Scattering theory of current and intensity noise correlations in conductors and wave guides

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(Received 16 June 1992)

The low-frequency current-fluctuation spectra of phase-coherent electron conductors are related to the scattering matrix of the conductor. Each contact of the conductor is connected to a thermal equilibrium electron reservoir. The current-current correlations of the conductor are compared with the intensity-intensity correlations of a photon wave guide with all its ports connected to blackbody radiation sources. Only two sources of noise are considered: (a) Fluctuations in the occupation numbers of the incident channels that reflect the thermal equilibrium fluctuations of the reservoir states and (b) a shot noise (or partition noise) that originates if a carrier can be scattered into more than one final state and is present even at zero temperature. The theory uses single-particle scattering states to build up multiparticle states with the proper symmetry. Second quantization provides an elegant treatment of this problem if annihilation and creation operators for both the input and output channels of the wave guide are introduced. At equilibrium, in the absence of transport, the correlations of flux fluctuations measured at two different contacts are negative for both fermions and bosons. Away from equilibrium, in the presence of a net flux, the fluctuations are related to transport coefficients which invoke products of four scattering matrices. The transport portion of the correlation of the flux fluctuations at two different contacts is negative for fermions but is positive for bosons. The transport portion (shot noise) is very sensitive to the transmission behavior of the wave guide. Both for fermions and for bosons completely open transmission channels give no contribution. In addition to two-terminal conductors, we consider four-probe conductors in high magnetic fields which have the property that carriers transmitted and reflected at a barrier reach separate contacts. We discuss a four-terminal experiment which explicitly shows that the correlation function in the presence of two particle sources is not an incoherent sum of correlations generated by particles originating in one of the sources but contains exchange terms due to the indistinguishability of identical particles. We discuss the conditions for such exchange terms to be sensitive to a quantum-mechanical phase and the possibility to tune this phase with the help of an Aharonov-Bohm flux.

I. INTRODUCTION

The investigation of time-dependent fluctuations in small conductors with contacts separated by a distance which is short when compared to a phase-breaking length is an interesting avenue of research: It is desirable to characterize the properties of a sample not only by its average time-independent transport characteristics but also by its kinematic properties. Due to the wavelike transport in small conductors and the preservation of phase coherence over large distances it is possible to ask about noise properties which are sensitive to the quantum-mechanical phase. For conductors which are so small that scattering inside the conductor can be taken to be elastic, Refs. 1 and 2 found a general relationship between the low-frequency noise properties and the scattering matrix for two- and multiterminal many-channel conductors. It is the purpose of this paper to present and discuss the technical details omitted in these earlier papers. To elucidate the role of statistics, the conductor, which can be viewed as a waveguide for electrons, is compared with a waveguide for photons with the same transmission properties.[2] The contacts of the waveguide (see Fig. 1) are treated as large equilibrium reservoirs of electrons or as black-body radiation sources of photons. The statistical assumptions are the same as those used to calculate the equilibrium density-density correlation functions in a gas of free carriers. A gas of identical (noninteracting) Fermi particles exhibits a hole, and a gas of identical bosons exhibits bunching in the density-density correlation function.[3] The current-current correlations which are of interest here are similarly obtained as sum over *exchange amplitudes*. The calculation becomes especially transparent if, in addition to the creation and annihilation operators for the incoming channels, a set of creation and annihilation operators for the outgoing channels is introduced.[4] In terms of both input and out-

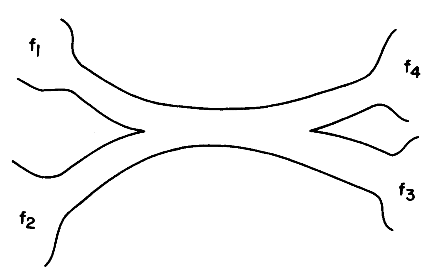

FIG. 1. Conductor (waveguide) with contacts connected to electron reservoirs (blackbody radiation sources).

put operators, the current at each probe can be expressed in terms of occupation number operators. In contrast, in terms of the operators which annihilate carriers in the incoming channels, the current operator cannot be expressed as a function of number operators. It is for this reason that exchange[2–7] plays a fundamental role in the calculation of current and flux correlation spectra.

The relationship of the current-current fluctuation spectra and the scattering matrix of the conductor[1,2] permits a treatment of both the noise which originates when carriers are incident from different contacts and are united in a portion of the conductor and of the noise which originates if a carrier stream is divided into two or more streams. This unification noise and partition noise is of fundamental interest:[8,9] The investigation of current-current fluctuations is the electronic analog of the study of intensity-intensity correlations (second-order coherence) in optics.[4–10] In the pioneering experiment[11] of Hanbury Brown and Twiss, a beam of light emanating from a single source is split into a "transmitted" beam and a "reflected" beam and the intensity correlation of the two beams is determined with the help of two detectors as shown in Fig. 2(a). Variations of this experiment include the measurement of the statistical properties of one of the beams only and include the use of more than one source. Thus the current or flux correlation experiments on waveguides provide an important alternative to studies of photon and electron correlations of beams of

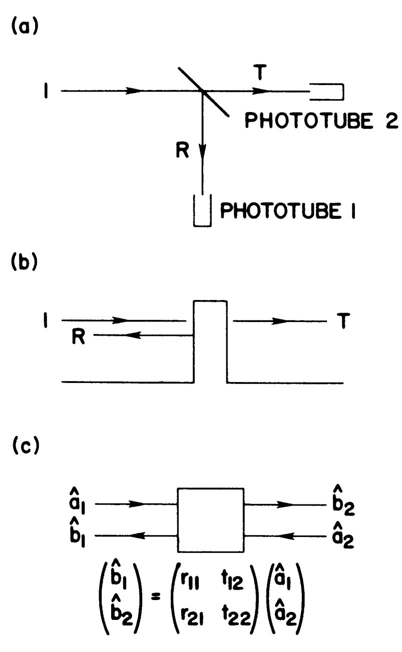

FIG. 2. (a) Intensity-intensity correlation measurement of a photon beam partitioned at a mirror into a transmitted (T) and reflected (R) beam. (b) Electron beam partitioned at a barrier into a transmitted (T) and reflected beam (R). (c) Second quantization representation of beam partitioning. The $\hat{a}$ operators annihilate carriers in the incoming channels and the $\hat{b}$ operators annihilate carriers in the outgoing channels. The $\hat{a}$ and $\hat{b}$ operators are connected by the scattering matrix **S**.

carriers in free space.[5–7] Since we deal with a two-particle effect which depends on the fact that there are more than two carriers present in a coherence volume, using electronic or optical waveguides might actually be of advantage: In a conductor the Fermi function is one for all states below the Fermi energy, and changes abruptly to zero in a narrow energy interval of width kT at the Fermi energy. Furthermore, the confinement of carriers to a "waveguide" prevents the spreading of waves. Since electrons are charged it is possible to couple a vector potential to the electron waves and to study the Aharonov-Bohm effect in a current-current correlation[6,2] or a density-density correlation.[12] Since second-order coherence is a two-particle effect it is especially interesting to ask whether there is a two-particle Aharonov-Bohm effect.[2,12]

The discussion reported here is closely related to recent theoretical[13–27] and experimental[28–33] work which addresses fluctuations in highly transmissive samples. For such conductors the more conventional theory[34,35] which treats transmission as a perturbation is not suitable. Initial work[13–16] has focused on two-terminal single-channel conductors, or has invoked assumptions which reduce the multichannel problem to an equivalent single-channel problem. The derivation presented below is general: We do not require a scattering matrix which can be represented as a composition of many single-channel scattering problems[13,15] nor do we need to search for a special basis of scattering states in which the scattering matrix is diagonal.[9] The discussion presented here treats fluctuations between different energy levels in a single quantum channel in the same manner as it treats fluctuations in a many-channel multiprobe conductor. Below we present a more detailed discussion of the questions raised in this brief Introduction.

A. Fluctuations in systems of indistinguishable particles

As is well known, the statistical properties of a quantum-mechanical system are a consequence of the *indistinguishability* of identical particles in a many-particle system.[3,36] The basic premise of indistinguishable particles leads to wave functions which must be either symmetric or antisymmetric if two particles are *exchanged.* (More exotic possibilities discussed sometimes for strictly two-dimensional systems will not be considered here.) A theory of fluctuations which wants to take the symmetry of many-particle states into account has to use Slater determinants, which is cumbersome, or a second quantization language, which is much more elegant and will be used here. A correlation function which is often discussed in solid-state textbooks[3] serves to illustrate the nature of fluctuations in a many-particle system. Consider for a moment a gas of noninteracting fermions or bosons in a three-dimensional space without any impurities. The average density $\langle n \rangle$ of the gas is spatially uniform. It is simply the sum of the absolute squares of the wave functions of occupied states. The correlation of the density fluctuations $\Delta n(\mathbf{r}) = n(\mathbf{r}) - \langle n \rangle$ of the equilibrium state is[3]

$$\langle \Delta n(\mathbf{r}_1)\Delta n(\mathbf{r}_2)\rangle = \langle n\rangle \delta(\mathbf{r}_1-\mathbf{r}_2)+\langle n\rangle \nu(r) , \quad (1.1)$$

where $\nu(r)$ is a function of distance $r=|\mathbf{r}_1-\mathbf{r}_2|$. For a Fermi gas $\nu(r)$ is negative. For a Bose gas $\nu(r)$ is positive. As $r \rightarrow 0$ the function $\nu(r)$ tends to a constant limit $\nu(r)=\mp\langle n\rangle/(2s+1)$ which is determined by the average density and the spin. For a Fermi gas at zero temperature, at distances which are large compared to the Fermi wavelength, $\nu(r)$ is[3]

$$\nu(r)=-(3/2\pi^2 k_F r^4)\cos^2(k_F r) . \quad (1.2)$$

The negative sign of the density-density correlation of a Fermi gas is a consequence of the fact that Eq. (1.1) has been calculated with the help of wave functions which change sign if any two particles are exchanged. The Pauli principle manifests itself like an interaction which pushes fermions away from one another and leads to a negative correlation function. In contrast for bosons the "interaction" is attractive and leads to a positive density-density correlation. $\nu(r)$ is a sum over exchange amplitudes of the form $\mp\psi_k^*(\mathbf{r}_1)\psi_p(\mathbf{r}_1)\psi_p^*(\mathbf{r}_2)\psi_k(\mathbf{r}_2)$ in which each wave function with momentum $\mathbf{k}$ or $\mathbf{p}$ occurs twice but with different arguments. We emphasize that through exchange every occupied state "couples" with every other occupied state. The resulting oscillatory structure in the exchange hole is often mistakenly labeled a Friedel oscillation.[37] In contrast to Friedel oscillations which are a consequence of single-particle interference, the oscillatory structure in the exchange hole is a consequence of the indistinguishability of the carriers. It is a two-particle effect. The oscillatory structure of the Fermi hole can be sensitive to an Aharonov-Bohm flux in a geometry in which there is no single-particle Aharonov-Bohm effect.[12] Below we extend the calculation of the exchange correlation to calculate the current or flux fluctuations in a conductor or waveguide.

B. Two-particle contributions to shot noise

Typically, in a conductor there are many different sources of noise. Here we consider only two sources of noise. First, at elevated temperatures thermal agitation causes fluctuations in the carrier stream incident from a contact into the conductor. Second, even at zero temperature and even in the zero-frequency limit (such that vacuum quantum fluctuations play no role) there is an additional source of noise due to the probabilistic scattering of carriers at an obstacle. For a Fermi system at zero temperature, the carrier stream incident on a scatterer with transmission probability T is noiseless. But the transmitted carrier stream and the reflected carrier stream are not noiseless. Eventually a carrier has to be either transmitted or reflected. No noise is generated by this mechanism if the transmission probability is 1 and no noise is generated if the transmission probability is zero. For one-dimensional scattering at an obstacle with transmission probability T a simple function which is zero for $T=0$ and for $T=1$ and is maximal in between is $T(1-T)$. Since this simple result[4,13,15,20] is central to this discussion it is worthwhile to discuss its origin.

Consider a state describing carriers incident from the left on the scatterer shown in Fig. 2(b). We assume that this state is characterized by well-defined quantum numbers (energy, momentum, and spin). We denote the occupation number of the incident state by n_I. We consider a series of repeated experiments in each of which this state is occupied with probability 1. Therefore, on the average $\langle n_I\rangle=1$ and the fluctuations $\Delta n_I=n_I-\langle n_I\rangle$ vanish. In particular, the mean square fluctuation of the occupation number of the incident state is $\langle(\Delta n_I)^2\rangle=0$. Consider now the occupation number of the transmitted state n_T and of the reflected state n_R. In contrast to a wave which is partially transmitted and partially reflected at an obstacle with transmission probability T, a particle must be either transmitted or reflected. Therefore each experiment can have only two outcomes: either $n_T=1$ and $n_R=0$ or $n_T=0$ and $n_R=1$. Therefore the correlation of the occupation numbers $\langle n_R n_T\rangle$ vanishes since for each event one of these numbers is zero. The average occupation numbers are $\langle n_T\rangle=T$ and $\langle n_R\rangle=R$. The average of the squares of the occupation numbers are $\langle(n_T)^2\rangle=T$ and $\langle(n_R)^2\rangle=R$. But these results immediately imply that the fluctuations $\Delta n_T=n_T-\langle n_T\rangle$ and $\Delta n_R=n_R-\langle n_R\rangle$ have average mean square fluctuations and correlations given by[4]

$$\langle(\Delta n_R)^2\rangle=\langle(\Delta n_T)^2\rangle=T(1-T) , \quad (1.3)$$

$$\langle\Delta n_R \Delta n_T\rangle=-T(1-T) . \quad (1.4)$$

We emphasize that the fluctuations given by Eqs. (1.3) and (1.4) occur for a well-defined initial state. The fluctuations given by Eqs. (1.3) and (1.4) are a consequence of probabilistic reflection and transmission (a wave phenomenon) and are a consequence of the fact that detectors register either a transmitted particle or a reflected particle (a particle phenomenon).

Let us briefly indicate how a quantum-mechanical derivation of Eqs. (1.3) and (1.4) proceeds. Obviously, we cannot appeal to a wave equation only but must use an approach which permits the discussion of both wavelike phenomena and particlelike phenomena. We use second quantization and introduce the operators $\hat{a}_i^\dagger$ and $\hat{a}_i$, $i=1,2$, which create and annihilate carriers in the incident states. Next we follow Loudon and introduce operators $\hat{b}_i^\dagger$ and $\hat{b}_i$, $i=1,2$ which create and annihilate carriers in the outgoing states.[4] Figure 2(a) indicates only one incident channel: a second incident channel is realized by a beam which transmits into phototube 2 and reflects into phototube 1. In Fig. 2(b) the two incident channels describe carriers arriving from the left and carriers arriving from the right. The two sets of operators [see Fig. 2(c)] satisfy the same commutation rules and are related by the scattering matrix of the conductor,

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix}=\begin{bmatrix}r_{11} & t_{12}\\ t_{21} & r_{22}\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix} . \quad (1.5)$$

The scattering matrix provides a unitary transformation of the a operators into the b operators. The single-particle state with the fluctuations given by Eqs. (1.3) and (1.4) is $|1\rangle=\hat{a}_1^\dagger|0\rangle$. The occupation probability of the transmitted state is now simply $\langle n_T\rangle=\langle 1|\hat{b}_2^\dagger\hat{b}_2|1\rangle$. The expectation value of the correlation of the occupation numbers $\langle n_R n_T \rangle$ is $\langle 1|\hat{b}_1^\dagger \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2|1\rangle = 0$. As an additional example let us consider a photon state incident which contains precisely n quanta. The incident state is now described by $|n\rangle = (n!)^{-1/2}(\hat{a}_1^\dagger)^n|0\rangle$. Now the correlation of the occupation numbers $\langle n_R n_T \rangle$ is nonvanishing and given by $\langle n|\hat{b}_1^\dagger \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2^\dagger|n\rangle = RTn(n-1)$. With $\langle n_R \rangle = nR$ and $\langle n_T \rangle = nT$ it is easily shown that $\langle (\Delta n_T)^2 \rangle = \langle (\Delta n_R)^2 \rangle = nRT$ and $\langle \Delta n_R \Delta n_T \rangle = -nRT$. Clearly, as long as we deal with a sequence of single incident states (containing one or many quanta) the expectation value of the correlation Eq. (1.4) is negative. An experiment which measures such a negative correlation for photons is reported in Ref. 38.

