according to the Fermi function $f(E)$, even in the presence of supercurrent flow.

Calculating the currents implied from Fig. 10, the total electronlike currents are given by integrating Eq. (B7) over the entire continuum, which is the same as Eq. (21). The hole currents are found to be equal to the electron currents, thereby giving a continuum current too large by a factor of 2. This factor of 2 is easily understood when we note that the dispersion law shown in Fig. 9 contains twice the number of states as the normal metal. We therefore correct for this excess number of quasiparticle states by dividing a factor of 2 into the final result, namely,

$$2I_c(\phi) = I^e_c(\phi) + I^h_c(\phi), \tag{B8}$$

obtaining Eq. (21).

We feel that the physics of Josephson current flow may be less transparent when working solely with the excitation spectrum. Since the excitation spectrum will not be populated at $T = 0$,[18] it is unclear how it can carry the required zero-temperature Josephson current. It was even noted in Ref. 18 that "this seems to prevent the calculation" of the Josephson current at $T = 0$. However, a disadvantage of working with the full dispersion curve is that one must also use the negative energy solutions of Eq. (3). Further, one must compute twice the number of transmission coefficients. Alternatively, one could compute the currents carried by only the electron branch (or only by the hole branch) of the dispersion curve, and still use both positive and negative energies.

APPENDIX C: SUPERCURRENT FROM FREE ENERGY

The free energy of a set of independent fermions in the grand canonical ensemble is

$$F = N\mu - kT\sum_E \ln\left(1 + e^{-E/kT}\right). \tag{C1}$$

The derivative of F with respect to the phase difference becomes a sum over the discrete energies $E_n(\phi)$ and integral over the continuum density of states $\rho(E,\phi)$

$$\frac{dF}{d\phi} = \sum_n \frac{dE_n}{d\phi} f(E_n) + \left(\int_{-\infty}^{-\Delta} + \int_{\Delta}^{\infty}\right) \left.\frac{d\rho(E,\phi)}{d\phi}\right|_E \times \ln\left(1+e^{-E/kT}\right) dE. \tag{C2}$$

The discrete spectrum therefore carries the Josephson current suggested in Sec. II, namely Eq. (10), where

$$I_d(\phi) = \frac{2e}{\hbar}\sum_n \left\{ \frac{dE_n^+(\phi)}{d\phi} f(E_n^+(\phi)) + \frac{dE_n^-(\phi)}{d\phi} f(E_n^-(\phi)) \right\}, \tag{C3}$$

letting us identify the current $I_n^{\pm}(\phi)$ carried by an Andreev level as

$$I_n^{\pm}(\phi) = \frac{2e}{\hbar}\frac{dE_n^{\pm}(\phi)}{d\phi}. \tag{C4}$$

Equation (C3) is computationally much simpler than obtaining the current from the quasiparticle wave functions $u(x)$ and $v(x)$, and shows how to easily compute $I_d(\phi)$ when the transmission amplitudes depend on the energy. Further, Eq. (C3) is identical to the current carried by the discrete spectrum from Ref. 13.

We now show how Eq. (15) for the current $I_n^{\pm}(\phi)$ carried by an Andreev level follows from Eq. (C4) combined with Eqs. (7) and (8). We wish to compute

$$\frac{dE_n^{\pm}}{d\phi} = \frac{dE_n^{\pm}}{d\alpha}\frac{d\alpha}{d\phi}. \tag{C5}$$

Equation (7) shows how the energy depends on the effective phase α. We find

$$\frac{dE_n^{\pm}}{d\alpha} = \frac{\partial E_n^{\pm}}{\partial \alpha} = \mp\frac{\hbar}{2e}\frac{ev_F}{L+2\xi(E_n^{\pm})}. \tag{C6}$$

On the other hand, Eq. (8) shows that $\alpha(E,\phi)$ is a function of both the energy and the phase difference. Therefore

$$d\alpha = \left.\frac{\partial \alpha}{\partial E}\right|_\phi dE + \left.\frac{\partial \alpha}{\partial \phi}\right|_E d\phi. \tag{C7}$$

Combining (C5), (C6), and (C7), we find

$$\frac{dE_n^{\pm}}{d\phi} = \frac{\partial E_n^{\pm}}{\partial \alpha}\left.\frac{\partial \alpha}{\partial \phi}\right|_E \frac{1}{\gamma}, \tag{C8}$$

where γ is the feedback factor in Eq. (15), namely

$$\gamma = 1 - \frac{\partial E_n^{\pm}}{\partial \alpha}\left.\frac{\partial \alpha}{\partial E}\right|_\phi. \tag{C9}$$

