Localized electrons on a lattice with incommensurate magnetic flux

Shmuel Fishman
Department of Physics, Technion, Israeli Institute of Technology, 32000 Haifa, Israel

Yonathan Shapir
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627-0011

Xiang-Rong Wang
School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455
and Department of Physics, The Hong Kong University of Science and Technology,
Clear Water Bay, Kowloon, Hong Kong

(Received 27 May 1992)

The magnetic-field effects on lattice wave functions of Hofstadter electrons strongly localized at boundaries are studied analytically and numerically. The exponential decay of the wave function is modulated by a field-dependent amplitude $J(t)=\prod_{r=0}^{t-1}2\cos(\pi\alpha r)$, where α is the magnetic flux per plaquette (in units of a flux quantum) and t is the distance from the boundary (in units of the lattice spacing). The behavior of $|J(t)|$ is found to depend sensitively on the value of α. While for rational values $\alpha=p/q$ the envelope of $J(t)$ increases as $2^{t/q}$, the behavior for α irrational ($q\to\infty$) is erratic with an aperiodic structure which drastically changes with α. For algebraic α it is found that $J(t)$ increases as a power law $t^{\beta(\alpha)}$ while it grows faster (presumably as $t^{\beta(\alpha)\ln t}$) for transcendental α. This is very different from the growth rate $J(t)\sim e^{\sqrt{t}}$ that is typical for cosines with random phases. The theoretical analysis is extended to products of the type $J^{\nu}(t)=\prod_{r=0}^{t-1}2\cos(\pi\alpha r^{\nu})$ with $\nu>0$. Different behavior of $J^{\nu}(t)$ is found in various regimes of ν. It changes from periodic for small ν to randomlike for large ν.

I. INTRODUCTION

The properties of noninteracting electrons on a lattice subjected to an external magnetic field have attracted much attention since the early works of Hofstadter,[1] Wannier,[2] and Azbel.[3] These works have focused primarily on the exotic spectral properties as a function of the electron's energy and the parameter $\alpha=\phi/\phi_0$, where ϕ is the magnetic flux per plaquette and $\phi_0=\hbar/e$ is the flux quantum. The spectrum has special scaling properties as a function of the commensurability q for $\alpha=p/q$ (rational) and becomes a Cantor set for incommensurate fluxes (α irrational). The wave function itself is extended in the q subbands for commensurate values of rational α.

The Hamiltonian describing the lattice electron is

$$\mathcal{H}=W\sum_i a_i^\dagger a_i+V\sum_{\langle ij\rangle} a_i^\dagger a_j e^{i\gamma_{ij}}+\text{c.c.} \tag{1.1}$$

The phase γ_{ij} is associated to the link $\langle ij\rangle$ between the nearest-neighbor sites in accordance with the "Peierls ansatz." Any gauge such that the sum around a plaquette

$$\sum_{\square}\gamma_{ij}=e\phi/\hbar \tag{1.2}$$

will do.

More recently[4-7] there has been growing interest in the effect of a magnetic field on localized electrons. The combined effects of lattice periodicity and magnetic flux create a very complex behavior in the spatial variation of the wave functions even in the absence of disorder.[4] This is the case when the electron's energy is deep inside a gap between the quasibands of the bulk eigenstates.[1-3] The electron may be localized at inhomogeneities such as the edge of the lattice (i.e., surface gap states) or at isolated impurities in an otherwise ordered bulk. Clearly these states decay exponentially going away from the inhomogeneity into the bulk. This exponential decay and particularly the "localization" length (associated with this exponential decay) will be affected by the application of an external magnetic field to the system.

In the present work we look at some basic aspects of this problem. While avoiding the full complexities of the related questions, we look at a simple (maybe the simpler) model in which the intricate salient features of this problem are predominant and can be addressed.

As is well known,[1-3] the two-dimensional (2D) tight-binding lattice electron problem reduces (in the Landau gauge) to that of a 1D electron hopping in a potential which varies as $\cos(k_\perp j+2\pi\alpha)$ where $k_\perp$ are the transverse momenta in the y direction, and may be taken to be zero by shifting the origin in the x direction.[1] The Schrödinger equation reduces then to the famous Harper equation

$$u_{n+1}+u_{n-1}+V_n u_n=(E-W)u_n\ , \tag{1.3}$$

where the diagonal potential is

$$V_n=\lambda\cos\pi\alpha n\ . \tag{1.4}$$

As discussed below, this model was generalized[8-10] to potentials of the type

$$V_n^\nu = \lambda \cos \pi \alpha n^\nu \ . \tag{1.5}$$

It can be used in order to analyze the transition from extended to localized wave functions in the proposed experimentally realizable layered systems[10,11] in which the distance between adjacent layered increases as n^ν (for $0<\nu<1$). The quantum transmission of such layered systems in the strong localization regime, for different values of ν, is also a part of our present investigations.