Emission of light by a blackbody or electrons emitted by an electron reservoir cannot be characterized as described above. We deal with a stream of carriers in a certain energy range and must consider the simultaneous incidence of particles with different energies. For simplicity let us consider two particles incident from the left with energies E and E' with occupation numbers $n(E)$ and $m(E')$. Instead of by $\hat{b}_1^\dagger(E)\hat{b}_1(E)\hat{b}_2^\dagger(E)\hat{b}_2(E)$ the correlation of the fluctuations in the reflected and transmitted stream is determined by $\hat{b}_1^\dagger(E)\hat{b}_1(E')\hat{b}_2^\dagger(E')\hat{b}_2(E)$. In the space of all single-particle states this operator has a vanishing expectation value. For the two-particle states the expectation value of this operator

$$\langle m(E'),n(E)|\hat{b}_1^\dagger(E)\hat{b}_1(E')\hat{b}_2^\dagger(E')\hat{b}_2(E)|n(E),m(E')\rangle$$

is given by $\mp r_{11}^*(E)t_{21}^*(E')r_{11}(E')t_{21}(E')n(E)m(E')$. Note that the factor which multiplies the occupation numbers is an exchange amplitude expressed in terms of transmission and reflection amplitudes. If the occupation numbers $n(E)$ and $m(E')$ are taken to be statistically independent[3] and are determined by the equilibrium distribution of the reservoir from which they are emitted, then after a repetition of many experiments the statistical average of $n(E)m(E')$ is $f(E)f(E')$. Here f is the Fermi distribution function or the Bose distribution function. Taking into account the contribution from the exchange amplitude which has E and E' interchanged and considering the limit $E' \to E$ we find for the correlation of the fluctuations of the occupation probabilities of the transmitted and the reflected beam

$$\langle \Delta n_R \Delta n_T \rangle = \mp 2T(E)R(E)f^2(E) . \tag{1.6}$$

In Eq. (1.6) the upper sign applies for Fermi statistics and the lower sign applies to Bose statistics (a convention which will be maintained throughout this paper). Note that this correlation is proportional to $T(1-T)$ independent of statistics. But equally important is that the correlation depends not on f but on f^2, indicating that we deal with a two-particle effect. For a system with low degeneracy $f \ll 1$ there is no correlation in the occupation number fluctuations of transmitted and reflected beams.

In a similar way, by calculating the quantum statistical expectation value of $\hat{b}_2^\dagger(E)\hat{b}_2(E')\hat{b}_2^\dagger(E')\hat{b}_2(E)$ and $\hat{b}_1^\dagger(E)\hat{b}_1(E')\hat{b}_1^\dagger(E')\hat{b}_1(E)$, we obtain the fluctuations in the transmitted beam and in the reflected beam,

$$\langle (\Delta n_T)^2 \rangle = 2Tf \mp 2T^2f^2 = 2Tf(1 \mp Tf) , \tag{1.7}$$

$$\langle (\Delta n_R)^2 \rangle = 2Rf \mp 2R^2f^2 = 2Rf(1 \mp Rf) . \tag{1.8}$$

Note that the sum of Eqs. (1.7) and (1.8) plus twice the correlation, Eq. (1.6), is equal to the occupation number fluctuations in the incident carrier stream, $\langle (\Delta n_I)^2 \rangle = 2f(1 \mp f)$. Not only are the average occupation numbers conserved at the scatterer, but the fluctuations are also conserved, $\Delta n_I = \Delta n_T + \Delta n_R$. For fermions the fluctuations given by Eqs. (1.6)–(1.8) are bounded since $f \le 1$. The fluctuations in the transmitted stream can be smaller or can exceed the fluctuations in the incident stream depending on whether f is smaller or exceeds $1/(1+T)$. Similarly, the fluctuations in the reflected stream can be smaller or can exceed the fluctuations in the incident stream depending on whether f is smaller or larger than $1/(1+R)$. For bosons the fluctuations given by Eqs. (1.6)–(1.8) can exceed every bound since for bosons f is not bounded. For bosons both the fluctuations in the transmitted and in the reflected stream are always *smaller* than the fluctuations in the incident stream. In summary, for single incident carriers the fluctuations of the occupation numbers in the transmitted and reflected beam are negative. For carriers incident from a thermal reservoir, this correlation, like the density-density correlation Eq. (1.1), changes sign as we change the symmetry of the wave functions. Below we are interested in the fluctuations of currents. In a two-terminal geometry fluctuations in the incident beam, fluctuations in the reflected beam, and a correlation between the two, contribute to the current. Only in special multiprobe geometries[1,9,17] is it possible to measure separately the fluctuations in the incident, transmitted, and reflected beam. In Sec. V of this paper we analyze an example given in Ref. 1 in more detail.

C. The problem and the solution

Next we state the problem to be solved more clearly and give the answer discussed in this paper. We consider a conductor connected to electron baths via a number of contacts labeled $\alpha = 1,2,3,\ldots$. Each contact is characterized by an equilibrium Fermi function $f_\alpha = 1/\{\exp[(E-\mu_\alpha)/kT]+1\}$, with a chemical potential μ_α. These electron baths act as emitters and absorbers of electrons.[39,41] For bosons we assume that the wave guide at each port is connected to black-body reservoirs which act as emitters and absorbers of radiation and are characterized by a thermal equilibrium distribution $f_\alpha = 1/\{\exp[(E-\mu_\alpha)/kT]-1\}$. In the steady state the (time-averaged) current $\langle I_\alpha \rangle$ at contact α is[42]

$$\langle I_\alpha \rangle = \frac{e}{h}\int dE \left[(M_\alpha - R_{\alpha\alpha})f_\alpha - \sum_\beta T_{\alpha\beta} f_\beta \right] . \tag{1.9}$$

Here $R_{\alpha\alpha} = \mathrm{Tr}(\mathbf{s}_{\alpha\alpha}^\dagger \mathbf{s}_{\alpha\alpha})$ is the total probability for reflection back into probe α and $\mathbf{s}_{\alpha\alpha}$ is a scattering matrix of dimension $M_\alpha \times M_\alpha$. In Eq. (1.9) $T_{\alpha\beta} = \mathrm{Tr}(\mathbf{s}_{\alpha\beta}^\dagger \mathbf{s}_{\alpha\beta})$ is the total probability for transmission from probe β into probe α and $\mathbf{s}_{\alpha\beta}$ is a scattering matrix of dimension $M_\alpha \times M_\beta$. We refer only to a few recent publications[43] to illustrate the application of Eq. (1.9).

Each spin degree of freedom (polarization) is treated separately. For photons, the energy flux per quantum channel in an energy interval dE is $dW=(1/h)h\nu(E)dE$. Thus Eq. (1.9) gives the photon energy flux at a port if $(e/h)\int dE$ is replaced by $(1/h)\int dEE$. Alternatively, we can compare the Fermi carrier flux and the boson carrier flux by taking $e=1$ in Eq. (1.9). To be brief, for the case of bosons, we will give the results in electrical units only.

In this paper we give a derivation of the low-frequency spectral density of the current fluctuations belonging to Eq. (1.9). The spectral density of current-current fluctuations $S_{\alpha\beta}(\omega)$ is determined by the quantum statistical expectation value of the Fourier transformed current operator $\hat{I}_\alpha(\omega)$ through the relation[3]

$$\tfrac{1}{2}\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')+\Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle \equiv 2\pi S_{\alpha\beta}(\omega)\delta(\omega+\omega') , \quad (1.10)$$

where $\Delta\hat{I}_\alpha(\omega)=\hat{I}_\alpha(\omega)-\langle\hat{I}_\alpha(\omega)\rangle$. We will demonstrate that for small frequencies the current operator is given by

$$\hat{I}_\alpha(\omega)=\frac{e}{\hbar}\int dE[\hat{\mathbf{a}}_\alpha^\dagger(E)\hat{\mathbf{a}}_\alpha(E+\hbar\omega)-\hat{\mathbf{b}}_\alpha^\dagger(E)\hat{\mathbf{b}}_\alpha(E+\hbar\omega)] . \quad (1.11)$$

Here $\hat{a}_\alpha^\dagger$ is a vector of the operators $\hat{a}_{\alpha m}^\dagger$, $m=1,2,\ldots,M_\alpha$, which create a carrier in the incoming channel m in probe α. Similarly, $\hat{\mathbf{b}}_\alpha$ is a vector of the operators $\hat{b}_{\alpha m}$, $m=1,2,\ldots,M_\alpha$, which annihilate a carrier in the outgoing channel m in probe α. In the zero-frequency limit the current operator Eq. (1.11) is just the difference of the occupation number operators of the incoming quantum channels and of the outcoming quantum channels. As in the two-channel example discussed above, the annihilation operators in the outgoing channels are linearly related to the annihilation operators in the incoming channels via the scattering matrix,

$$\hat{\mathbf{b}}_\alpha=\sum_\beta \mathbf{s}_{\alpha\beta}\hat{\mathbf{a}}_\beta . \quad (1.12)$$

Inserting Eq. (1.12) and its self-adjoint into Eq. (1.11) gives

$$\hat{I}_\alpha(\omega)=(e/\hbar)\int dE \sum_{\beta\gamma}\hat{\mathbf{a}}_\beta^\dagger(E)\mathbf{A}_{\beta\gamma}(\alpha,E,E+\hbar\omega)\hat{\mathbf{a}}_\gamma(E+\hbar\omega) , \quad (1.13)$$

where we have introduced the matrix[1,2,22]

$$\mathbf{A}_{\beta\gamma}(\alpha,E,E+\hbar\omega)=\mathbf{1}_\alpha\delta_{\alpha\beta}\delta_{\alpha\gamma}-\mathbf{s}_{\alpha\beta}^\dagger(E)\mathbf{s}_{\alpha\gamma}(E+\hbar\omega) . \quad (1.14)$$

Here $\mathbf{1}_\alpha$ is a unit matrix with the dimensions equal to the number channels in lead α. This term stems from the incident channels, whereas the product of the scattering matrices arises from outgoing channels. Note that Eq. (1.13) is the full current operator and not the operator for the fluctuation away from the average. Through Eq. (1.13) the evaluation of the current fluctuations is reduced to the evaluation of the quantum statistical expectation values of products of four creation and annihilation input operators. Instead of subtracting the average current from the total current we can evaluate the deviation of this product from its average value. The quantum statistical expectation value of a product of four a operators away from its average is given by

$$\langle \hat{a}_{\alpha m}^\dagger(E)\hat{a}_{\beta n}(E')\hat{a}_{\gamma k}^\dagger(E'')\hat{a}_{\delta l}(E''')\rangle-\langle\hat{a}_{\alpha m}^\dagger(E)\hat{a}_{\beta n}(E')\rangle\langle\hat{a}_{\gamma k}^\dagger(E'')\hat{a}_{\delta l}(E''')\rangle = \delta(E-E''')\delta(E'-E'')\delta_{\alpha\delta m l}\delta_{\beta\gamma n k}f_\alpha(E)[1\mp f_\gamma(E')] . \quad (1.15)$$

In the zero-frequency limit, using Eqs. (1.10)–(1.15) we find for the spectral densities of the current fluctuations $\langle\Delta I_\alpha\Delta I_\beta\rangle_\omega\equiv\Delta\nu S_{\alpha\beta}(\omega)$, measured in a frequency interval $\Delta\nu$,

$$\langle\Delta I_\alpha\Delta I_\beta\rangle=2\frac{e^2}{h}\Delta\nu\sum_{\gamma\delta}\int dE\,\mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \times f_\gamma(E)[1\mp f_\delta(E)] . \quad (1.16)$$

Equation (1.16) is the basic result of this paper: It gives the low-frequency limit of the spectral densities belonging to Eq. (1.9) in terms of the scattering matrix of the conductor. It is understood that the scattering matrices and thus the matrix $\mathbf{A}$ are evaluated at energy E. For $\alpha=\beta$ we find from Eq. (1.16) the mean squared current fluctuations at probe α. For $\alpha\neq\beta$ we find from Eq. (1.16) the *correlation* of the fluctuations at two contacts.

Equation (1.16) shows that the low-frequency noise is determined by two types of *noise conductances*. We show that at equilibrium only the terms $(e^2/h)\mathrm{Tr}(\mathbf{s}_{\alpha\beta}^\dagger\mathbf{s}_{\alpha\beta})$ which are bilinear in the scattering matrix survive. These bilinear terms are identical to the transport coefficients in Eq. (1.9) which govern the average current. We find a *generalized Nyquist-Johnson relation* which relates the mean squared current fluctuations to the diagonal conductances of Eq. (1.6) and relates the current-current correlations to the off-diagonal conductances[1] of Eq. (1.9). Away from equilibrium, in the presence of transport, Eq. (1.16) gives noise conductances which are products of four scattering matrices,[1,2] $(e^2/h)\mathrm{Tr}(\mathbf{s}_{\alpha\gamma}^\dagger\mathbf{s}_{\alpha\delta}\mathbf{s}_{\beta\delta}^\dagger\mathbf{s}_{\beta\gamma})$. These noise conductances are interesting since they establish correlations between currents measured at two contacts α and β due to carriers emitted by contacts γ and δ. Clearly, we expect such correlations if a single source ($\gamma=\delta$) simultaneously radiates into two contacts ($\alpha \neq \beta$). But for $\gamma \neq \delta$ such a transport coefficient correlates carriers emanating from differing contacts. This is surprising since the two contacts are reservoirs which are mutually incoherent. If all indices are different such a transport coefficient is, in general, not real but depends on the phases of the scattering matrix. Conductors which permit the measurement of such noise conductances have been discussed in Refs. 2 and 12 and an additional example is discussed in Sec. V.

D. Related work

Much work in solid-state physics has treated fluctuations in systems in which tunneling can be treated as a perturbation.[34,35] In these works the transmission probability is taken to be small compared to 1. Khlus[13] recognized that an extension of this work was needed in situations in which the transmission probability is equal or comparable to 1. Since we require the contacts to be at equilibrium, the theory presented here is also a weak coupling theory. But it is a weak coupling theory only in the sense that the overall transmission probability $\mathrm{Tr}(\mathbf{t}_{\alpha\beta}^{\dagger}\mathbf{t}_{\alpha\beta})$ is small when compared to the number of modes M_α in reservoir α. This leaves open the possibility that some of the transmission probabilities $T_{\alpha\beta nm}$ are comparable or equal to 1. Khlus treats a metallic point contact with a barrier (an oxide layer) and finds noise conductances proportional to $T(1-T)$ and proportional to T^2. In his work Khlus uses a Keldysh approach. The discussion given below is simpler.

The need to present a fluctuation theory which accompanies Eq. (1.9) was recognized by the author following work which established and discussed the magnetic field symmetry[42] of the transport coefficients of Eq. (1.9). Onsager and Casimir[44] derive the symmetry of the transport coefficients from the microreversibility properties of correlation functions, i.e., from the fluctuation properties of the system. In contrast, in Ref. 42 this symmetry was obtained directly from the microreversibility property of the scattering matrix. The importance of such symmetry considerations is illustrated by recent work on spin glasses and recent work on high-T_c conductors.[45] It is interesting to ask if within the framework of a scattering approach we can develop a fluctuation theory.

Landauer[14] presented a discussion of equilibrium fluctuations in a one-channel conductor based on trains of clocked current pulses incident on the conductor. The discussion was extended to explicitly include shot noise in a paper by Landauer and Martin.[20] Shot noise in many-channel conductors was discussed by Martin and Landauer.[9] The strategy adopted in this work is to transform the many-channel many-lead problem in such a way that it can be mapped on the single-channel problem.[14] Many-channel conductors are treated by searching a special basis in which transmission through the sample is nonmixing and can be considered to be a set of independent one-channel conductors. A similar strategy is applied to discuss many terminal conductors. At its core this discussion considers fluctuations of occupation numbers of (real valued) current pulses. The discussion presented here treats the noise starting from multiparticle wave functions: the derivation proceeds by summing exchange amplitudes.

The discussion presented in this paper is closely related to the work of Lesovik[15] and Yurke and Kochanski.[16] Lesovik, in a brief but interesting paper, has discussed noise in a quantum point contact. He assumes that differing channels are not mixed by the point contact. Yurke and Kochanski[16] have investigated the momentum noise of single and multiple barriers in the single-channel approximation. Using a second quantization they present a step by step derivation of their results. Our discussion of the single-channel case differs from this work only in that we make use of Eq. (1.5). We also find it convenient to use $\hat{b}$ operators and $\hat{a}$ operators which create particles in a small energy range rather than a small momentum range. In first quantization the scattering matrix, Eq. (1.5), relates current amplitudes and not amplitudes of wave functions. A quasiclassical discussion of fluctuations, especially, with a view to ballistic transport has been presented by Beenakker and van Houten.[18] They also address the purely classical limit.