Equation (8) gives

$$\left.\frac{\partial \alpha}{\partial E}\right|_\phi = \left(\frac{R}{\Delta \sin(\alpha)}\right)\left(\frac{L-2a}{\xi_0}\right)\sin\left(\frac{L-2a}{\xi_0}\frac{E}{\Delta}\right), \tag{C10}$$

so that combining (C10) with (24), (C6), (C8), (C9), and (C4) yields Eq. (15).

The continuum contribution to the supercurrent in Eq. (C2) requires the derivative of the continuum density of states with respect to the phase difference

$$\left.\frac{d\rho(E,\phi)}{d\phi}\right|_E = \left.\frac{d\rho(E,\alpha)}{d\alpha}\right|_E \left.\frac{\partial \alpha}{\partial \phi}\right|_E. \tag{C11}$$

Equation (C11) implies that we can take the result (17) for $I_c(\phi)$ in the clean junction, replace ϕ by α, then multiply the integrand by (24) to obtain Eq. (21) for $I_c(\phi)$ in the dirty junction. Even though we exploit the continuum contribution of Eq. (C2) to better understand the formulas presented in Sec. II, we do not immediately see the full correspondence between the continuum contribution[13] in Eq. (C2) and the transmission formalism[18] for computing supercurrents carried in the continuum. However, we feel the two methods can probably be shown to be identical.

APPENDIX D: APPROXIMATE ANDREEV LEVELS IN A LONG JUNCTION

In this appendix we obtain approximate expressions for the Andreev levels in a long SNS junction with an impurity. Equation (8) can be recast as

$$\cos^2(\alpha/2) = 1 - T\sin^2(\phi/2) - R\sin^2[(L-2a)E/2\Delta\xi_0]. \tag{D1}$$

For $E \ll \Delta$, Eq. (7) becomes

$$E^{\pm} \simeq \left(\frac{\hbar v_F}{2L^*}\right)(2\pi n - \pi \mp \alpha), \tag{D2}$$

with $L^* = L + 2\xi_0$. Taking the impurity exactly in the center of the junction, $a = L/2$, we immediately obtain from Eqs. (D1) and (D2)

$$E^{\pm} \simeq \pi\left(\frac{\hbar v_F}{2L^*}\right)\left\{2n-1 \mp \frac{2}{\pi}\cos^{-1}[\sqrt{1-T\sin^2(\phi/2)}]\right\}. \tag{D3}$$

We therefore find the Andreev energy gap near $E = 0$ and $\phi = \pi$ as

$$E_{\text{gap}} \simeq 2\pi\left(\frac{\hbar v_F}{2L^*}\right)\left(1 - \frac{2}{\pi}\cos^{-1}\sqrt{R}\right), \tag{D4}$$

which becomes $E_{\text{gap}} \simeq (2\sqrt{R})(\hbar v_F/L^*)$ for small R. For $T \to 1$, $E \simeq 0$, and near $\phi \simeq \pi$, we can expand Eq. (D3) as

$$E^{\pm} \simeq \pm 2\left(\frac{\hbar v_F}{2L^*}\right)\sqrt{1-T\sin^2(\phi/2)}. \tag{D5}$$

The slope of the Jospehson current versus phase at $\phi = \pi$ is then approximately

$$\left.\frac{dI}{d\phi}\right|_{\phi=\pi} \simeq \left(\frac{ev_F}{L^*}\right)\left(\frac{T}{2\sqrt{R}}\right). \tag{D6}$$

For an impurity which is not centered in the junction, and taking $E \ll \Delta$ and $L-2a \ll \xi_0$, we can approximate Eq. (D1) as

$$\cos^2(\alpha/2) \simeq 1 - T\sin^2(\phi/2) - R((L-2a)E/2\Delta\xi_0)^2. \tag{D7}$$

Solving Eq. (D7) together with Eq. (D2) for $E^{\pm}$ merely replaces $L^* \to L_{\rm eff}$ in Eqs. (D5) and (D6), where $L_{\rm eff}^2 = (L+2\xi_0)^2 + R(L-2a)^2$. This very approximate analysis confirms Eqs. (34) and (35) in Sec. IV.

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