In the localized regime the decay of the wave function may be studied by looking at the probability of an electron localized at the origin to tunnel to another site a distance t away, $|I(t)|^2$, where $I(t)=\langle \psi_0^*(t)\psi_0(0)\rangle$ is the related Green's function [$\psi_0(r)$ is the wave function localized at the origin]. Our results will be derived within the "directed paths" approximation[7] which becomes better as the localization (=decay) length becomes smaller. This will be the case when the hopping matrix element V is much smaller than the on-site energy W and the energy E is far away from the band $W\pm 2V$ of the extended bulk eigenstates ($E=0$ is a convenient choice which fulfills this requirement).

Preliminary results of investigations which go beyond this approximation and include "returning loops" are reported in the last section.

The Green's function $G(r)$ may then be expressed as a sum over paths, and since each step has an amplitude of V/W (for $E=0$) the leading contribution comes from the shortest, hence directed, paths. If only these are kept one has[4]

$$|I(t)|^2 = \left[\frac{V}{W} \right]^{2t} |J(t)|^2 \ , \tag{1.6}$$

where $(V/W)^{2t}$ is responsible for the strong exponential decay and $J(t)$ contains all the interference effects. In the absence of a magnetic field $J(t)$ is just the total number of paths going from the origin to the final site which increases as 2^{2t} [note that since $V \ll 2W$, $I(t)$ still strongly decays exponentially with t]. In the presence of a magnetic field $J(t)$ has a very complex behavior as a function of the flux α and the distance t. $J(t)$ also depends explicitly on the geometry and the one utilized in most of the works is that of a square lattice with the origin and final sites being along the diagonal (this choice is a natural one; similar calculations, however, may be carried for any locations of these sites). For the surface realizations this choice means that the edge is in the $[\bar{1},1]$ direction (the distance along this direction will hereafter be denoted x), while the direction perpendicular to the edge is $[1,1]$ (and the distance from the surface into the bulk in this direction is t).

The $J(x,t)$ for consecutive t are related by a transfer matrix T:

$$J(x,t+1) = \sum_{x'} T_{t,t+1}(x,x') J(x',t) \ . \tag{1.7}$$

The solution relies on the diagonalization of T, and the technical calculations are given elsewhere[4] and will not be repeated here. The important feature is that in the presence of magnetic fields a gauge may be chosen such that the matrices which will depend explicitly on t will, nevertheless, commute and therefore may be diagonalizable simultaneously. The eigenvectors are the transverse waves $e^{ik_\perp x}$ with $k_\perp = \pm m\pi/L$ ($L \gg t$ is a large width cutoff). So the localized state at (0,0) may be decomposed into transverse Fourier components (x direction) and each component (with $k_\perp$) will "propagate" independently into the bulk in the t direction. It is thus very natural to define the quantity $J(t,k_\perp)$ for each Fourier component. In previous calculations it was found for a given $k_\perp$ and $\alpha=\phi/\phi_0$ that

$$|J_\alpha(t,k_\perp)|^2 = \prod_{r=0}^{t-1} |2\cos(\pi\alpha r - k_\perp)|^2 \ . \tag{1.8}$$

The asymptotic behavior of these products do determine the decay into the bulk of the surface gap states, which in the transverse direction along the surface have the same dependence as the eigenvectors, i.e., $e^{ik_\perp x}$.

To find $G(x,t)$ for an electron localized at a site we still need to sum factors like $J_\alpha(k_\perp,t)e^{ik_\perp x}$ over all $k_\perp$. The asymptotic behavior in real space will be determined by that of $J_\alpha(k_\perp,t)$ for large t as we observed numerically. The closed form of $G(x,t)$ requires the knowledge of the exact amplitude and phase of all $J_\alpha(k_\perp,t)$ and is beyond the scope of the present paper.

The behavior of products as in Eq. (1.8) is very sensitive to the values of α. For any rational $\alpha=p/q$ it may be shown that $|J(t)|$ increases as $2^{t/q}$. That naturally raises the question what will be the behavior as $q\to\infty$. This and related questions are the subject of the present paper.

Preliminary numerical investigation for $\alpha=(\sqrt{5}-1)/2$ (the golden mean) has exhibited $J_\alpha(x,t)$ in the form of bounded aperiodic fluctuations. Although an exponential behavior where q is the scale is ruled out, other possibilities like powers of t of $\ln t$ are still possible.