The frequency dependence of the noise spectra has been addressed by several authors.[13,15,19,22–25]. In a brief paper[22] we have pointed out that the current-fluctuation spectra can in general not be expressed in terms of transmission and reflection probabilities even for a one-channel conductor at equilibrium. The spectra are sensitive to the reflection and transmission amplitudes. Although the low-frequency limit of the noise spectra is the main topic of this paper, our derivation gives all the necessary technical steps which lead to the frequency-dependent noise spectra of Ref. 22. At equilibrium the frequency-dependent noise spectrum is via a fluctuation dissipation theorem, which includes zero point quantum fluctuations, connected to a frequency-dependent conductance. A derivation and discussion of this conductance is given by Büttiker and Thomas.[23]

Two recent experiments by Li *et al.*[28,29] in a resonant double barrier and in a quantum point contact found shot noise much below the naively expected value. Both experiments deal with transmission through conductors in which some channels are completely open ($T=1$) or are completely closed ($T=0$). However, their experiment on the double barrier might not be related to coherent transport[28] but might instead reflect the incoherent addition of shot noise in series resistors.[26] Their interesting data on the quantum point contact are obtained after subtracting from the raw data a $1/f$-like component of the noise spectrum. Reference 1 and subsequent discussions[17] investigated shot noise in the quantized Hall regime. Carrier transmission along edge states provides another example of a transmission channel with probability of 1. Indeed there are experiments which exhibit reduced shot noise. Reference 1 and subsequent papers proposed experiments where the transmission properties of the conductor can be changed with the help of a gate voltage. An experiment in such a conductor was carried out by Washburn *et al.*[31] They measured the mean square fluctuations of the voltage difference between two voltage contacts on either side of a barrier produced with the help of a gate. Measurements were carried out both in the integer quantized Hall regime as well as in the fractional quantized Hall regime. None of these experiments measures a correlation of currents or voltages. The only work on electrical conductors which measures a correlation between fluctuations at different contacts is by Kil,[32] who in his thesis has studied the correlation between longitudinal and Hall voltages in a quantized Hall conductor.

Highly transmissive channels are not found only in the resonant tunneling, ballistic, or in the high magnetic field transport regime. Beenakker and Büttiker[26] evaluated the shot noise for metallic diffusive conductors which are much longer than a mean free path. A fraction of the quantum channels remains open even in such a disordered system and gives rise to shot noise which is smaller than full shot noise by a factor $\frac{1}{3}$. The effect of inelastic scattering on the shot noise is discussed with the help of side branches[46] (voltage probes). The effect of inelastic scattering in conjunction with voltage fluctuations which prevent local charge pileup leads to the absence of shot noise in a macroscopic conductor.

The discussion presented here assumes that the potential of the conductor does not fluctuate as a consequence of external time-dependent variations of the distribution of dopants and other background charges which determine the electrostatic potential affecting the conduction schemes. Fluctuations which change the potential can be very important especially at low frequencies. Recently a number of studies have appeared which investigate the telegraphic noise in quantum point contacts.[33] Simple models have been discussed by Dekker *et al.*,[33] Liefrink *et al.*,[33] and independently by the author.[47] Effects of gate voltage fluctuations on the conduction electrons have been investigated by Hekking *et al.*[48] In metallic diffusive systems the fluctuations in positions of impurities lead to $1/f$ noise. Theory and experiment are reviewed by Feng and Lee.[49] To observe the noise sources discussed in this work it is, therefore, desirable to perform measurements at frequencies above the range where $1/f$ fluctuations are observed. That of course requires an analysis of the thermal and shot noise beyond[22] the low-frequency limit emphasized in this work.

Despite the more difficult aspects of fluctuations in electrical conductors which stem from the interaction of the electrons among themselves and with background charges, the high degeneracy which can be achieved in conductors should make them better candidates to observe multiparticle effects than the proposed experiments with electron beams in a vacuum.[6,7] Even with the best available electron sources at best only one in a hundred possible states in a beam is occupied. On the other hand, in a conductor the current per quantum channel in an energy interval eV is $I=(e/h)eV$. At temperatures so low that the scattering processes can be neglected, the phase coherence length is only determined by the width of the energy interval and is given by $\tau_\phi = h/\mathrm{eV}$. Therefore the number of carriers per available state is $(I/e)\tau_\phi = 1$.

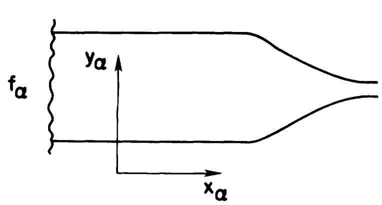

FIG. 3. Widening of contact into a reservoir with a large density of states. x_α and y_α represent a local coordinate frame in reservoir α.

II. THE SPECTRAL DENSITY OF CURRENT-CURRENT CORRELATIONS

A. Scattering states

In this section we formulate the scattering problem. Figure 1 taken from Ref. 2 shows conductors with a number of leads which in turn are connected to electron reservoirs. The electron reservoirs are taken to be conductors without elastic scattering. To be specific, we invoke the following model of a reservoir: We assume that at each port the conductor widens[50,51] into a wide but perfect conductor (see Fig. 3). We assume that the Hamiltonian in the wide portion is separable: electron motion can be decomposed into motion along the conductor and motion transverse to the conductor.[52] Our results will eventually be independent of the detailed properties of the states in the reservoirs. For simplicity we assume that the states in reservoir α are given by $\exp(ik_{\alpha m}x_\alpha)\phi_{\alpha m}(y_\alpha)$, $m=1,2,\ldots,M_\alpha$. Here x_α and y_α are local Cartesian coordinates in reservoir α. The component of the wave vector parallel to the conductor is $k_{\alpha m}$. For simplicity we assume that the transverse eigenfunction $\chi_{\alpha m}$ with energy $E_{\alpha m}(0)$ is independent of $k_{\alpha m}$. The kinetic energy associated with longitudinal motion is $\hbar^2k^2/2m^*$ with a longitudinal effective mass m^*. The total energy, including transverse and longitudinal motion, is

$$E_{\alpha m}(k) = E_{\alpha m}(0) + \hbar^2 k^2/2m^* \ . \tag{2.1}$$

Each transverse state provides a channel for electron propagation from the reservoir to the conductor (positive velocity) and away from the conductor (negative velocity). The total number of such quantum channels with threshold energy $E_{\alpha m}(0)$ smaller than the Fermi energy $E_{F\alpha}$ in reservoir α is M_α.

Reflection and transmission of carriers at the conductor are described by the scattering states $\psi_{\alpha m}$. The scattering state $\psi_{\alpha m}$ consists of a wave $\exp(ik_{\alpha m}x_\alpha)\phi_{\alpha m}(y_\alpha)$ which is incident in reservoir α in channel m. Here $k_{\alpha m}$ is the wave vector at energy E that is a solution of $E=E_{\alpha m}(k)$. This incident wave typically generates reflected waves in all channels m of reservoir α, and generates transmitted waves in all the channels of all the other reservoirs. The complete wave in reservoir α is[53]

$$\psi_{\alpha m}(\alpha)=\sum_n [\delta_{mn} e^{ik_{\alpha m}x_\alpha}\phi_{\alpha m}(y_\alpha) + (v_{\alpha m}/v_{\alpha n})^{1/2} s_{\alpha\alpha nm} \times e^{-ik_{\alpha n}x_\alpha}\phi_{\alpha n}(y_\alpha)] , \quad (2.2)$$

and in all the reservoirs $\beta\neq\alpha$ is of the form

$$\psi_{\alpha m}(\beta)=\sum_n (v_{\alpha m}/v_{\beta n})^{1/2} s_{\beta\alpha nm} e^{-ik_{\beta n}x_\beta}\phi_{\beta n}(y_\beta) . \quad (2.3)$$

The amplitudes $s_{\beta\alpha nm}$ are elements of the scattering matrix **S** (see also Appendix A). Below we will frequently invoke the scattering matrices $\mathbf{s}_{\beta\alpha}$ which relate the current amplitudes of the waves incident in reservoir α to the outgoing waves in reservoir β.

The scattering states given by Eqs. (2.2) and (2.3), together with possible bound states[54] in the interior of the conductor, form a complete set of mutually orthogonal states. This is known as the completeness theorem in scattering theory.[55] Constructive proofs of the orthogonality and completeness of these states have also been given.[56] The completeness of the states is important because later on we want to exhaust all possible fluctuations of the system. With this in mind, we note that the most general wave incident on the conductor is given by an arbitrary superposition of these scattering states,

$$\Psi(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\sum_{\alpha m}\int dk_{\alpha m}\psi_{\alpha m}(k_{\alpha m},\mathbf{r})a_{\alpha m}(k_{\alpha m})c^{-i\omega_{\alpha m}(k)t} . \quad (2.4)$$

Here $a_{\alpha m}(k)$ are the amplitudes which characterize the incident wave and $\omega_{\alpha m}(k)=E_{\alpha m}/\hbar$. Before proceeding, we note that Eq. (2.4) describes waves which are incident not only from a particular reservoir but from all the reservoirs simultaneously. The phases of the amplitudes $a_{\alpha m}(k)$ determine not only the phase relationship of differing waves in the same reservoir, but also the phase relationship of waves incident from differing reservoirs.

B. The field operator

To proceed we shall now consider Eq. (2.4) not as a wave packet with complex amplitudes $a_{\alpha m}(k)$ but as a second quantization operator acting on a Fock space. The Fock space is a direct sum of N-particle Hilbert spaces. The operator which we need is denoted by $\hat{\Psi}$ and is obtained from Eq. (2.4) by replacing the amplitude $a_{\alpha m}(k)$ by an operator $\hat{a}_{\alpha m}(k)$ which annihilates a carrier incident in the scattering state $\psi_{\alpha m}(k_{\alpha m})$. (Since we have assumed that the phase of the scattering state is fixed by the convention that the incident state is $e^{ik_{\alpha m}x_\alpha}$, we could invoke an arbitrary energy-dependent phase factor $e^{i\phi_{\alpha m}}$ multiplying each $\hat{a}_{\alpha m}$. However, all expectation values invoke pairs of complex conjugate scattering states, and hence such phase factors are unimportant for what follows). The operator $\hat{\Psi}$ is called the Fermi field or the Bose field operator. For our problem we find it more convenient to change from the integral in k space in Eq. (2.4) to an energy integration, and to introduce annihilation and creation operators which are suitably normalized for such a representation. We introduce the annihilation and creation operators $\hat{a}_{\alpha m}(E)=\hat{a}_{\alpha m}(k)/[\hbar v_{\alpha m}(k)]^{1/2}$ and $\hat{a}^\dagger_{\alpha m}(E)=\hat{a}^\dagger_{\alpha m}(k)/[\hbar v_{\alpha m}(k)]^{1/2}$ which obey the commutation relations

$$[\hat{a}^\dagger_{\alpha m}(E),\hat{a}_{\beta n}(E')]_\pm=\delta_{\alpha\beta}\delta_{mn}\delta(E-E') , \quad (2.5)$$

where the index $\pm$ denotes the anticommutator (for fermions) and the commutator for bosons. With these annihilation operators the Fermi field (or Bose field) is

$$\hat{\Psi}(\mathbf{r},t)=\sum_{\alpha m}\int\frac{dE_{\alpha m}}{[hv_{\alpha m}(E_{\alpha m})]^{1/2}}\psi_{\alpha m}(E_{\alpha m},\mathbf{r}) \times\hat{a}_{\alpha m}(E_{\alpha m})e^{-i\omega_{\alpha m}t} . \quad (2.6)$$

Note that $1/hv_{\alpha m}(E)$ is just the one-dimensional density of states of the quantum channel m in probe α at energy E. It is possible to use an energy representation of the field operator for the following reason: Since Eqs. (2.4) and (2.6) include only scattering states which describe a wave incident on the waveguide in a single quantum channel, the energy uniquely specifies the scattering state.

The field operator acts on many-particle states denoted by $|\sigma\rangle$ which are specified by occupation numbers $\sigma_{\alpha m}(E)$ for each incident channel m in every probe α of the conductor.

C. Quantum expectation values and statistical averages

Since every observable of our system can be expressed in terms of the field operator, and since the field operator is expressed with the help of annihilation operators, all expectation values of the system are known if the expectation values of products of the creation and annihilation operators are known. The quantum-mechanical expectation value of $\hat{a}^\dagger_{\alpha m}(E)\hat{a}_{\beta n}(E')$ is given by

$$\langle\sigma|\hat{a}^\dagger_{\alpha m}(E)\hat{a}_{\beta n}(E')|\sigma\rangle=\delta(E-E')\delta_{\alpha\beta}\delta_{mn}\sigma_{\alpha m}(E) . \quad (2.7)$$

To evaluate fluctuations, we need to calculate expectation values of products of four a operators,[3]

$$\langle\sigma|\hat{a}^\dagger_{\alpha m}(E)\hat{a}_{\beta n}(E')\hat{a}^\dagger_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle . \quad (2.8)$$

The expectation value of this product is nonzero only if it contains two pairs of operators $\hat{a}^\dagger_{\alpha m}(E)$ and $\hat{a}_{\alpha m}(E)$ with the same indices and arguments. We get a contribution from *normal pairing*, i.e., if $\alpha=\beta$, $\gamma=\delta$, $m=n$, $k=l$, $E=E'$, and $E''=E'''$. In addition we get a contribution from *exchange pairing*, $\alpha=\delta$, $\beta=\gamma$, $m=l$, $n=k$, $E=E'''$, and $E'=E''$. For the case of exchange pairing we notice that

$$\hat{a}^\dagger_{\alpha m}(E)\hat{a}^\dagger_{\beta n}(E')\hat{a}^\dagger_{\beta n}(E'')\hat{a}_{\alpha m}(E''') =\hat{a}^\dagger_{\alpha m}(E)\hat{a}_{\alpha m}(E''')[\delta(E'-E'')\mp\hat{a}^\dagger_{\beta n}(E'')\hat{a}_{\beta n}(E')] . \quad (2.9)$$

Thus the expectation value of the products of four operators is

$$=\delta(E-E''')\delta(E'-E'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}\sigma_{\alpha m}(E)[1\mp\sigma_{\gamma k}(E'')]$$
$$+\delta(E-E')\delta(E''-E''')\delta_{\alpha\beta nm}\delta_{\gamma\delta kl}\sigma_{\alpha n}(E)\sigma_{\gamma m}(E'') \ . \quad (2.10)$$

We have not considered the possibility that the indices and arguments of all four operators are identical. We deal with a continuum of states and the measure of these terms is insignificant compared to those considered above. We note that the last term in Eq. (2.10) is just $\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')|\sigma\rangle\langle\sigma|\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle$ and hence, instead of Eq. (2.10) we can also state that

$$\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle-\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')|\sigma\rangle\langle\sigma|\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle$$
$$=\delta(E-E''')\delta(E'-E'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}\sigma_{\alpha m}(E)[1\mp\sigma_{\gamma k}(E'')] \ . \quad (2.11)$$

Equations (2.7) and (2.11) are quantum-mechanical expectation values evaluated for states specified by a specific set of occupation numbers. To study fluctuations we need to consider an ensemble of states $|\sigma\rangle$ and weigh each state properly. We then need to find the statistical average, denoted by $\langle\ \rangle_s$, of the quantum-mechanical expectation values. We have assumed that the contacts are thermal equilibrium reservoirs. Since the annihilation and creation operators are related to the incident channels their statistics must reflect the equilibrium statistical properties of the reservoirs. As in the calculation of the density-density correlation (discussed in Sec. I A), we assume that the occupation numbers at different probes, for different quantum channels, and for the same quantum channel at different energies are statistically independent. Thus the statistical average of the occupation probability $\sigma_{\alpha m}(E)$ is determined by the distribution function of reservoir α, $\langle\sigma_{\alpha m}(E)\rangle_s=f_\alpha(E)$, independent of the channel index. The statistical average of the quantum expectation value, Eq. (2.7), is

$$\langle\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')|\sigma\rangle\rangle_s=\delta(E-E')\delta_{\alpha\beta mn}f_\alpha(E) \ . \quad (2.12)$$

Similarly, since the occupation probabilities at differing energies in the same channel and at the same energy in different channels are uncorrelated $\langle\sigma_{\alpha m}(E)\sigma_{\beta n}(E')\rangle_s=f_\alpha(E)f_\beta(E')$, using Eq. (2.8), we find on the statistical average

$$\langle\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle\rangle_s$$
$$=\delta(E-E''')\delta(E'-E'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}f_\alpha(E)[1\mp f_\gamma(E'')]+\delta(E-E')\delta(E''-E''')\delta_{\alpha\beta mn}\delta_{\gamma\delta kl}f_\alpha(E)f_\gamma(E'') \ . \quad (2.13)$$

The statistical average of Eq. (2.9) is

$$\langle\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle\rangle_s-\langle\langle\sigma|\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\beta n}(E')|\sigma\rangle\rangle_s\langle\langle\sigma|\hat{a}^{\dagger}_{\gamma k}(E'')\hat{a}_{\delta l}(E''')|\sigma\rangle\rangle_s$$
$$=\delta(E-E''')\delta(E'-E'')\delta_{\alpha\delta ml}\delta_{\beta\gamma nk}f_\alpha(E)[1\mp f_\gamma(E')] \ . \quad (2.14)$$

Remarkably Eq. (2.14) establishes correlations between differing channels $n\neq m$ within the same probe and establishes correlations between differing channels in differing probes $\alpha\neq\gamma$. These correlations are of infinite range. In reality Eq. (2.14) applies only within a coherence volume. But within such a volume, on a fundamental level, all quantum channels are correlated. Whether or not this correlation has a physical manifestation depends on the properties of the observable (particle density, current density, total current) for which the fluctuations are calculated. We emphasize that Eq. (2.14) is not merely a statement about the fluctuations of the occupation probabilities of differing channels: For the correlation of the occupation operator $\hat{n}_{\alpha m}=\hat{a}^{\dagger}_{\alpha m}\hat{a}_{\alpha m}$, we find from Eq. (2.14) with $\alpha=\beta$, $m=n$, and $\gamma=\delta$, $k=l$,

$$\langle\langle\hat{n}_{\alpha m}\hat{n}_{\gamma k}\rangle\rangle_s-\langle\langle\hat{n}_{\alpha m}\rangle\rangle_s\langle\langle\hat{n}_{\gamma k}\rangle\rangle_s=0 \ .$$

The occupation probabilities of differing channels are statistically uncorrelated. Equation (2.14), however, predicts correlations and is obviously a deeper statement. These additional correlations are quantum mechanical in origin and are a consequence of particle exchange. A gedanken experiment which investigates the Aharonov-Bohm effect in a density-density correlation of two quantum channels emanating from incoherent reservoirs is the subject of Ref. 12.