Another asymptotic behavior we should consider arises if the phases of the cosines in the product are random. Then the behavior is that of a product of random variables. The typical (though not the average) behavior is $e^{\langle \ln J\rangle} \sim e^{c\sqrt{t}}$. It should be noted that this behavior corresponds, in the original lattice model, to a magnetic flux which is uniform in the x direction but changes randomly from one row to the next in the t direction (so-called "random rods").

The initial motivation for the generalization of the potential V_n^ν [Eq. (1.5)] to $\nu\neq 1$ came from the field of quantum chaos. The kicked rotor model was mapped on the tight-binding model with the diagonal potential[12]

$$\tilde{V}_n^\nu = \tan \pi \alpha n^2 \ . \tag{1.9}$$

It was argued that it behaves like a random potential, leading to localization of the eigenstates of the corresponding model. In the field of quantum chaos it explains the quantal suppression of chaos. It was argued that the sequence (1.9) is pseudorandom since if n changes by 1, the phase changes by $2\pi\alpha n$ which is a large number. Since the tangent depends only on the phase (modπ) it depends on a small fraction of a large number. Therefore, if α is irrational it is related to remote digits in a representation of an irrational number and therefore can be considered pseudorandom. The theoretical investigation of this issue motivated the introduction[8] of the tight-binding model (1.3) with the potential (1.5). The parameter ν controls the degree of pseudorandomness that increases with ν. It was found that the asymptotic behavior of sums of the form

$$S_N^\nu = \sum_{n=1}^{N} V_n^\nu \tag{1.10}$$

is of crucial importance for the understanding of localization. Sums of this form[13–15] are of great importance for the present work as well. The pseudorandom properties of the sequences (1.5) were classified in the framework of standard tests for pseudorandomness.[16] One of the conclusions of these investigations is that they are very different for various regimes of ν. For $\nu>2$ the behavior of the sequences is very similar to that of the corresponding random ones, namely to sequences where the phase is truly random. For $1<\nu<2$ the asymptotic growth of the sums (1.10) is $S_N^\nu \sim \sqrt{N}$, as for random ones, but the growth takes place[8,9] in narrow regions in n. For $0<\nu<1$ the difference between consecutive terms approaches zero for large n and the sequences do not resemble random sequences at all. The physically important cases $\nu=1$ and $\nu=2$ are bordering cases between different regimes. Although the generalization (1.5) of the usual Harper equation was proposed[8–10] for purely theoretical reasons, it was suggested to be relevant for plasmon dynamics in artificially constructed superlattices.[10,11] Effects of such potentials on modulated waveguides in the microwave regime may be investigated as well.[17]

In the present paper the asymptotic behavior of $J(t)$ is investigated. For this purpose the sum

$$A_n = \ln|J(t=N)| \tag{1.11}$$

is investigated in Sec. II. It turns out to be very different from the corresponding quantity where the phases are random. It can be generalized to arbitrary ν as

$$A_N^\nu = \ln|J^\nu(t=N)| = \ln\left[\prod_{r=0}^{N-1} 2|\cos\pi\alpha r^\nu|\right] . \tag{1.12}$$

In the end of Sec. II it is shown that a small perturbation in ν around 1 yields a drastic change in the behavior of A_N^ν. In Sec. III the behavior of the sums (1.10) is studied in the regimes $0<\nu<\frac{1}{2}$, $\frac{1}{2}<\nu<1$, $1<\nu<2$, and $\nu\geq 2$. In each of these regimes one finds different behavior that also differs from the one found for $\nu=1$. The results are summarized in Sec. IV.

II. THE SUMS A_N FOR THE INCOMMENSURATE PHASE ($\nu=1$)

In this section we will estimate the sum of (1.11), namely

$$A_N = \sum_{n=0}^{N-1} \ln 2|\cos(\pi\phi_n)| , \tag{2.1}$$

with

$$\phi_n = \alpha n . \tag{2.2}$$

Of particular interest will be the difference between this sum and the corresponding random sum, namely the sum where the phase ϕ_n is replaced by a random variable that is uniformly distributed in the interval $[-1,1]$. In order to estimate the sum (2.1) we will exploit the fact that each term is periodic in ϕ_n. It is easy to see that

$$\ln|\sin\theta| = -\ln 2 - \sum_{m=1}^{\infty} \frac{1}{m}\cos 2m\theta , \tag{2.3}$$

implying

$$\ln|\cos\pi\phi_n| = -\ln 2 - \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\cos(2\pi\phi_n m) . \tag{2.4}$$

The original sum (2.1) reduces to the form

$$A_N = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{n=0}^{N-1} \cos(2\pi\phi_n m) . \tag{2.5}$$

If the phases ϕ_n are random

$$\langle A_N \rangle = 0 , \tag{2.6}$$

while

$$\langle (A_N)^2 \rangle = \frac{N}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{12}N , \tag{2.7}$$

where $\langle\ \rangle$ denote averages over the realizations of the random phases.