In the remaining part we will not distinguish quantum-mechanical expectation values and statistical averages, but use the symbol $\langle\ \rangle$ to denote both of these procedures.

D. Current operators

With the help of Eq. (2.6) the current density operator can now be expressed as

$$\hat{\mathbf{j}}(\mathbf{r},t)=\frac{\hbar}{2mi}[\hat{\Psi}^{\dagger}\nabla\hat{\Psi}-(\nabla\hat{\Psi}^{\dagger})\hat{\Psi}] \ . \quad (2.15)$$

Our principal aim is not to calculate current densities somewhere in the interior of the conductor, but the total current entering the conductor at a contact. The total current in probe α is

$$\hat{I}_\alpha(t)=\int dy_\alpha \hat{j}_{x_\alpha}(\mathbf{r},t) \ , \quad (2.16)$$

where dy_α denotes the integral over the cross section of reservoir α (see Fig. 3). In the total current operator $\hat{I}_\alpha$ we have only kept the time as an argument, anticipating that deep in the reservoir $\hat{I}$ is independent of the location x_α of the cross section.[57] To evaluate the total current we insert the field operator into Eq. (2.16). This leads to the evaluation of current matrix elements[1,22,58,59] evaluated in probe α, invoking a scattering state emanating from a channel in probe β and a scattering state emanating from a channel in probe γ. Instead of proceeding this way, we now transform the field operator into a more convenient form. To do this we introduce operators $\hat{b}_{\alpha m}$ which annihilate a carrier in an *outgoing* channel m in probe α. The $\hat{b}$ and $\hat{a}$ operators satisfy the same commutation relations and are related to one another by a unitary transformation,

$$\hat{b}_{\alpha m}=\sum_{\beta n} s_{\alpha\beta mn}\hat{a}_{\beta n} , \tag{2.17}$$

where $s_{\alpha\beta mn}$ is an element of the scattering matrix. Equation (1.12) given in the Introduction states the same connection between the a and b operators in a matrix notation. Now in each probe the field operator takes the simple form

$$\begin{aligned}\hat{\Psi}(\mathbf{r},t)=&\sum_m \int \frac{dE_{\alpha m}}{[hv_{\alpha m}(E_\alpha m)]^{1/2}}\chi^{+}_{\alpha m}(x_\alpha,y_\alpha)\\ &\times\hat{a}_{\alpha m}(E_{\alpha m})e^{-i\omega_{\alpha m}t}\\ &+\sum_m \int \frac{dE_{\alpha m}}{[hv_{\alpha m}(E_{\alpha m})]^{1/2}}\chi^{-}_{\alpha m}(x_\alpha,y_\alpha)\\ &\times\hat{b}_{\alpha m}(E_{\alpha m})e^{-i\omega_{\alpha m}t} ,\end{aligned} \tag{2.18}$$

where

$$\chi^{\sigma}_{\alpha m}(E,x,y)=\exp(i\sigma kx)\phi_{\alpha m}(y) \tag{2.19}$$

is an abbreviation for the asymptotic incoming ($\sigma=+$) or outgoing ($\sigma=-$) wave in probe α in channel m. In Eq. (2.17) the first summation is over all the incident channels in probe α and the second summation is over all the outgoing channels in probe α. Now if we express the current operator in terms of the field operator, we must evaluate matrix elements of the form

$$\begin{aligned}I^{\sigma_1\sigma_2}_{mn}(E,E+\hbar\omega)\equiv\frac{e\hbar}{2mi}\int dy_\alpha \Bigg[&\chi^{\sigma_1\dagger}_{m}(E)\frac{d\chi^{\sigma_2}_{n}(E+\hbar\omega)}{dx}\\ &-\frac{d\chi^{\sigma_1\dagger}_{m}(E)}{dx}\\ &\times\chi^{\sigma_2}_{n}(E+\hbar\omega)\Bigg] ,\end{aligned} \tag{2.20}$$

with $\sigma_1=+,-$ and $\sigma_2=+,-$. In Eq. (2.20) we have omitted the index α since both wave functions are states of probe α and the matrix element is evaluated in probe α. The wave vector in the mth quantum channel at energy E is denoted by k and determined by $E=E_m(0)+\hbar^2k^2/2m$. The wave vector in channel n at energy $E+\hbar\omega$ is denoted by q and is determined by $E+\hbar\omega=E_n(0)+\hbar^2q^2/2m$. A simple calculation gives

$$\begin{aligned}I^{\sigma_1\sigma_2}_{mn}(E,E+\hbar\omega)=&\delta_{mn}\frac{e\hbar}{2m}(\sigma_1k+\sigma_2q)\\ &\times\exp[-i(\sigma_1k-\sigma_2q)x] .\end{aligned} \tag{2.21}$$

Note that the matrix elements are nonvanishing only for wave functions belonging to the same quantum channel. The δ function is a consequence of the orthogonality of the transverse wave functions belonging to differing quantum channels, $\int dy\, \phi_m(y)\phi_n(y)=\delta_{mn}$. In the limit $\omega\to 0$ we have $k=q$. The right-hand side becomes independent of x and is nonvanishing only if the matrix element invokes states with velocities which are identical not only in magnitude but also have the same sign σ,

$$I^{\sigma_1\sigma_2}_{mn}(E,E)=\delta_{\sigma_1\sigma_2}\delta_{mn}ev_n(k) . \tag{2.22}$$

It is useful to note that for small frequencies the departure of Eq. (2.21) away from Eq. (2.22) is very small.[22] For small frequencies the difference $k-q$ can be expanded in powers of ω. Taking $q=k+\Delta q$, we find $\Delta q=\omega/v(E)$. For v of the order of a metallic Fermi velocity (10^8 cm/sec) a frequency of 10^{12} Hz is needed to bring the wavelength v/ω down to 1 μm. Similarly, if the prefactor is evaluated for $\sigma_1=-\sigma_2$, the prefactor is of the order $(\hbar\omega/E_F)v_F$. Thus for a considerable range of frequencies we can find a good approximation to the current by evaluating the current-matrix elements in the zero-frequency limit. In the expression for the current operator the matrix elements $I^{\sigma_1\sigma_2}_{mn}(E,E+\hbar\omega)$ always occur together with the velocity factors $[hv_m(E_m)]^{-1/2}$ and $[hv_m(E_m+\hbar\omega)]^{-1/2}$. In the low-frequency limit of interest here, we can also evaluate both of these velocities at the same energy. Thus we evaluate the current operator with the help of the following expression:

$$\frac{I^{\sigma_1\sigma_2}_{mn}(E,E+\hbar\omega)}{h[v_m(E_m)v_m(E_m+\hbar\omega)]^{1/2}}=\delta_{\sigma_1\sigma_2}\delta_{mn}\frac{e}{h} . \tag{2.23}$$

We reemphasize, that although Eq. (2.23) is an approximation for $\hbar\omega\neq 0$, it is an exact expression in the zero-frequency limit. Using Eq. (2.23) we find the following expression for the current operator at reservoir α:

$$\begin{aligned}\hat{I}_\alpha(t)=\frac{e}{h}\sum_m\int dE\,dE'[&\hat{a}^{\dagger}_{\alpha m}(E)\hat{a}_{\alpha m}(E')\\ &-\hat{b}^{\dagger}_{\alpha m}(E)\hat{b}_{\alpha m}(E')]\\ &\times\exp[i(E-E')t/\hbar] .\end{aligned} \tag{2.24}$$

If we write all the a and b operators as vectors with as many components as there are channels in the probe we can also express Eq. (2.24) in the form

$$\begin{aligned}\hat{I}_\alpha(t)=\frac{e}{h}\int dE\,dE'[&\hat{\mathbf{a}}^{\dagger}_{\alpha}(E)\hat{\mathbf{a}}_{\alpha}(E')-\hat{\mathbf{b}}^{\dagger}_{\alpha}(E)\hat{\mathbf{b}}_{\alpha}(E')]\\ &\times\exp[i(E-E')t/\hbar] .\end{aligned} \tag{2.25}$$

Now we again make use of Eq. (2.17) to find an expression of the current operator which invokes only the a operators. In matrix notation,

$$\hat{I}_\alpha(t)=\frac{e}{h}\int dE\, dE' \sum_{\beta\gamma} \hat{\mathbf{a}}_\beta^\dagger(E)\,\mathbf{A}_{\beta\gamma}(\alpha,E,E')\hat{\mathbf{a}}_\gamma(E') \times\exp[i(E-E')t/\hbar] \qquad (2.26)$$

with a matrix[1,2,22] $\mathbf{A}_{\beta\gamma}(\alpha,E,E')$ given by Eq. (1.14). We note that the current operator is not a function of occupation operators only if expressed in terms of the a operators. The elements $A_{\beta\gamma mn}(\alpha)$ can be understood as current-matrix elements evaluated in reservoir α of a scattering state $\psi_{\beta m}$ that is incident in channel m in lead β with energy E and a scattering state incident in channel n in lead γ with energy E'. We emphasize the universal character of these results: Eqs. (2.24)–(2.26) are independent of the particular properties of the states in the reservoir but depend only on the number of available channels.

E. Average currents

Before proceeding to evaluate current fluctuations we use Eq. (2.26) to evaluate the average currents. Using Eq. (2.10) we find for the average current

$$\langle \hat{I}_\alpha \rangle=\frac{e}{h}\sum_{\beta m}\int dE\; A_{\beta\beta mm}(\alpha)f_\beta(E)$$
$$=\frac{e}{h}\sum_{\beta}\int dE\,\mathrm{Tr}[\,\mathbf{A}_{\beta\beta}(\alpha)]f_\beta(E)\;. \qquad (2.27)$$

But for $\alpha=\beta$ we find from Eq. (2.27) that

$$\mathrm{Tr}[\,\mathbf{A}_{\alpha\alpha}(\alpha)]=\mathrm{Tr}(\mathbf{1}_\alpha-\mathbf{s}_{\alpha\alpha}^\dagger\mathbf{s}_{\alpha\alpha})\equiv M_\alpha-R_{\alpha\alpha}\;, \qquad (2.28)$$

where $M_\alpha\equiv\mathrm{Tr}(\mathbf{1}_\alpha)$ is the number of quantum channels in reservoir α and $R_{\alpha\alpha}\equiv\mathrm{Tr}(\mathbf{r}_{\alpha\alpha}^\dagger\mathbf{r}_{\alpha\alpha})$ is the total probability for reflection for carriers incident in probe α. For $\beta\neq\alpha$ we find from Eq. (2.27)

$$\mathrm{Tr}[\,\mathbf{A}_{\beta\beta}(\alpha)]=-\mathrm{Tr}(\mathbf{s}_{\alpha\beta}^\dagger\mathbf{s}_{\alpha\beta})\equiv-T_{\alpha\beta}\;, \qquad (2.29)$$

where the total probability for transmission is $T_{\alpha\beta}=\mathrm{Tr}(\mathbf{t}_{\alpha\beta}^\dagger\mathbf{t}_{\alpha\beta})$. Thus Eq. (3.1) gives the average incident current at a probe in terms of total reflection and transmission probabilities, i.e., Eq. (1.9). Our discussion has provided a simple derivation of this basic transport law using second quantization and the statistical assumptions discussed in Sec. II C. A number of discussions which have used formal linear-response theory[57,58] to derive Eq. (1.9) must at some point evaluate current-matrix elements. Nevertheless, the relationship between the current-matrix elements and the scattering matrix, expressed with the help of Eq. (1.14), seems not to have been noticed in any of these works.[57,58]

If the chemical potentials μ_α at the differing contacts differ only by a small amount, we can expand the distribution functions away from the equilibrium chemical potential μ, $f_\alpha=-(df/dE)(\mu_\alpha-\mu)$ and instead of Eq. (1.9) we find

$$\langle I_\alpha \rangle=(e/h)\int dE(-df/dE) \times\left[(M_\alpha-R_{\alpha\alpha})\mu_\alpha-\sum_\beta T_{\alpha\beta}\mu_\beta\right]\;. \qquad (2.30)$$

Here we have taken into account that current conservation requires $M_\alpha=R_{\alpha\alpha}+\sum_\beta T_{\alpha\beta}$ and that, consequently, Eq. (2.30) does not in an explicit way depend on μ. Equation (2.30) can be used to calculate the resistances $\mathcal{R}_{\alpha\beta,\gamma\delta}\equiv(V_\gamma-V_\delta)/I$. Here the first pair of indices denote the current source and sink and the second pair of indices denotes the probes which are used to measure voltages. $\mathcal{R}$ is a four-probe resistance if all indices differ from one another. It is a two-probe resistance if the first and second pairs of indices are identical. These resistances[42] obey the reciprocity symmetry $\mathcal{R}_{\alpha\beta,\gamma\delta}(B)=\mathcal{R}_{\gamma\delta,\alpha\beta}(-B)$.

F. Current-current fluctuation spectra

To evaluate the current-fluctuation spectra we calculate the Fourier amplitude

$$\hat{I}_\alpha(\omega)=\int dt\,\exp(i\omega t)\hat{I}_\alpha(t) \qquad (2.31)$$

of the current operator. Using Eq. (2.26) and performing the integral over time gives a δ function $2\pi\delta(E-E'+\hbar\omega)$. Carrying out the integration with respect to E' we find Eq. (1.13). The factor 2π has been absorbed into the prefactor by replacing h with $\hbar$. We can now use Eq. (1.13) and evaluate the spectral density of the current fluctuations with the help of Eq. (1.10). First consider the expectation value $\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle$. Using Eq. (1.13) twice, we find

$$\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle=\frac{e^2}{\hbar^2}\int dE\,dE'\sum_{\gamma\delta\epsilon\zeta}\langle \hat{\mathbf{a}}_\gamma^\dagger(E)\,\mathbf{A}_{\gamma\delta}(\alpha,E,E+\hbar\omega)\hat{\mathbf{a}}_\delta(E+\hbar\omega)\hat{\mathbf{a}}_\epsilon^\dagger(E')\,\mathbf{A}_{\epsilon\zeta}(\beta,E',E'+\hbar\omega')\hat{\mathbf{a}}_\zeta(E'+\hbar\omega')\rangle\;. \qquad (2.32)$$

Note that we have again used the operator of the total current. We compensate for that by invoking only the fluctuations of the product of the a operators away from the average. Invoking Eq. (2.14) now gives two δ functions $\delta(E-E'-\hbar\omega')\delta(E+\hbar\omega-E')$. Integrating with respect to E' gives

$$\langle \Delta\hat{I}_\alpha(\omega)\Delta\hat{I}_\beta(\omega')\rangle=\frac{e^2}{\hbar}\int dE\sum_{\gamma\delta}\mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha,E,E+\hbar\omega)\,\mathbf{A}_{\delta\gamma}(\beta,E+\hbar\omega,E)]f_\gamma(E)[1\mp f_\delta(E+\hbar\omega)]\delta(\omega+\omega')\;. \qquad (2.33)$$

Comparing with Eq. (1.10), we find a contribution to the spectral density given by

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_\omega = \Delta\nu \frac{e^2}{\hbar} \int dE \sum_{\gamma\delta} \mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha,E,E+\hbar\omega)\,\mathbf{A}_{\delta\gamma}(\beta,E+\hbar\omega,E)]f_\gamma(E)[1-f_\delta(E+\hbar\omega)]\ . \tag{2.34}$$

Next we consider the expectation value of the two current operators with their order interchanged. Instead of Eq. (2.33) we now have

$$\langle \Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle = \frac{e^2}{\hbar^2}\int dE\,dE' \sum_{\gamma\delta\epsilon\zeta} \langle \hat{\mathbf{a}}_\gamma^\dagger(E)\,\mathbf{A}_{\gamma\delta}(\beta,E,E+\hbar\omega')\hat{\mathbf{a}}_\delta(E+\hbar\omega')\hat{\mathbf{a}}_\epsilon^\dagger(E')\,\mathbf{A}_{\epsilon\zeta}(\alpha,E',E'+\hbar\omega)\hat{\mathbf{a}}_\zeta(E'+\hbar\omega)\rangle\ . \tag{2.35}$$

Taking the statistical average gives rise to two δ functions $\delta(E-E'-\hbar\omega)\delta(E+\hbar\omega'-E')$. Integrating with respect to E' gives

$$\langle \Delta\hat{I}_\beta(\omega')\Delta\hat{I}_\alpha(\omega)\rangle = \frac{e^2}{\hbar}\int dE \sum_{\gamma\delta} \mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\beta,E,E-\hbar\omega)\,\mathbf{A}_{\delta\gamma}(\beta,E-\hbar\omega,E)]f_\gamma(E)[1\mp f_\delta(E-\hbar\omega)]\delta(\omega+\omega')\ . \tag{2.36}$$

Let us now rename the energy variable in Eq. (2.36), $E\to E'$, and after that let us introduce the new variable $E=E'-\hbar\omega$. If we also rename the summation indices $\gamma\to\delta$ and $\delta\to\gamma$ and take into account that we can interchange the order of the matrices under the trace, we find that the second term gives a contribution to the spectral density

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_\omega = \Delta\nu \frac{e^2}{\hbar} \int dE \sum_{\gamma\delta} \mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha,E,E+\hbar\omega)\,\mathbf{A}_{\delta\gamma}(\beta,E+\hbar\omega,E)]f_\delta(E+\hbar\omega)[1\mp f_\gamma(E)]\ . \tag{2.37}$$

Combining both contributions we find a spectral density

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_\omega = \Delta\nu \frac{e^2}{\hbar} \int dE \sum_{\gamma\delta} \mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha,E,E+\hbar\omega)\,\mathbf{A}_{\delta\gamma}(\beta,E+\hbar\omega,E)] \times\{f_\gamma(E)[1\mp f_\delta(E+\hbar\omega)]+f_\delta(E+\hbar\omega)[1\mp f_\gamma(E)]\}\ . \tag{2.38}$$

Equation (2.38) is the most general result of this paper. It can be used to investigate the frequency dependence of the spectral densities.[22] It is worthwhile to stress that the order of the currents of Eq. (2.38) matters: With a little algebra it is easily shown that

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_\omega = \langle \Delta I_\beta \Delta I_\alpha \rangle_{-\omega}\ . \tag{2.39}$$

Only in the zero-frequency limit, which will be our next subject, is $\langle \Delta I_\alpha \Delta I_\beta \rangle \equiv \langle \Delta I_\beta \Delta I_\alpha \rangle$.

G. Low-frequency fluctuations

In the zero-frequency limit we obtain from Eq. (2.38) a spectral density

$$\langle \Delta I_\alpha \Delta I_\beta \rangle = \Delta\nu \frac{e^2}{h} \sum_{\gamma\delta} \int dE\, \mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha,E,E)\,\mathbf{A}_{\delta\gamma}(\beta,E,E)] \times\{f_\gamma(E)[1\mp f_\delta(E)] + f_\delta(E)[1\mp f_\gamma(E)]\}\ . \tag{2.40}$$

For $\alpha=\beta$ the terms in Eq. (2.40) proportional to $f_\gamma(E)[1\mp f_\delta(E)]$ and the terms proportional to $f_\delta(E)[1\mp f_\gamma(E)]$ each give the same contribution to the fluctuation spectrum. For $\alpha=\beta$ we can simply relabel the summation indices γ and δ to show this. The case $\alpha\neq\beta$ is less trivial. Clearly, the terms bilinear in the functions, $f_\gamma f_\delta$ and $f_\delta f_\gamma$, each give the same contribution. But according to Eq. (B8) of Appendix B, the terms linear in f are also identical. Therefore, in the zero-frequency limit, the current-fluctuation spectra are given by Eq. (1.16).

The mean square currents and the cross correlations are not completely independent from one another. In the low-frequency limit of interest here, flux is conserved. It is conserved not only on the average but also for the fluctuations, $\sum_\alpha \Delta I_\alpha = 0$. Consequently, the fluctuations given by Eq. (1.16) must obey the "sum rule,"

$$\left\langle \left[\sum_\alpha \Delta I_\alpha\right]^2\right\rangle = \sum_\alpha \langle (\Delta I_\alpha)^2\rangle + \sum_{\alpha\beta(\alpha\neq\beta)} \langle \Delta I_\alpha \Delta I_\beta\rangle = 0\ . \tag{2.41}$$

Equation (2.41) is a consequence of Eq. (B2) of Appendix B. Whether we deal with Fermi statistics or whether we deal with Bose-Einstein statistics the sum of all the cross correlations $\alpha\neq\beta$ must be negative to compensate for the positive mean square fluxes. Since the mean square currents are necessarily positive, the correlations of fluxes at differing terminals, $\langle \Delta I_\alpha \Delta I_\beta \rangle$, must as a rule be negative. Clearly flux conservation tends to make correlations between currents at differing ports negative: an increase in flux at one terminal must be compensated by a decrease in current at another terminal. Below we show that for Fermi statistics the correlation of currents at differing ports is indeed always negative (or at best zero). In contrast, for Bose-Einstein statistics the "rule" can be broken: under special circumstances it is possible to have correlations between currents at differing terminals which are positive. Our discussion below sheds some light on the conditions which are needed to break the rule for a Bose system. First we proceed to show that in thermal equilibrium all cross correlations both for Fermi systems and Bose systems are indeed negative.

III. EQUILIBRIUM FLUCTUATIONS

A. Equilibrium current fluctuations

At equilibrium all the distribution functions f_α are at the same chemical potential and hence identical. Taking this into account the spectral density of the current fluctuations is, according to Eq. (1.16),

$$\langle I_\alpha I_\beta \rangle = 2\frac{e^2}{h}\Delta\nu \int dE\, f(E)[1 \mp f(E)] \times \sum_{\gamma\delta} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha,E,E)\mathbf{A}_{\delta\gamma}(\beta,E,E)] \ . \tag{3.1}$$

Using Eqs. (A3) and (A4) it can be shown that

$$\sum_{\gamma\delta} \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta}\mathbf{s}_{\beta\gamma}) = \delta_{\alpha\beta}\mathrm{Tr}(\mathbf{1}_\alpha) \ .$$

Therefore in Eq. (3.1) only terms which are bilinear in the scattering matrix remain. For $\alpha=\beta$ Eq. (3.1) yields a mean squared current in the frequency interval $\Delta\nu$ at probe α,

$$\langle (I_\alpha)^2 \rangle = 4\Delta\nu kT(e^2/h)\int dE(-df/dE) \times \mathrm{Tr}(\mathbf{1}_\alpha - \mathbf{s}^\dagger_{\alpha\alpha}\mathbf{s}_{\alpha\alpha}) \ , \tag{3.2}$$

where we have used that $f(1\mp f) = -kT(df/dE)$. Taking into account that $\mathrm{Tr}(\mathbf{1}_\alpha - \mathbf{s}^\dagger_{\alpha\alpha}\mathbf{s}_{\alpha\alpha}) \equiv M_\alpha - R_{\alpha\alpha}$ we find[1]

$$\langle (I_\alpha)^2 \rangle = 4\Delta\nu kT(e^2/h)\int dE(-df/dE)[M_\alpha - R_{\alpha\alpha}] \ . \tag{3.3}$$

Alternatively, using current conservation [see Appendix A, Eq. (A5)], we can also express the equilibrium mean squared fluctuations at a terminal as

$$\langle (I_\alpha)^2 \rangle = 4\Delta\nu kT(e^2/h)\int dE(-df/dE)\left[\sum_{\beta(\neq\alpha)} T_{\alpha\beta}\right] \ . \tag{3.4}$$

The low-frequency fluctuations at a terminal are determined by the sum of all transmission probabilities permitting transmission into probe α. Equation (3.4) for the case of a two-terminal conductor reduces to the Johnson-Nyquist noise formula[14,1,2]

$$\langle I^2 \rangle = 4\Delta\nu kTG \ , \tag{3.5}$$

where $G=(e^2/h)\int dE(-df/dE)T$, where $T=\mathrm{Tr}(\mathbf{t}^\dagger\mathbf{t})$ is the Landauer conductance. The currents at differing terminals are in general correlated. From Eq. (3.1) we find with the help of Eq. (B7)

$$\langle I_\alpha I_\beta \rangle = -2\Delta\nu kT(e^2/h)\int dE(-df/dE)(T_{\alpha\beta}+T_{\beta\alpha}) \ . \tag{3.6}$$

The correlations between the fluctuating currents at differing probes are determined by the transmission probabilities which link the two probes. If we compare Eqs. (3.3) and (3.6) with Eq. (1.9) and take into account that $\sum_{\beta(\neq\alpha)}T_{\alpha\beta} = M_\alpha - R_{\alpha\alpha}$, we see that the current fluctuations are related to the symmetrized transport coefficients. Equations (3.3) and (3.6) are, therefore, a manifestation of the fluctuation dissipation theorem.

B. Equilibrium voltage fluctuations

To discuss voltage fluctuations it is useful to consider the fluctuating currents as Langevin forces in the general equation relating the chemical currents and the chemical potentials,

$$I_\alpha = (e/h)\int dE(-df/dE) \times \left[(M_\alpha - R_{\alpha\alpha})\mu_\alpha - \sum_\beta T_{\alpha\beta}\mu_\beta\right] + \delta I_\alpha \ . \tag{3.7}$$

Here δI_α is a fluctuating current with spectral densities given by Eq. (1.16) in the presence of transport or by Eqs. (3.4) and (3.6) at equilibrium. In the discussion given above we have assumed that the chemical potentials are held fixed (independent of time). If the terminals of the conductor are not connected by a zero-impedance external circuit, we can use Eq. (3.7) to ask about chemical potential fluctuations. We want to invert Eq. (3.7) to find the voltages as functions of the currents. A complication arises since the matrix of transport coefficients in Eq. (3.7) has one zero eigenvalue due to current conservation. We can, therefore, not invert, Eq. (3.7). Instead we must consider voltage differences. We chose one of the potentials as a reference potential (ground). The voltage difference between an arbitrary contact and the reference voltage V_β is

$$(V_\alpha - V_\beta) = \sum_\gamma \mathcal{R}_{\gamma\beta,\alpha\beta}(I_\gamma - \delta I_\gamma) \ . \tag{3.8}$$

In Eq. (3.8) $\mathcal{R}_{\gamma\beta,\alpha\beta}$ is the three-terminal resistance which gives the voltage difference between α and β for a current I_γ incident in contact γ and taken out at the reference probe β.

To be specific, we consider the case where all connections between probes exhibit infinite impedance. In particular, any voltmeter connecting two probes exhibits an infinite impedance. Thus the current at all terminals is zero, i.e., we consider $I_\alpha(t)=0$ for all α. Now let the index α label all the probes except the reference probe. This set of equations expresses the voltage differences as a function of the currents in terms of a resistance matrix whose diagonal elements $\mathcal{R}_{\alpha\beta,\alpha\beta}$ are the two-terminal resistances of the multiprobe conductor and whose off-diagonal elements are three-terminal resistances of the conductor. We can now ask: What are the fluctuation densities belonging to these equations? Instead of presenting a calculation, we use the following argument. The current and voltages are thermodynamically conjugate variables. Therefore the mean square voltage fluctuations must be related to the diagonal elements of the resistance matrix and the correlations of the voltage differences are related to the symmetrized off-diagonal transport coefficients. Thus we find for the spectral density of the mean square voltage fluctuations,[1]

$$\langle (V_\alpha - V_\beta)^2 \rangle = 4\Delta\nu kT \mathcal{R}_{\alpha\beta,\alpha\beta} . \tag{3.9}$$

We emphasize that in general, $\mathcal{R}_{\alpha\beta,\alpha\beta}$ is a two-terminal resistance of a multiprobe conductor. This resistance depends on the presence of all the other contacts. Similarly, the correlation of the voltage fluctuations is related to the symmetrized off-diagonal elements of the resistance matrix

$$\langle (V_\alpha - V_\beta)(V_\gamma - V_\beta) \rangle = 2\Delta\nu kT(\mathcal{R}_{\alpha\beta,\gamma\beta} + \mathcal{R}_{\gamma\beta,\alpha\beta}) . \tag{3.10}$$

Note that both voltage differences in Eq. (3.9) are measured with respect to the same reference potential. The fluctuations are determined by a symmetrical combination of three-terminal resistances.

Next we would like to know the correlation function of two voltage differences $V_\alpha - V_\beta$ and $V_\gamma - V_\delta$ measured across two completely different pairs of terminals. (All indices differ from one another.) This correlation function can be obtained from Eq. (3.10) by noting that voltages (and hence resistances) are additive. We replace $V_\gamma - V_\delta$ by $V_\gamma - V_\beta + V_\beta - V_\delta$. Thus the four-terminal correlation is a sum of two three-terminal correlations

$$\begin{aligned} \langle (V_\alpha - V_\beta)(V_\gamma - V_\delta) \rangle &= \langle (V_\alpha - V_\beta)(V_\gamma - V_\beta) \rangle \\ &\quad - \langle (V_\alpha - V_\beta)(V_\delta - V_\beta) \rangle . \end{aligned} \tag{3.11}$$

Using Eq. (3.10) we find

$$\begin{aligned} &\langle (V_\alpha - V_\beta)(V_\gamma - V_\delta) \rangle \\ &\quad = 2\Delta\nu kT(\mathcal{R}_{\alpha\beta,\gamma\beta} + \mathcal{R}_{\gamma\beta,\alpha\beta} - \mathcal{R}_{\alpha\beta,\delta\beta} - \mathcal{R}_{\delta\beta,\alpha\beta}) . \end{aligned} \tag{3.12}$$

But $\mathcal{R}_{\alpha\beta,\delta\beta} = -\mathcal{R}_{\alpha\beta,\beta\delta}$ and $\mathcal{R}_{\alpha\beta,\gamma\beta} + \mathcal{R}_{\alpha\beta,\beta\delta} = \mathcal{R}_{\alpha\beta,\gamma\delta}$, and similarly $\mathcal{R}_{\delta\beta,\alpha\beta} = -\mathcal{R}_{\beta\delta,\alpha\beta}$ and $\mathcal{R}_{\gamma\beta,\alpha\beta} + \mathcal{R}_{\beta\delta,\alpha\beta} = \mathcal{R}_{\gamma\delta,\alpha\beta}$. Therefore the correlation between voltage differences measured across two pairs of leads is given by[1]

$$\langle (V_\alpha - V_\beta)(V_\gamma - V_\delta) \rangle = 2\Delta\nu kT(\mathcal{R}_{\alpha\beta,\gamma\delta} + \mathcal{R}_{\gamma\delta,\alpha\beta}) . \tag{3.13}$$

The correlation is determined by a symmetrized four-terminal resistance. If two indices coincide, the correlation is determined by a symmetrized three-terminal resistance. For $\alpha = \gamma$ and $\beta = \delta$, Eq. (3.13), reduces to Eq. (3.9).

Equation (3.9) was tested in an experiment by Washburn *et al.*,[31] in a gated conductor in the quantized Hall regime. Equation (3.13) can be compared with an experiment by Kil[32] in which he measures the correlation of longitudinal and Hall voltages in a Hall bar geometry. There are many additional questions which would deserve a discussion, especially the correlation of voltage and current fluctuations if the conductor is part of a more general external impedance circuit.