Therefore, if the phase ϕ_n is random one expects that the typical size of A_N will be of the order $\sqrt{N}$. If, on the other hand, ϕ_N is given by (2.1) the behavior is completely different. In this case, the sum over n is just a geometric series, namely

$$\sum_{n=0}^{N-1} \cos 2\pi\alpha n m = \frac{1}{2}\left[\frac{\sin\pi\alpha(2N-1)m}{\sin\pi\alpha m} + 1\right] , \tag{2.8}$$

leading to

$$A_N = \tilde{A}_N + \tfrac{1}{2}\ln 2 , \tag{2.9}$$

with

$$\tilde{A}_N = -\tfrac{1}{2}\sum_{m=1}^{\infty} \frac{(-1)^m}{m}\frac{\sin\pi\alpha(2N-1)m}{\sin\pi\alpha m} . \tag{2.10}$$

The sum $\tilde{A}_N$ is dominated by the terms where the denominators are small. These are terms where m is equal to q_k—a denominator of the rational approximant of order k to α. The method that will be used is closely related to the one that was applied by Berry[18] for a different problem. Let α be an irrational number, and the elements of its continued fraction expansion be a_0, $a_1 \cdots a_i \cdots$ and let its rational approximants of order k be

$$\alpha_k = p_k/q_k , \tag{2.11a}$$

where p_k and q_k are integers. The error in this approximant is

$$\epsilon_k = \alpha - \alpha_k \ . \tag{2.11b}$$

For almost all irrational numbers (Ref. 19, p. 78; Ref. 20, p. 169)

$$|\epsilon_k| \sim \frac{\bar{c}_2}{q_k^2 \ln q_k} \ , \tag{2.12}$$

while for irrational numbers with continued fractions with bounded elements a_i (Ref. 19, p. 44)

$$|\epsilon_k| \sim \frac{c_2}{q_k^2} \ , \tag{2.13}$$

where $\bar{c}_2$ and c_2 are constants independent of k. For the golden mean $\alpha = \sqrt{5} - 1/2$, for example $c_2 = 1/\sqrt{5}$ (Ref. 19, p. 41; Ref. 20, p. 163).

The sum (2.10) can be estimated as

$$\tilde{A}_n = -\tfrac{1}{2} \sum_k \frac{(-1)^{q_k}}{q_k} \frac{\sin \pi \epsilon_k (2N-1)}{\sin \pi \epsilon_k} \ . \tag{2.14}$$

For irrational numbers with continued fractions with bounded elements the approximation (2.13) holds, resulting in

$$\tilde{A}_N = -\tfrac{1}{2} \sum_k \frac{(-1)^{q_k}}{c_2 \pi} \sin \left[\frac{\pi(2N-1)}{q_k} c_2 \right] \ . \tag{2.15}$$

We turn now to estimate the sum $\tilde{A}_N$ of (2.15) for large N. The contributions from the terms with large values of q_k can be approximated by a convergent integral. This motivates to split the sum in two parts, namely

$$\tilde{A}_N = -\frac{1}{2c_2\pi} [\tilde{A}_N^{(1)} + \tilde{A}_N^{(2)}] \ , \tag{2.16}$$

where

$$\tilde{A}_N^{(1)} = \sum_{k=0}^{k_N} (-1)^{q_k} \sin \left[\frac{\pi(2N-1)}{q_k} c_2 \right] \tag{2.17}$$

and

$$\tilde{A}_N^{(2)} = \sum_{k=k_N+1}^{\infty} (-1)^{q_k} \sin \left[\frac{\pi(2N-1)}{q_k} c_2 \right] \ . \tag{2.18}$$

The condition

$$q_{k_N} = N \tag{2.19}$$

determines k_N. We first turn to evaluate the sum $\tilde{A}_N^{(2)}$.