IV. TRANSPORT FLUCTUATIONS

A. Zero-temperature limit

We now consider current fluctuations in the presence of a steady current. Two or more reservoirs connected to the conductor are at different chemical potentials. It is useful to consider first the zero-temperature limit fluctuations of a Fermi system in the presence of transport. (For a Bose system this simple limit cannot be discussed without addressing the Bose condensation transition.) Consider Eq. (1.16). Since $f_\alpha(1-f_\alpha) = -kTdf_\alpha/dE$ all terms in Eq. (1.16) in which the Fermi functions occur in this manner vanish. We can thus restrict the sum over the probe indices in Eq. (1.16) to $\gamma \neq \delta$. But for $\gamma \neq \delta$ the $\mathbf{A}$ matrices are $\mathbf{A}_{\gamma\delta}(\alpha) = -\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\gamma\delta}$. In the zero-temperature limit, we thus obtain from Eq. (1.16)

$$\begin{aligned} \langle \Delta I_\alpha \Delta I_\beta \rangle = 2\frac{e^2}{h}\Delta\nu \sum_{\gamma\delta,\gamma\neq\delta} \int dE \, \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta}\mathbf{s}_{\beta\gamma}) \\ \times f_\gamma(E)[1-f_\delta(E)] . \end{aligned} \tag{4.1}$$

Here the Fermi functions are step functions $f_\alpha(E) = 1 - \Theta(E - \mu_\alpha)$, which are equal to 1 for energies E below the chemical potential of the contact and are equal to zero for energies above this chemical potential. From Eq. (4.1) it is seen that the mean squared spectral densities

$$\begin{aligned} \langle \Delta I_\alpha \Delta I_\alpha \rangle = 2\frac{e^2}{h}\Delta\nu \sum_{\gamma\delta,\gamma\neq\delta} \int dE \, \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\alpha\delta}\mathbf{s}_{\alpha\gamma}) \\ \times f_\gamma(E)[1-f_\delta(E)] \end{aligned} \tag{4.2}$$

are determined by noise conductances

$$G_{\gamma\delta}(\alpha,\alpha) \equiv (e^2/h)\mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\alpha\delta}\mathbf{s}_{\alpha\gamma}) \tag{4.3}$$

which are real. Each scattering matrix occurs together with its adjoint. The cross correlations $\alpha \neq \beta$ are negative. To show this we consider the terms linear in f and quadratic in f in Eq. (4.1) separately. Consider first the terms linear in f. We notice that $\sum_\delta \mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta} = 0$. Therefore $\sum_{\delta\neq\gamma} \mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta} = -\mathbf{s}_{\alpha\gamma}\mathbf{s}^\dagger_{\beta\gamma}$. Thus the linear terms in Eq. (4.2) give a contribution proportional to $-\sum_\gamma \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\gamma}\mathbf{s}^\dagger_{\beta\gamma}\mathbf{s}_{\beta\gamma})f_\gamma(E)$ to the cross correlation. Since the Fermi functions at $kT=0$ are either zero or one this is equal to $-\sum_\gamma \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\gamma}\mathbf{s}^\dagger_{\beta\gamma}\mathbf{s}_{\beta\gamma})f_\gamma(E)f_\gamma(E)$, i.e., the diagonal terms quadratic in f omitted in Eq. (4.1) through restriction of the sum. Therefore Eq. (4.1) for $\alpha \neq \beta$ is equal to

$$\begin{aligned} \langle \Delta I_\alpha \Delta I_\beta \rangle = -2\frac{e^2}{h}\Delta\nu \sum_{\gamma\delta} \int dE \, \mathrm{Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta}\mathbf{s}_{\beta\gamma}) \\ \times f_\gamma(E)f_\delta(E) . \end{aligned} \tag{4.4}$$

Note that the noise conductances

$$G_{\gamma\delta}(\alpha\beta) \equiv (e^2/h)\mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\beta\delta}\mathbf{s}_{\beta\gamma}) \tag{4.5}$$

are not real. However, since $G_{\delta\gamma}(\alpha\beta)=G^{\dagger}_{\gamma\delta}(\alpha\beta)$ also occurs in the expression with the same weight as $G_{\gamma\delta}(\alpha\beta)$, the expression for the spectral density, Eq. (4.4), is real. Equation (4.1) is a key result of Ref. 1. Examples in which the noise conductances are not vanishing have been discussed in Refs. 2 and 12 and an additional simple example is given in Sec. V of this work. An alternative derivation of Eq. (4.1) is a major subject of Ref. 9.

B. Equilibriumlike and transport fluctuations

Next we consider fluctuations in the presence of a steady flux of Fermi or Bose carriers at an elevated temperature. We split the noise spectra into an equilibriumlike portion labeled by the index "eq" and a transport contribution labeled by the index "tr,"

$$\langle \Delta I_\alpha \Delta I_\beta \rangle = \langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{eq}} + \langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{tr}} \,. \tag{4.6}$$

It is convenient to *extend* the notion of "equilibrium fluctuations" in the following way. We calculate the mean square fluctuations at contact α as if all contacts were connected to a reservoir at chemical potential μ_α,

$$\langle (I_\alpha)^2 \rangle = 4\Delta\nu (e^2/h) \int dE\, f_\alpha (1 \mp f_\alpha)[M_\alpha - R_{\alpha\alpha}] \,. \tag{4.7}$$

In the equilibrium cross correlations we take into account that the Fermi functions at contact α and β are in general not the same, and write

$$\langle I_\alpha I_\beta \rangle = -2\Delta\nu k T (e^2/h) \int dE [T_{\alpha\beta} f_\beta (1 \mp f_\beta) + T_{\beta\alpha} f_\alpha (1 \mp f_\alpha)] \,. \tag{4.8}$$

The equilibriumlike fluctuations, Eqs. (4.2) and (4.3), obey current conservation,

$$\left\langle \left[\sum_\alpha \Delta I_\alpha \right]^2 \right\rangle_{\mathrm{eq}} = \sum_\alpha \langle (\Delta I_\alpha)^2 \rangle_{\mathrm{eq}} + \sum_{\alpha\beta(\alpha\neq\beta)_{\mathrm{eq}}} \langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{eq}} = 0 \,. \tag{4.9}$$

The transport fluctuations are now calculated by subtracting Eq. (4.7), respectively, Eq. (4.8) from the full result, Eq. (1.16). Using the results of Appendix B, especially Eqs. (B7) and (B9), we find

$$\langle (\Delta I_\alpha)^2 \rangle_{\mathrm{tr}} = 2\Delta\nu (e^2/h) \int dE \left[\sum_\gamma T_{\alpha\gamma}(f_\gamma - f_\alpha) \pm M_\alpha f_\alpha^2 \mp \sum_{\gamma,\delta} f_\gamma f_\delta \mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\alpha\delta}\mathbf{s}_{\alpha\gamma}) \right] , \tag{4.10}$$

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{tr}} = \mp 2\Delta\nu (e^2/h) \int dE \sum_{\gamma,\delta} f_\gamma f_\delta \mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\beta\delta}\mathbf{s}_{\beta\gamma}) \,. \tag{4.11}$$

The transport fluctuations also obey current conservation,

$$\left\langle \left[\sum_\alpha \Delta I_\alpha \right]^2 \right\rangle_{\mathrm{tr}} = \sum_\alpha \langle (\Delta I_\alpha)^2 \rangle_{\mathrm{tr}} + \sum_{\alpha\beta(\alpha\neq\beta)_{\mathrm{tr}}} \langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{tr}} = 0 \tag{4.12}$$

but $\langle (\Delta I_\alpha)^2 \rangle_{\mathrm{tr}}$ is not always positive. We emphasize that our partioning of the fluctuations into an "equilibrium" portion and a "transport" portion is not unique and mainly a matter of convenience.

C. The sign of flux-flux correlations

Equation (4.11) indicates that the transport correlations change sign if we switch from fermions to bosons. To show that a unique sign is associated with the statistics only, we need to show that the sum of all terms on the right-hand side of Eq. (4.11) is indeed positive. To show this, we use the cyclic property of the trace and rewrite the noise conductances in Eq. (4.11) as $\mathrm{Tr}(\mathbf{s}_{\beta\gamma}\mathbf{s}^{\dagger}_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\beta\delta})$. Next we use the linearity property of the trace and obtain for Eq. (4.11)

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_{\mathrm{tr}} = \mp 2\Delta\nu (e^2/h) \times \int dE\, \mathrm{Tr} \left[\left[\sum_\gamma f_\gamma \mathbf{s}_{\beta\gamma}\mathbf{s}^{\dagger}_{\alpha\gamma} \right] \times \left[\sum_\delta f_\delta \mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\beta\delta} \right] \right] . \tag{4.13}$$

Now Eq. (4.13) is the trace of the product of two matrices which are adjoint. But the trace of a product of a matrix with its adjoint is positive. We have therefore demonstrated the following: For a Fermi system both the equilibrium cross correlations and the transport cross correlations are negative. Therefore quite generally a Fermi system will exhibit negative cross correlations. On the other hand, for a Bose system the equilibrium cross correlations are negative, whereas the transport correlations are positive. This leaves open the possibility that there are nonequilibrium situations in which one or more cross correlations are positive.[2,5,11]

Equation (4.13) also provides a convenient way of evaluating Eq. (4.11). We notice that each sum in Eq. (4.13) is zero if the Fermi or Bose functions f are the same in all reservoirs, since $\mathbf{S}$ is unitary. Therefore in Eq. (4.13) we can replace f_γ by $f_\gamma - f_a$ and f_δ by $f_\delta - f_b$ where f_a and f_b are arbitrary energy-dependent functions. Hence instead of Eq. (4.11) or Eq. (4.13) we obtain[2]

$$\langle \Delta I_\alpha \Delta I_\beta \rangle_{\rm tr} = \mp 2\Delta\nu(e^2/h) \times \int dE \sum_{\gamma,\delta} (f_\gamma - f_a)(f_\delta - f_b) \times {\rm Tr}(\mathbf{s}^\dagger_{\alpha\gamma}\mathbf{s}_{\alpha\delta}\mathbf{s}^\dagger_{\beta\delta}\mathbf{s}_{\beta\gamma}) \ . \quad (4.14)$$

We now go on to illustrate these results by discussing a few examples.

V. EXAMPLES

A. Two-terminal conductors

Consider a conductor connecting two large reservoirs. First, we consider again the zero-temperature Fermi case. In the presence of a voltage drop $eV=\mu_1-\mu_2$ a current $\langle I\rangle=(e/h)T|eV|$ with $T={\rm Tr}(\mathbf{t}^\dagger\mathbf{t})$ is impressed on the conductor. In a two-probe conductor, current conservation requires that $\Delta I_1=-\Delta I_2$. Consequently the mean square flux fluctuations are the same at either probe. Moreover, the mean square current fluctuations are equal to $-\langle \Delta I_1 \Delta I_2\rangle$. Therefore we can use Eq. (4.13) to calculate the mean square current. Choosing $f_a=f_b=f_2$ only the term with indices $\alpha=1, \beta=2, \gamma=1, \delta=1$ is nonvanishing. Therefore we obtain for the mean square current

$$\langle (\Delta I)^2\rangle = 2\Delta\nu(e^2/h)\int dE(f_1-f_2)^2{\rm Tr}(\mathbf{s}^\dagger_{11}\mathbf{s}_{11}\mathbf{s}^\dagger_{21}\mathbf{s}_{21}) \ . \quad (5.1)$$

Now if we use the more transparent notation of transmission matrices $\mathbf{t}_{21}=\mathbf{s}_{21}$ and $\mathbf{t}_{12}=\mathbf{s}_{12}$ and reflection matrices $\mathbf{r}_{11}=\mathbf{s}_{11}$ and $\mathbf{r}_{22}=\mathbf{s}_{22}$, we obtain[1,2]

$$\langle (\Delta I)^2\rangle = 2\Delta\nu(e^2/h)|eV|{\rm Tr}(\mathbf{r}^\dagger_{11}\mathbf{r}_{11}\mathbf{t}^\dagger_{21}\mathbf{t}_{21}) \ . \quad (5.2)$$

To arrive at Eq. (5.2) we have also assumed that the scattering matrices vary slowly with energy and that in the energy range of interest the scattering matrices can be taken at the Fermi energy. The integral over the energy is then determined by the energy dependence of the Fermi functions alone. But at zero temperature $\int dE(f_1-f_2)^2$ is just equal to the absolute value of the voltage drop eV.

In terms of the matrix elements $t_{21,nm}$ and $r_{11,nm}$ where n and m label different modes in the probes we find

$$ {\rm Tr}(\mathbf{r}^\dagger_{11}\mathbf{r}_{11}\mathbf{t}^\dagger_{21}\mathbf{t}_{21}) = \sum_{k,l,m,n} r^*_{kn}r_{km}t^*_{lm}t_{ln} \ . \quad (5.3)$$

In Eq. (5.3) we have omitted the probe indices for simplicity and have kept only the channel or mode indices. Each term in Eq. (5.3) with a set of unequal indices $klmn$ can be understood as a coupling term for carriers injected in channel n and transmitted into channel l and carriers injected into channel m and reflected into channel k. The coupling between these two scattering channels, which on the statistical average are decoupled, occurs via time-reversed reflection from channel l to channel n and transmission from channel l to channel m.

The matrices $\mathbf{t}^\dagger_{21}\mathbf{t}_{21}$ and $\mathbf{r}^\dagger_{11}\mathbf{r}_{11}$ are Hermitian and commute. Therefore they can be diagonalized simultaneously. Let T_n, $n=1,\ldots,M_1$ be the eigenvalues of $\mathbf{t}^\dagger_{21}\mathbf{t}_{21}$. Then $R_n=1-T_n$ are the eigenvalues of $\mathbf{r}^\dagger_{11}\mathbf{r}_{11}$. In terms of these eigenvalues the mean square current fluctuations in a two-port conductor at $kT=0$ are[1]

$$\langle (\Delta I)^2\rangle = 2(e^2/h)\Delta\nu|eV|\sum T_n(1-T_n) \ . \quad (5.4)$$

A result of this form was found by Khlus[13] and Lesovik[15] for conductors which do not mix channels and where T_n is the transmission probability of channel n. Our derivation of Eq. (5.4) shows that this result is also valid for a scatterer which mixes channels if the T_n are taken to be the eigenvalues of $\mathbf{t}^\dagger\mathbf{t}$.

In the important case of a quantized conductance such as is found in point contacts,[60] Eqs. (5.2)–(5.4) predict that there is no shot noise at a plateau, i.e., when there are only completely open or completely closed modes.[15] Shot noise with an oscillatory amplitude has indeed been observed.[29] Full shot noise, corresponding to uncorrected electron transfer, is obtained only if all the eigenvalues T_n are small compared to I. In this case, $1-T_n\approx 1$ and Eq. (5.4) gives $\langle (\Delta I)^2\rangle = 2e\Delta\nu I$.

Conductors in a high magnetic field, under conditions where transport is dominated by edge states,[61] provide another example of transmission channels with unit transmission probabilities.[61] In the plateau region our theory predicts no shot noise.[1,31] More surprisingly, Eq. (5.4) predicts reduced noise even for conductors which are completely disordered. For a metallic diffusive conductor which is much longer than an elastic scattering length, but much shorter than a phase-breaking length, Ref. 26 found that the disorder-averaged noise is only $\frac{1}{3}$ of the full shot noise. The square of the transmission probability of an ensemble of conductors ($\langle\ \rangle_e$) is $\langle T_n^2\rangle_e = \frac{2}{3}\langle T_n\rangle_e$. Hence $\langle\langle(\Delta I)^2\rangle\rangle_e = \frac{1}{3}2e\Delta\nu\langle I\rangle_e$ with $\langle I\rangle_e = (e^2/h)(Nl/L)|V|$, where N is the number of quantum channels, l the elastic length, and L the length of the conductor.

To compare results for differing statistics, we now consider a two-port waveguide at elevated temperatures. For the fluctuations at port I we obtain from Eq. (4.7) and Eq. (4.10)

$$\langle (\Delta I)^2\rangle = 2\Delta\nu(e^2/h)\int dE \sum_n [T_n f_1(1\mp f_1) + T_n f_2(1\mp f_2) \pm R_n T_n (f_1-f_2)^2] \ . \quad (5.5)$$

For Fermi systems Eq. (5.5) is the zero-frequency limit of a result obtained by Lesovik.[15] The discussion presented here emphasizes that it is a general result if the T_n are taken to be the eigenvalues[2,20,9] of $\mathbf{t}^\dagger\mathbf{t}$. For Bose systems Eq. (5.5) was given in Ref. 2 and seems to be novel. Note that the shot noise appears with a negative sign for a Bose system. As discussed in Ref. 27 as we move away from equilibrium, there are situations when the noise is actually smaller than at equilibrium. Apparently, a scatterer in a Bose system breaks up large fluctuations: transmission and reflection at a barrier diminishes the photon bunching.