The factor $(-1)^{q_k}$ introduces a rapid oscillation in the terms of the sum. However, locally the number of even values of q_k is not equal to the number of odd values. Actually, the number of odd values of q_k is twice the number of even values as demonstrated in Appendix A. Therefore this factor does not result in local cancellations. It may change the value of the sum but not its asymptotic N dependence. We will ignore this factor in the evaluation of $\tilde{A}_N^{(2)}$. If $(-1)^{q_k}$ is ignored the sum can be approximated by an integral. For this purpose we note that

$$q_k \sim e^{\delta k} \ , \tag{2.20}$$

where δ depends in general on α. For the golden mean

$$\delta = -\ln \alpha \ , \tag{2.21}$$

while for generic numbers [almost all irrational numbers for which Eq. (2.12) applies]

$$\delta = \frac{\pi^2}{12 \ln 2} \ . \tag{2.22}$$

For the numbers with bounded a_i like the golden mean, the sum behaves as

$$\tilde{A}_N^{(2)} \sim \int_{k_N}^{\infty} dk \, \sin(2\pi N c_2 / q_k) \sim \frac{1}{\delta} \int_{\bar{q}}^{\infty} \frac{dq}{q} \sin(2\pi N/q) \ , \tag{2.23}$$

where $\bar{q} = q_{k_N}/c_2$ and k_N are related via (2.21) and (2.19). This integral is convergent. It can be easily estimated with the help of the change of variable

$$x = \frac{1}{q} \ , \tag{2.24}$$

leading to

$$\tilde{A}_N^{(2)} \sim \frac{1}{\delta} \int_0^{1/\bar{q}} dx \frac{\sin 2\pi N x}{x} \ . \tag{2.25}$$

Since

$$\lim_{N \to \infty} \frac{\sin 2\pi N x}{x} = \pi \delta(x) \tag{2.26}$$

$\tilde{A}_N^{(2)}$ behaves as a constant in the limit $N \to \infty$.

The error in the estimate of the sum $\tilde{A}_N^{(2)}$ by an integral is of the order of

$$E_N = \sum_{k=k_N+1}^{\infty} \frac{c_2\pi(2N-1)}{q_k} = \frac{c_2\pi(2N-1)e^{-\delta(k_N+1)}}{1-e^{-\delta}} \tag{2.27}$$

that is bounded following the definition (2.19) of k_N and (2.20). Therefore the sum $\tilde{A}_N^{(2)}$ is bounded with a bound that is finite in the limit $N \to \infty$. We turn now to estimate the behavior of $\tilde{A}_N^{(1)}$, that turns out to dominate the sum $\tilde{A}_N$ of (2.15). For this purpose it is rewritten in the form

$$\tilde{A}_N^{(1)} = \sum_{m=0}^{k_N} (-1)^{q_k} \sin 2\pi \mu \gamma^m \ , \tag{2.28}$$

where

$$2\mu = c_2(2N-1)e^{-\delta k_N} \ , \tag{2.29}$$

$$\gamma = e^{\delta} \ , \tag{2.30}$$

and

$$k = k_N - m \ . \tag{2.31}$$

Note that μ approaches a constant in the limit $N \to \infty$.

Typically μ is an irrational number. For integer values of γ, the sequence

$$\phi^{\mu}(m)=(\mu\gamma^{m})\text{mod } 1 \quad (2.32)$$

can be considered random, since it is a shift along the digits of an irrational number, in the base γ. If γ is not an integer it is hard to believe that $\phi^{\mu}(m)$ will be less random (although we do not claim to justify it rigorously). In what follows it will be assumed that $\{\phi^{m}(m)\}$ is a random uncorrelated sequence. With this assumption $\tilde{A}_N^{(1)}$ takes the form

$$\tilde{A}_N^{(1)}=\sum_{m=0}^{k_N}(-1)^{q_k}\sin 2\pi\phi^{\mu}(m) \ . \quad (2.33)$$

Assuming that $\{\phi^{\mu}(m)\}$ is random one finds that the variance of $\tilde{A}_N^{(1)}$ is $\frac{1}{2}k_N\sim \ln N/2\delta$. Hence the typical magnitude of $\tilde{A}_N^{(1)}$ is of the order of $\sqrt{\ln N}$. Therefore $\tilde{A}_{N^{(1)}}$ is the dominant contribution to $\tilde{A}_N$ of (2.15) and $\tilde{A}_N^{(2)}$ is negligible for large N. Because of the randomness of $\{\phi^{\mu}(m)\}$ the sums $\tilde{A}_N^{(1)}$ and $\tilde{A}_N$ look like a random walk in one dimension.

It is instructive to define $\tilde{A}_N^{\max}$ as the maximal value that $|\tilde{A}_{N'}|$ may take for $N'\le N$. From the random-walk property of $\tilde{A}_N^{(1)}$ one finds that for large N

$$\tilde{A}_N^{\max}\sim k_N\sim \ln N \ . \quad (2.34)$$

[It is easy to show that the typical interval between two consecutive N''s for which $\tilde{A}_{N'}^{(1)}>C\ln N'(C^2<\delta/2)$ is $\Delta N\le N'^{2C^2\delta}(\pi\ln N'/\delta)^{1/2}<<N$.]

For generic α, where (2.12) holds the sum, (2.15) should be replaced by

$$\tilde{A}_N=-\frac{1}{2}\sum_k\frac{(-1)^{q_k}\ln q_k}{\bar{c}_2\pi}\sin\left[\frac{\pi(2N-1)\bar{c}_2}{q_k\ln q_k}\right] \ . \quad (2.35)$$

The sum can be split into two parts as was done in (2.16) and sums similar to (2.17) and (2.18) are obtained with each term multiplied by $\ln q_k$ and c_2 replaced by $\bar{c}_2$. The sum $\tilde{A}_N^{(2)}$ is approximated by an integral with an error of the order (2.27) that is bounded.

Considerations similar to those leading to (2.25), with the change of variable

$$x=\frac{\bar{c}_2}{q\ln q} \ , \quad (2.36)$$

lead for large $\bar{q}$ to

$$\tilde{A}_N^{(2)}\sim -\frac{1}{\delta}\int_0^{\bar{c}_2/\bar{q}\ln\bar{q}}dx\frac{\ln x}{x}\sin 2\pi N x \ . \quad (2.37)$$

Here $\bar{q}=q_{k_N}=N$. Since for large N the upper limit of the integral is much smaller than $1/N$ the integral can be estimated as

$$\tilde{A}_N^{(2)}\sim -\frac{2\pi N}{\delta}\int_0^{\bar{c}_2/\bar{q}\ln\bar{q}}dx\,\ln x\sim -\frac{2\pi}{\delta}\frac{N}{\bar{q}}=\frac{2\pi}{\delta} \ , \quad (2.38)$$

which is bounded. Again the leading contribution results from $\tilde{A}_N^{(1)}$ that now takes the form

$$\tilde{A}_N^{(1)}=\delta\sum_{m=0}^{k_N}(-1)^{g_k}k\sin\left[\frac{2\pi\mu\gamma^m}{\delta k}\right] \ , \quad (2.39)$$

where μ, γ, and k are defined by (2.29)–(2.31). Following the argument presented after Eq. (2.32), the sequence $\tilde{A}_N^{(1)}$ can be considered random with the variance of the order of $\frac{1}{2}(\delta k_N)^2\sim\frac{1}{2}(\ln N)^2$. The typical values of this sequence are therefore of the order of $\ln N$, while the maximal value of $|\tilde{A}_{N'}|$ for $N'\le N$ is

$$\tilde{A}_N^{\max}\sim k_N^2\sim(\ln N)^2 \ . \quad (2.40)$$

We conclude that the typical growth of A_N with N is $\ln N$ and not $\sqrt{N}$ (as if the original phases ϕ_n were random). For irrational numbers with continued fractions with bounded elements the typical growth is of the order $\sqrt{\ln N}$. The maximal value that the sum takes grows as $(\ln N)^2$ and $\ln N$, respectively, in these cases.

In Fig. 1 A_N and $A_N^{\max}$ as a function of N are depicted for $\alpha=\frac{1}{2}(\sqrt{5}-1)$ and $\alpha=1/e$. The values of A_N are of the order unity and are very different from the values expected for random sequencing. It is hard to see the expected growth of A_N because of its erratic nature. Since $\tilde{A}_N^{\max}$ is a monotonically increasing sequence, its growth is systematic and obvious. Note the difference between $\ln N$ and a faster growth consistent with $(\ln N)^2$, found for irrationals with bounded and unbounded elements of the continued fractions. These results are in accord with the assumptions that were made on the phases of the sine in (2.33) and (2.39).

In Fig. 1(b) we see that in addition to the continuous increase of $A_N^{\max}$ with $\ln N$, there are also discontinuous "jumps" which are periodically spaced and of similar heights. These jumps are compounded by contributions from two consecutive N's. When we looked instead to the series $B_N=\ln\pi_{r=1}^N|\sin(\pi\alpha r)|$ with the same $\alpha=(\sqrt{5}-1)/2$ we observe a continuous increase with $\ln N$ [but with a different slope which may be understood from the absence of the factors $(-1)^m$ in Eq. (2.10)] but without any discontinuous jumps. For other rational α's other structures of periodic jumps were seen. On the other hand no periodic structure is seen in these jumps for nonalgebraic α's [Fig. 1(d)]. This is clearly a commensurability effect connected with the continuous fraction expansion of α. So far we do not have a more quantitative understanding of these discontinuities.

All computations were done in double precision. To check possible effects of the finite precision we also ran the same program in single precision (and with both degrees of precision on a different machine) without discernible differences. So we are confident that the structure observed is real and is not an artifact of the finite precision.

In this section we examined so far the behavior of A_N for $\nu=1$; namely the expression (2.2) for the phase ϕ_n^{ν} was used. In the next section values of $\nu\neq 1$ will be investigated. As a first step in that direction we explore the stability of the $\nu=1$ behavior for slight deviations of ν from this value.