Let us now apply Eq. (5.5) to the following situation: We assume that f_2 is zero at energies for which f_1 is nonzero. In a conductor such a situation occurs if the applied voltage is so large that transmission occurs only from left to right. In optical experiments we can assume that we have a hot source only at probe 1, and that at probe 2 there is a cold detector whose radiation can be neglected. From Eq. (5.5) we obtain

$$\langle(\Delta I)^2\rangle = 2\Delta\nu(e^2/h)\sum_n \int dE[T_n f_1(1\mp f_1) \pm T_n(1-T_n)f_1^2] = 2\Delta\nu(e^2/h)\sum_n \int dE[T_n f_1(1\mp T_n f_1)] \;. \quad (5.6)$$

Equation (5.6) is the current noise caused by the occupation number fluctuations given by Eq. (1.7). The Bose version of Eq. (5.6) resembles a result obtained long ago by Hanbury Brown and Twiss.[11] This early work does not address waveguide structures: In place of the transmission probabilities this result contains the area of the aperture of the photo detector multiplied by the quantum efficiency of the detector.

A theory which treats tunneling perturbatively, to first order in the transmission probabilities T_n, misses the terms $T_n^2(f_1-f_2)^2$ in Eq. (5.5). If we further assume that the transmission probabilities are energy independent, as is often assumed in such calculations, the perturbation theory[34,35] predicts

$$\langle(\Delta I)^2\rangle = 2\Delta\nu(e^2/h)\sum T_n eV\coth(eV/2k_BT) = 2e\Delta\nu I\coth(eV/2k_BT) \;. \quad (5.7)$$

According to perturbation theory the crossover from thermal noise (proportional to k_BT) to shot noise (proportional to eV) is universal. The crossover is independent of the properties of the tunneling barrier and occurs at a voltage $eV \approx 2k_BT$. On the other hand, the full result, Eq. (5.5), for thermal energies and voltages which are small compared to the Fermi energy is

$$\langle(\Delta I)^2\rangle = 2\Delta\nu(e^2/h) \times \sum_n [2k_BTT_n^2 + R_nT_neV\coth(eV/2k_BT)] \quad (5.8)$$

as given by Khlus[13] and discussed in more detail by Martin and Landauer.[9] According to Eq. (5.8) the crossover from thermal noise to shot noise depends in a sensitive way on the transmission behavior of the conductor. At a conductance plateau of a quantum point contact, or at a conductance plateau of a high field Hall conductor the noise remains thermal (unless the applied voltage is strong enough to cause a breakdown of quantization). The metallic diffusive conductors[26] discussed above have an ensemble-averaged temperature and voltage dependence given by

$$\langle\langle(\Delta I)^2\rangle\rangle_e = \tfrac{2}{3}\Delta\nu\langle G\rangle_e[4k_BT + eV\coth(eV/2k_BT)] \quad (5.9)$$

which shows a crossover from thermal to shot noise at a voltage $eV \approx 4k_BT$.

In a two-terminal conductor the reflected beam always reaches the same contact as the incident carrier beam. To investigate situations in which the reflected beam is separated from the incident beam we now consider a multiterminal conductor.

B. Separation of transmitted and reflected currents in a four-terminal conductor

Figure 4 shows a confined two-dimensional electron gas in a high magnetic field. At magnetic fields which lead to the quantized Hall effect the only extended states at the Fermi surface are "edge states," the quantum-mechanical analogs of classical skipping orbits. Backscattering from one edge of the sample to another edge of the sample is suppressed.[61] Motion along edge states on one side of the sample is immune to disorder and unidirectional. Even forward scattering from one edge state to another (on the same sample side) is small due to the smoothness of the potential compared to the magnetic length, the wide separation between edge states, and the much reduced phase space.[63] Figure 4 is adapted from Ref. 1 and shows, for simplicity only, a single edge state. Under these conditions scattering from one edge of the conductor to the other must be introduced by external means. Backscattering can be introduced with the help of a gate across the conductor which permits depletion of the electron density[31] or with the help of a split gate which permits the formation of a narrow constriction[59] between the two-dimensional regions to the left and right. Here we assume that carriers at the constriction have a probability T for transmission and a probability R for reflection. The scattering matrix for the conductor of Fig. 4 is of the form

$$\begin{bmatrix} 0 & s_{12} & 0 & s_{14} \\ 0 & 0 & s_{23} & 0 \\ 0 & s_{32} & 0 & s_{34} \\ s_{41} & 0 & 0 & 0 \end{bmatrix} , \quad (5.10)$$

where $s_{14}=r$, $s_{34}=t$, $s_{32}=r'$, $s_{12}=t$ form elements of a

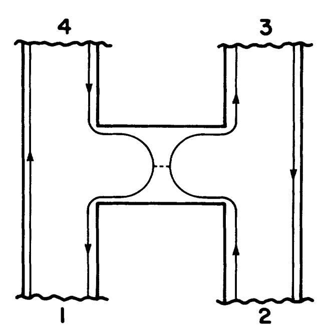

FIG. 4. Four-probe quantum Hall conductor. An edge state follows the boundaries and in the center of the conductor is partially transmitted and reflected with the help of a split gate. In this experiment the noise of incident, transmitted, and reflected carriers and the correlation between these can be separately measured.

2×2 scattering matrix and have the absolute squares $T=|t|^2$, $R=|r|^2\equiv|r'|^2$. The remaining scattering elements in Eq. (5.10) describe motion along edge states without backscattering and are given by $s_{41}=\exp(i\phi_1)$, $s_{23}=\exp(i\phi_2)$ with phases determined by the effective path length between these contacts.

First, we discuss the zero-temperature limit, recalling the results of Ref. 1 (see also Ref. 17). Let us assume that $\mu_1=\mu_4>\mu_2=\mu_3$. Carriers from contact 4 impinging on the constriction are there either transmitted into contact 3 or reflected into contact 1. At $kT=0$ there is no noise in the incident carrier stream, $\Delta I_4=0$. (For the same reason, $\Delta I_2=0$.) Thus the only currents which fluctuate are the currents at probe 1 and at probe 3. From current conservation $\Delta I_1+\Delta I_3=0$. The cross correlation is most simply found from Eq. (4.12). Reference 1 found

$$\langle(\Delta I_1)^2\rangle=\langle(\Delta I_3)^2\rangle=-\langle\Delta I_1\Delta I_3\rangle$$
$$=2\Delta\nu(e^2/h)TR(\mu-\mu_0)\ . \qquad (5.11)$$

Equations (5.11) correspond to the zero-temperature limit of Eqs. (1.6)–(1.8).

Next, let us investigate the fluctuations at elevated temperatures. We assume, as above, that the distribution functions in contacts 1 and 4 are the same $f\equiv f_1=f_2$, and the distribution functions in reservoirs 2 and 3 are also identical $f_0\equiv f_2=f_3$. With a little algebra we find from Eqs. (4.7) and (4.10) the mean squared fluctuations

$$\langle(\Delta I_1)^2\rangle=2\Delta\nu(e^2/h)\int dE[f(1\mp f)+Rf(1\mp f)+Tf_0(1\mp f_0)\pm RT(f-f_0)^2]\ , \qquad (5.12)$$

$$\langle(\Delta I_2)^2\rangle=\langle(\Delta I_4)^2\rangle=4\Delta\nu(e^2/h)\int dEf_0(1\mp f_0)\ , \qquad (5.13)$$

$$\langle(\Delta I_3)^2\rangle=2\Delta\nu(e^2/h)\int dE[f_0(1\mp f_0)+Rf_0(1\mp f_0)+Tf(1\mp f)\pm RT(f-f_0)^2]\ . \qquad (5.14)$$

The fluctuations at contacts 2 and 4 are equilibrium fluctuations with a contribution $f_0(1\mp f_0)$ arising from the incident channel and an identical contribution arising from the outgoing channel.

At contact 1 we have four contributions: an equilibrium contribution arising from the channel with transmission $T=1$ connecting contacts 1 and 4, equilibrium contributions determined by R and T from transmission into contact 1 of carriers emanating in contacts 4 and 2, and transport fluctuations associated with current partitioning at the split gate. A similar interpretation holds for the fluctuations at contact 4. The correlations of the fluctuations at differing contacts are

$$\langle\Delta I_2\Delta I_4\rangle=0\ , \qquad (5.15)$$

$$\langle\Delta I_1\Delta I_2\rangle=-2\Delta\nu(e^2/h)\int dE\ Tf_0(1\mp f_0)\ , \qquad (5.16)$$

$$\langle\Delta I_3\Delta I_4\rangle=-2\Delta\nu(e^2/h)\int dE\ Tf(1\mp f)\ , \qquad (5.17)$$

$$\langle\Delta I_2\Delta I_3\rangle=-2\Delta\nu(e^2/h)\int dE[f_0(1\mp f_0)+Rf_0(1\mp f_0)]\ , \qquad (5.18)$$

$$\langle\Delta I_1\Delta I_4\rangle=-2\Delta\nu(e^2/h)\int dE[f(1\mp f)+Rf(1\mp f)]\ , \qquad (5.19)$$

$$\langle\Delta I_1\Delta I_3\rangle=\mp2\Delta\nu(e^2/h)\int dE\ RT(f-f_0)^2\ . \qquad (5.20)$$

As can be seen the sum of the mean squared currents and twice the sum of all correlations is zero due to flux conservation. There is neither an equilibrium nor a transport correlation between the currents at probes 2 and 4. The correlations given by Eqs. (5.16)–(5.19) are equilibriumlike. They are negative irrespective of statistics. The correlation between 1 and 3 is transportlike and changes sign if we change statistics.

It is quite remarkable that in the conductor examined here, the cross correlation Eq. (5.20), even at elevated temperatures, contains only a transport effect. The cross correlation is not "contaminated" by thermal fluctuations. Thermal fluctuations in a correlation of fluxes between two contacts require direct transmission between these two contacts. But in the conductor of Fig. 4 there is no direct transmission from contact 1 to 3 or from contact 3 to 1.

The results presented here for the waveguide differ from the fluctuations one would measure in beams reflected and transmitted at a mirror. For a beam reflected at a mirror one would find fluctuations in the occupation number given by Eq. (1.8) instead of the more complex result given by Eq. (5.12). The difference stems from the fact that at a port of a waveguide we have as many channels leaving as are entering. At elevated temperatures the channels entering a port also contribute to the fluctuations. If we consider a strongly biased situation $\mu-\mu_0\gg kT$ with $\mu\equiv\mu_1=\mu_4$, $\mu_0\equiv\mu_2=\mu_3$ we can neglect the term $Tf_0(1\mp f_0)$ in Eq. (5.12) but the equilibrium term $f(1\mp f)$ arising from the channel connecting contacts 1 and 4 cannot be neglected. But apart from this equilibrium contribution $\langle(\Delta I_1)^2\rangle$ is a measurement of the fluctuations in the reflected "beam."

If the conductor contains, in addition to the edge channel which is partially reflected and partially transmitted, a number of channels which are completely transmitted and/or a number of channels which are completely reflected and if these channels away from the barrier are not mixed, they only contribute equilibriumlike fluctuations which are easily added to the fluctuations given above.

In the example discussed above it was possible to express all the fluctuations in terms of the absolute values of the scattering matrix elements. Below we discuss an example in which this is not possible and which displays exchange effects in a more explicit manner.

C. Correlations between fluxes from mutually incoherent reservoirs

In this work we have emphasized a calculation of fluctuations from exchange amplitudes. Here we discuss a possible experiment to demonstrate exchange effects in correlations in a very explicit manner. We are interested in the correlation between the currents at two terminals using the two remaining terminals as current sources. We consider the four-terminal conductor shown in Fig. 5. (Different examples have been discussed in Refs. 2 and 12). It represents a quantum dot coupled to four leads. The quantum dot contains one circulating edge state which at each port is via transmission and reflection probabilities $t_i = t_i'$, and r_i, r_i' coupled to the edge state of the contacts. (The quantities without a prime describe transmission and reflection of carriers which approach the lead from inside the conductor.) Any four-terminal conductor can be used for this experiment and our particular example is chosen only for illustrative purposes. The experiment consists in measuring the correlation function between the fluctuating currents at probes 2 and 4, for example. In experiment A a steady-state current is incident only from probe 3. We take $f \equiv f_1$ and $f_0 \equiv f_2 = f_3 = f_4$ and measure $\langle \Delta I_2 \Delta I_4 \rangle$. In experiment B a steady-state current is incident only from probe 3. We take $f \equiv f_3$ and $f_0 \equiv f_1 = f_2 = f_4$ and measure $\langle \Delta I_2 \Delta I_4 \rangle$. In experiment C a steady-state current is incident from both probe 1 and probe 3. We take $f \equiv f_1 = f_3$ and $f_0 = f_2 = f_4$. The theory predicts that the correlation function measured in experiment C is not just the incoherent sum of

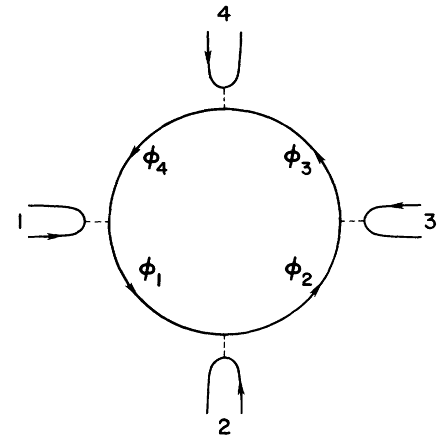

FIG. 5. Quantum dot with circulating edge state coupled to four contacts.

the correlation functions measured in experiments A and B but in addition contains contributions from exchange terms.

For experiment C we obtain from Eq. (4.13) for the correlations of the currents at terminals 2 and 4

$$\langle \Delta I_2 \Delta I_4 \rangle_{\mathrm{tr}} = \mp 2\Delta\nu (e^2/h) \int dE (f - f_0)^2 [\mathrm{Tr}(s_{21}^\dagger s_{21} s_{41}^\dagger s_{41}) + \mathrm{Tr}(s_{23}^\dagger s_{23} s_{43}^\dagger s_{43}) + \mathrm{Tr}(s_{21}^\dagger s_{23} s_{43}^\dagger s_{41}) + \mathrm{Tr}(s_{23}^\dagger s_{21} s_{41}^\dagger s_{43})] \; . \tag{5.21}$$

The first two terms in Eq. (5.21) represent the contribution to the cross correlation due to carriers emanating from source 1 (experiment A) and due to carriers emanating from source 3 (experiment B). The last two terms in Eq. (5.21) are the interesting terms: They arise due to both the carriers emanating from sources 1 and 3 and are a consequence of the fact that at a given terminal we cannot distinguish from which of the contacts the carriers have been emitted. We can thus, in an obvious notation, write Eq. (5.21) in the form

$$\langle \Delta I_2 \Delta I_4 \rangle_{\mathrm{C}} = \langle \Delta I_2 \Delta I_4 \rangle_{\mathrm{A}} + \langle \Delta I_2 \Delta I_4 \rangle_{\mathrm{B}} \mp 2\Delta\nu (e^2/h) \int dE (f - f_0)^2 [\mathrm{Tr}(s_{21}^\dagger s_{23} s_{43}^\dagger s_{41}) + \mathrm{Tr}(s_{23}^\dagger s_{21} s_{41}^\dagger s_{43})] \; . \tag{5.22}$$

The presence of the last two terms in Eq. (5.22) prevents the representation of experiment C in terms of an incoherent sum of the correlations measured in experiments A and B.

Let us evaluate these terms for the specific example considered here. The scattering matrix is found easily: In particular we find (see Refs. 64–66)

$$s_{21} = t_2 t_1 \exp(i\phi_1)/Z \; , \tag{5.23}$$

$$s_{31} = t_3 r_2 t_1 \exp[i(\phi_1 + \phi_2)]/Z \; , \tag{5.24}$$

$$s_{41} = t_4 r_3 r_2 t_1 \exp[i(\phi_1 + \phi_2 + \phi_3)]/Z \; . \tag{5.25}$$

Here the denominator Z is a function which takes into account that a carrier can complete many cycles on the circular edge state before exiting the sample,

$$Z = [1 - (R_1 R_2 R_3 R_4)^{1/2} \exp(i\chi)] \; , \tag{5.26}$$

where

$$\chi = \sum_{i=1}^{i=4} (\phi_i + \Delta\phi_i) \tag{5.27}$$

is the total accumulated phase in one cycle. The total phase is the sum of all phases accumulated during traversal from one contact to another and is the sum of all phases $\Delta\phi_i$ accumulated due to reflection at the contacts.

Consider the term $\mathrm{Tr}(s_{21}^\dagger s_{23} s_{43}^\dagger s_{41})$. Inserting the transmission probabilities, calculated as described above, gives

$$\mathrm{Tr}(s_{21}^\dagger s_{23} s_{43}^\dagger s_{41}) = T_1 T_2 T_3 T_4 (R_1 R_2 R_3 R_4)^{1/2} \times \exp(i\chi)/|Z|^4 \; . \tag{5.28}$$

The last term in Eq. (5.22) is just the complex conjugate of Eq. (5.28). Note that the last two terms are not simply a product of absolute squares of scattering matrix elements. Combining these two terms we obtain

$$\langle \Delta I_2 \Delta I_4 \rangle_{\rm C} = \langle \Delta I_2 \Delta I_4 \rangle_{\rm A} + \langle \Delta I_2 \Delta I_4 \rangle_{\rm B} \mp 4\Delta\nu (e^2/h) \int dE (f - f_0)^2 T_1 T_2 T_3 \times T_4 (R_1 R_2 R_3 R_4)^{1/2} \times \cos(\chi)/|Z|^4 \ . \quad (5.29)$$

In the presence of two incident currents we find a contribution to the correlation function which depends on a quantum-mechanical phase χ in an oscillatory manner.