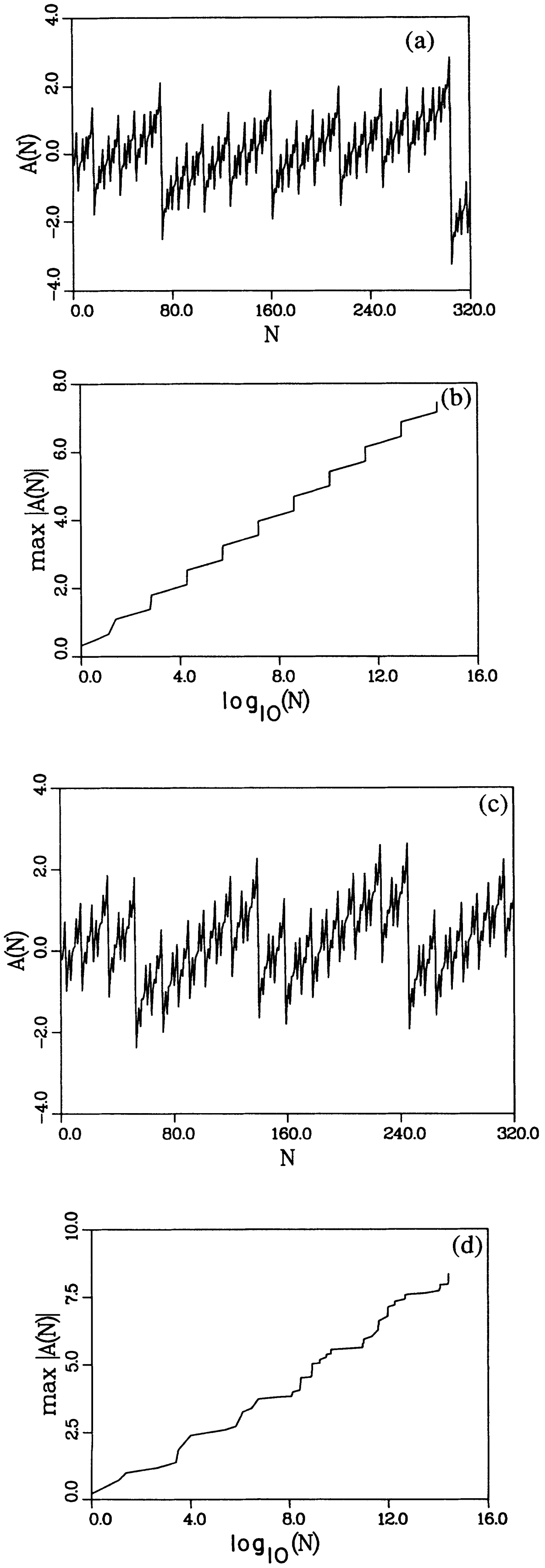

FIG. 1. $\nu=1$ in (a) A_N for $\alpha=\frac{1}{2}(\sqrt{5}-1)$, (b) max A_N for $\alpha=\frac{1}{2}(\sqrt{5}-1)$, (c) A_N for $\alpha=1/e$, and (d) max A_N for $\alpha=1/e$.

We therefore consider the case $\nu=1+\epsilon$ and attempt to find the deviation:

$$\Delta A_N^{\epsilon}(\alpha)=A_N^{1+\epsilon}(\alpha)-A_N^{1}(\alpha) \ . \tag{2.41}$$

A divergence in the $\Delta A_N^{\epsilon}(\alpha)$ as $N\to\infty$ for arbitrarily small ϵ will indicate the instability of the $\nu=1$ behavior.

For small ϵ we expand,

$$\begin{aligned} A_N^{1+\epsilon}(\alpha) &= \sum_{r=1}^{N} \ln|\cos(\pi\alpha r^{1+\epsilon})| \\ &= \sum_{r=1}^{N} \ln|\cos\{\pi\alpha r(1+\epsilon\ln r\}\}| \\ &= \sum_{r} \ln|\cos(\pi\alpha r)|+\epsilon\pi\alpha r\ln r|tg(\pi\alpha r)| \ . \end{aligned} \tag{2.42}$$

Thus from (2.26) we have

$$\Delta A_N^{\epsilon}(\alpha)=\epsilon\pi\alpha\sum_{r=1}^{N} r\ln r|tg(\pi\alpha r)| \ .$$

It is already clear that this sum will diverge at least as $N^2\ln N$. However, the presence of the $tg(\pi\alpha r)$ may yield a stronger divergence.