Next we consider the situation in which reservoirs 1 and 2 take the role of current contacts. In experiment C we have incident currents from both of these contacts. Reservoirs 1 and 2 have the distribution function f and reservoirs 3 and 4 are characterized by the distribution function f_0. We then study the correlation function between terminals 3 and 4. In experiment A we have an incident current only in probe 1 and in experiment B we have an incident current only in probe 2. The correlation function is given by Eq. (4.13) [or Eq. (5.21) with the substitutions $3 \rightarrow 2$ and $2 \rightarrow 3$],

$$\langle \Delta I_3 \Delta I_4 \rangle_{\rm C} = \langle \Delta I_2 \Delta I_4 \rangle_{\rm A} + \langle \Delta I_2 \Delta I_4 \rangle_{\rm B} \mp 2\Delta\nu (e^2/h) \int dE (f - f_0)^2 [\mathrm{Tr}(s_{31}^\dagger s_{32} s_{42}^\dagger s_{41}) + \mathrm{Tr}(s_{32}^\dagger s_{31} s_{41}^\dagger s_{42})] \ . \quad (5.30)$$

We find the following result:

$$\langle \Delta I_2 \Delta I_4 \rangle_{\rm C} = \langle \Delta I_2 \Delta I_4 \rangle_{\rm A} + \langle \Delta I_2 \Delta I_4 \rangle_{\rm B} \mp 4\Delta\nu (e^2/h) \int dE (f - f_0)^2 T_1 T_2 T_3 \times T_4 R_2 R_3 /|Z|^4 \ . \quad (5.31)$$

In contrast to Eq. (5.29) a quantum-mechanical phase does not appear explicitly. While Eq. (5.29) vanishes if only one of the reflection probabilities vanishes (i.e., the probe becomes strongly coupled to the dot), the correlation given by Eq. (5.31) remains nonzero even if $R_1 = R_4 = 0$. To elucidate the difference between the two results it is useful to represent the exchange terms $s^\dagger s s^\dagger s$ graphically. In Fig. 6 $s_{\alpha\beta}$ is represented as a full line describing propagation from contact β to α along the circular edge state. In Fig. 6 $s_{\alpha\beta}^\dagger$ is represented as a broken line describing holelike propagation from contact β to α in the opposite direction along the circular edge state. The four paths always form a closed loop. Quantum mechanically such a closed loop must have a phase associated with it. In Fig. 6(a) the loop traces the entire circular edge state and is associated with a phase χ. In Fig. 6(b) the loop covers only a portion of the circular edge state and the phase associated with this loop is zero (or a multiple of 2π). In our simple example the phase χ which appears also occurs in the single-particle transmission coefficient as soon as the transmission probabilities to the contacts are all smaller than one. The interesting question is, whether it is possible to obtain fluctuation loops, as shown in Fig. 6(a), which are associated with phases which do not occur in single-particle transport coefficients and are therefore characteristic of a two-particle effect. All examples which we have examined only revealed phases which also occur in single-particle transport coefficients. Since the phase χ is proportional to the magnetic field enclosed by the circular edge state, Eq. (5.29) presents an example of an Aharonov-Bohm effect in a correlation function.[2,6]

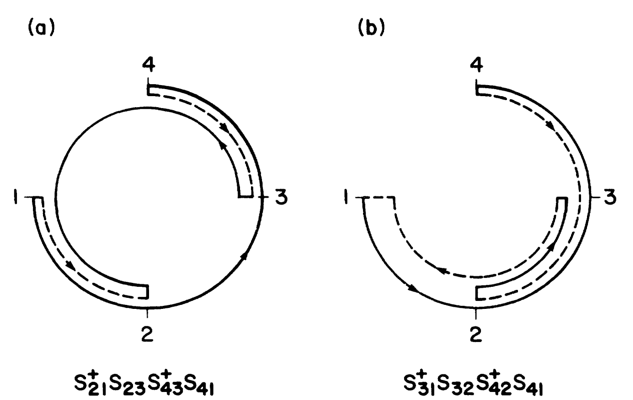

FIG. 6. (a) For carriers incident in both contacts 1 and 3 the current-current correlation between contacts 2 and 4 exhibits an exchange term which corresponds to excitation of the entire circulating loop. The phase of the exchange term is χ. (b) For carriers incident in contacts 1 and 2 the current-current correlation at contacts 3 and 4 exhibits exchange terms which correspond to the excitation of only part of the circulating edge state. The phase of the exchange term is zero.

VI. DISCUSSION

In this work we have presented a calculation of the noise spectral density for multichannel, multiterminal conductors and have compared these results with the intensity-intensity correlations of a photon wave guide. This juxtaposition of Fermi correlations and Bose correlations, clearly, has its limits: Electrons are not only fermions but carry charge. Through their charge electrons interact not only among themselves but also with positive background charges. Such interactions can give rise to collective behavior leading to results which might differ considerably from those presented in this work. Even for theories which include interactions[27,67] the discussion presented in this paper will hopefully provide a useful point of reference.

What are the experimental conditions under which a current-current correlation given by Eq. (1.16) can be measured? The correlations determined by Eq. (1.16) are for a conductor which is small compared to a phase-breaking length. Typically in an experiment macroscopic wires which are long compared to an inelastic length connect the small sample to a measuring apparatus. Will such a wire faithfully transmit a current fluctuation from the output contact of the small sample to the input of the detector? It could be that the answer is yes for the mean squared currents: The mean squared currents at two cross sections along the wire at r_1 and r_2 could be identical $\langle [\Delta I(r_1)]^2\rangle = \langle [\Delta I(r_2)]^2\rangle$ but might the correlation $\langle \Delta I(r_1)\Delta I(r_2)\rangle$ vanish if the distance between the two cross sections is larger than an inelastic length? To investigate the effect of inelastic scattering on shot noise, a simple model was analyzed by Beenakker and Büttiker.[26] In this model inelastic scattering is introduced with the help of a side branch[2,44] leading away to an electron reservoir. The correlation is indeed lost if the current at this side probe is allowed to fluctuate freely.[2,26] On the other hand, if the side probe is treated like a voltage probe (infinite impedance) the intrinsic current fluctuations are balanced by voltage fluctuations such that the net fluctuating current at the side probe vanishes. The conductor remains charge neutral and *the current correlation is preserved even over distances which are large compared to an inelastic length.* Current conservation is the key to avoid a loss of the correlation. A wire which remains locally charge neutral will preserve the correlation over *macroscopic distances.*[26] Inelastic scattering in conjunction with voltage fluctuations which prevent charge buildup provide another mechanism for the reduction of shot noise.

In any experiment, it will therefore not be easy to distinguish differing mechanisms for the reduction of shot noise. Similarly, a negative current-current correlation cannot immediately be attributed to the two-particle effects discussed here.

ACKNOWLEDGMENTS

Many stimulating discussions with R. Landauer and Th. Martin have helped to sharpen my views. I have also profited from H. Thomas and C. W. J. Beenakker with whom I have collaborated on certain extensions of the theory presented here.

APPENDIX A: PROPERTIES OF THE SINGLE-PARTICLE SCATTERING MATRIX

In this appendix we collect a number of properties of the single-particle scattering matrix $\mathbf{S}$ and the submatrices $\mathbf{s}_{\alpha\beta}$ which are frequently used in this paper. Since current is conserved $\mathbf{S}$ must be a unitary matrix,

$$\mathbf{S}^{\dagger}\mathbf{S}=\mathbf{S}\mathbf{S}^{\dagger}=\mathbf{1} \tag{A1}$$

and since the Hamiltonian is invariant, if momenta and magnetic field are reversed simultaneously,

$$\mathbf{S}^{*}(-\mathbf{B})=\mathbf{S}^{-1}(\mathbf{B}) \ . \tag{A2}$$

According to Eq. (A1), $\mathbf{S}^{-1}(\mathbf{B})=\mathbf{S}^{\dagger}(\mathbf{B})$ and hence

$$\mathbf{S}^{T}(\mathbf{B})=\mathbf{S}(-\mathbf{B}) \ , \tag{A3a}$$

where the upper index T denotes the transposed matrix. For the submatrices $\mathbf{s}_{\alpha\beta}$ the unitary property of $\mathbf{S}^{\dagger}\mathbf{S}=\mathbf{1}$ implies

$$\sum_{\alpha} \mathbf{s}_{\alpha\beta}^{\dagger}\mathbf{s}_{\alpha\gamma}=\mathbf{1}_{\beta}\delta_{\beta\gamma} \ , \tag{A3b}$$

where $\mathbf{1}_{\alpha}$ is a unit matrix of dimension $M_{\alpha}=\mathrm{Tr}(\mathbf{1}_{\alpha})$. Similarly, $\mathbf{S}\mathbf{S}^{\dagger}=\mathbf{1}$ implies

$$\sum_{\alpha} \mathbf{s}_{\beta\alpha}\mathbf{s}_{\gamma\alpha}^{\dagger}=\mathbf{1}_{\beta}\delta_{\beta\gamma} \ . \tag{A4}$$

In terms of the total reflection and transmission probabilities, $R_{\alpha\alpha}\equiv\mathrm{Tr}(\mathbf{s}_{\alpha\alpha}^{\dagger}\mathbf{s}_{\alpha\alpha})$ and $T_{\alpha\beta}=\mathrm{Tr}(\mathbf{s}_{\alpha\beta}^{\dagger}\mathbf{s}_{\alpha\beta})$, Eqs. (A3b) and (A4) state the conservation of current

$$M_{\alpha}=R_{\alpha\alpha}+\sum_{\beta} T_{\alpha\beta} \ , \tag{A5}$$

$$M_{\alpha}=R_{\alpha\alpha}+\sum_{\beta} T_{\beta\alpha} \ . \tag{A6}$$

APPENDIX B: PROPERTIES OF THE MATRIX A

A number of properties of the matrix $\mathbf{A}$ are given below. The matrix $\mathbf{A}$ is related to the current-matrix elements[1] and in this work was shown to relate the occupation number operators in a given probe to the creation and annihilation operators of the incoming channels. First, we emphasize that $\mathbf{A}$ is not by itself an observable, but the adjoint of $\mathbf{A}_{\beta\gamma}(\alpha)$ is the matrix $\mathbf{A}_{\gamma\beta}(\alpha)$ with α and γ interchanged,

$$\mathbf{A}_{\beta\gamma}^{\dagger}(\alpha)=\mathbf{1}_{\alpha}\delta_{\alpha\beta}\delta_{\alpha\gamma}-\mathbf{s}_{\alpha\gamma}^{\dagger}\mathbf{s}_{\alpha\beta}=\mathbf{A}_{\gamma\beta}(\alpha) \ . \tag{B1}$$

Nevertheless, $\mathbf{A}$ has a number of interesting properties which stem from the properties of the single-particle scattering matrix $\mathbf{S}$. Using Eq. (A3b) we find

$$\sum_{\alpha} \mathbf{A}_{\beta\gamma}(\alpha)=\sum_{\alpha}[\mathbf{1}_{\alpha}\delta_{\alpha\beta}\delta_{\alpha\gamma}-\mathbf{s}_{\alpha\beta}^{\dagger}\mathbf{s}_{\alpha\gamma}]$$
$$=\mathbf{1}_{\beta}\delta_{\beta\gamma}-\mathbf{1}_{\beta}\delta_{\beta\gamma}=0 \ . \tag{B2}$$

The $\mathbf{A}$ matrices enter the expressions for the fluctuations in the combination

$$\mathbf{A}_{\gamma\delta}(\alpha)\,\mathbf{A}_{\delta\gamma}(\beta)=[\mathbf{1}_{\alpha}\delta_{\alpha\gamma}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{\beta\delta}-\delta_{\beta\gamma}\delta_{\beta\delta}\mathbf{s}_{\alpha\gamma}^{\dagger}\mathbf{s}_{\alpha\delta} -\delta_{\alpha\gamma}\delta_{\alpha\delta}\mathbf{s}_{\beta\delta}^{\dagger}\mathbf{s}_{\beta\gamma}+\mathbf{s}_{\alpha\gamma}^{\dagger}\mathbf{s}_{\alpha\delta}\mathbf{s}_{\beta\delta}^{\dagger}\mathbf{s}_{\beta\gamma}] \ . \tag{B3}$$

Let us consider summation over one of the lower indices of the trace of $\mathbf{A}_{\gamma\delta}(\alpha)\,\mathbf{A}_{\delta\gamma}(\beta)$,

$$\sum_{\gamma}\mathrm{Tr}[\,\mathbf{A}_{\gamma\delta}(\alpha)\,\mathbf{A}_{\delta\gamma}(\beta)]=\mathrm{Tr}[\mathbf{1}_{\alpha}\delta_{\alpha\delta}\delta_{\beta\alpha}\delta_{\beta\delta}-\delta_{\beta\delta}\mathbf{s}_{\alpha\beta}^{\dagger}\mathbf{s}_{\alpha\delta} -\delta_{\alpha\delta}\mathbf{s}_{\beta\delta}^{\dagger}\mathbf{s}_{\beta\alpha}+\mathbf{1}_{\alpha}\delta_{\alpha\beta}\mathbf{s}_{\alpha\delta}\mathbf{s}_{\beta\delta}^{\dagger}] \ . \tag{B4}$$

To obtain the last term we have made use of the cyclic property of the trace and have used Eq. (A4). If we carry out the summation over δ we find

$$\sum_{\gamma,\delta} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] = \mathrm{Tr}[2 1_{\alpha}\delta_{\beta\alpha} - \mathbf{s}^{\dagger}_{\alpha\beta}\mathbf{s}_{\alpha\beta} - \mathbf{s}^{\dagger}_{\beta\alpha}\mathbf{s}_{\beta\alpha}] , \tag{B5}$$

where we have used Eq. (A3b). For $\alpha=\beta$ the right-hand side of Eq. (B5) is equal to $2M_{\alpha} - 2\,\mathrm{Tr}(\mathbf{r}^{\dagger}_{\alpha\beta}\mathbf{r}_{\alpha\beta})$ which is $2(M_{\alpha} - R_{\alpha\alpha})$. For $\alpha \neq \beta$ the right-hand side of Eq. (B5) is equal to $-\mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\beta}\mathbf{s}_{\alpha\beta}) - \mathrm{Tr}(\mathbf{s}^{\dagger}_{\beta\alpha}\mathbf{s}_{\beta\alpha})$ which is equal to $-(T_{\alpha\beta}+T_{\beta\alpha})$. Next consider the sum of the trace of the two $\mathbf{A}$ matrices over δ but weighted by functions f_{δ}. For $\alpha \neq \beta$ we find

$$\sum_{\delta} f_{\delta} \left[\sum_{\gamma} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \right] = -\mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\beta}\mathbf{s}_{\alpha\beta}) f_{\beta} - \mathrm{Tr}(\mathbf{s}^{\dagger}_{\beta\alpha}\mathbf{s}_{\beta\alpha}) f_{\alpha} . \tag{B6}$$

Similarly, we can show that for $\alpha \neq \beta$

$$\sum_{\gamma} f_{\gamma} \left[\sum_{\delta} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \right] = -\mathrm{Tr}(\mathbf{s}^{\dagger}_{\alpha\beta}\mathbf{s}_{\alpha\beta}) f_{\beta} - \mathrm{Tr}(\mathbf{s}^{\dagger}_{\beta\alpha}\mathbf{s}_{\beta\alpha}) f_{\alpha} , \tag{B7}$$

and hence

$$\sum_{\delta} f_{\delta} \left[\sum_{\gamma} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \right] = \sum_{\gamma} f_{\gamma} \left[\sum_{\delta} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \right] . \tag{B8}$$

For $\alpha=\beta$ we find, instead of Eq. (B6),

$$\sum_{\delta} f_{\delta} \left[\sum_{\gamma} \mathrm{Tr}[\mathbf{A}_{\gamma\delta}(\alpha)\mathbf{A}_{\delta\gamma}(\beta)] \right] = \mathrm{Tr}(1_{\alpha} - 2\mathbf{s}^{\dagger}_{\alpha\alpha}\mathbf{s}_{\alpha\alpha}) f_{\alpha} + \sum_{\delta} \mathrm{Tr}(\mathbf{s}_{\alpha\delta}\mathbf{s}^{\dagger}_{\alpha\delta}) f_{\delta} . \tag{B9}$$

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