To explore this effect we concentrate on the points r such that $2\alpha r\approx 2n+1$. Near these points 2α is approximated by the continued fraction p_k/q_k so that $|2\alpha-p_k/q_k|\sim\delta_k$, with $\delta_k\sim 1/q_k^2\ln q_k$. Evidently the singular points will be the $r=q_k$ for which p_k is odd. Near these points

$$tg\left[(n+\tfrac{1}{2})\pi+\frac{\pi\delta_k}{2}\right]\sim\frac{2}{\pi\delta_k} \ . \tag{2.43}$$

Inserting to $\Delta A_N^{\epsilon}(\alpha)$, we obtain

$$\begin{aligned} \Delta A_N^{\epsilon}(\alpha) &\sim \sum_k{}' q_k\ln q_k\frac{2q_k^2\ln q_k}{\pi q_k} \\ &\sim \sum_k{}'(q_k\ln q_k)^2 \ , \end{aligned} \tag{2.44}$$

where $\sum'$ means the sum only over these k for which p_k is odd.

The asymptotic behavior is obtained by transforming to an integral as before:

$$\Delta A_N^{\epsilon}(\alpha)\sim\epsilon N^2(\ln N)^2 \ . \tag{2.45}$$

For the nongeneric rational numbers like the Golden mean there will be one less power of $\ln N$, hence $\Delta A_N^{\epsilon}(\text{G.M.})\sim N^2\ln N$. In any case, the $\nu=1$ point is unstable to any small deviation in ν since the limits $N\to\infty$ and $\epsilon\to 0$ do not commute.

In the following section the sums A_N^{ν} will be investigated for various values of ν.

III. THE SUMS A_N^{ν} FOR VARIOUS VALUES OF $\nu\neq 1$

In this section the sums

$$A_N^{\nu}=\sum_{n=0}^{N-1}\ln 2|\cos\pi\phi_n^{\nu}| \tag{3.1}$$

with

$$\phi_n^\nu = \alpha n^\nu \tag{3.2}$$

will be investigated.

With the help of (2.4), which is independent of the form of ϕ_n^ν, it can be written in the form

$$A_N^\nu = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} S_N^\nu(\alpha m) , \tag{3.3}$$

where

$$S_N^\nu(\alpha) = \sum_{n=0}^{N-1} \cos 2\pi \alpha n^\nu . \tag{3.4}$$

The sum A_N^ν is dominated by the $S_N^\nu(\alpha m)$ with the lowest values of m. The general form of these sums does not depend strongly on m. Sums of the form (3.4) were encountered in earlier work.[8] From the experience of these investigations it is known that for $\nu \neq 1$ there are the following distinct regimes:[8-10,13-16] $0<\nu<1$, $1<\nu<2$, $\nu>2$, and $\nu=2$. We will investigate the sums in these regimes separately. It turns out that the regime $0<\nu<1$ should be divided into two subregimes and the behavior of the sum (3.3) for $0<\nu<\frac{1}{2}$ differs from the one found for $\frac{1}{2}<\nu<1$.

A. The regime $\nu<1$

For $\nu<1$ the terms in the sums $S_N^\nu(\alpha)$ of (3.4) vary slowly with n for sufficiently large n. Since each term in the sum is of order unity the sum $S_N^\nu(m\alpha)$ can be approximated by an integral in the regime where the difference between two consecutive terms is less than unity. This requires

$$2\pi \alpha m \nu \ll n^{1-\nu} . \tag{3.5}$$

The terms in the sum (3.3) can be considered as points (n,m) on a two-dimensional lattice, with the restrictions $0<n<N-1$ and $1<m<\infty$. We define a line in the (n,m) plane

$$2\pi \alpha m \nu = c_1 n^{1-\nu} \tag{3.6}$$

so that on one side of this line (3.5) is satisfied and the sums (3.4) can be replaced by integrals, while on the other side this is impossible and a different approximation is required. c_1 in (3.6) is of order unity. The sum of (3.3) can be divided into two parts so that in each part all terms are on one side of (3.6), namely

$$A_N^\nu = -B_n^\nu - D_N^\nu , \tag{3.7}$$

where

$$B_N^\nu = \sum_{m=1}^{M} \frac{(-1)^m}{m} \sum_{n=n^*(m)}^{N-1} \cos 2\pi \alpha m n^\nu \tag{3.8}$$

and

$$D_N^\nu = \sum_{n=0}^{N-1} \sum_{m=m^*(n)}^{\infty} \frac{(-1)^m}{m} \cos 2\pi \alpha m n^\nu . \tag{3.9}$$

The limits of the summations are

$$m^*(m) = \mathrm{Int}[c_1 n^{1-\nu}/2\pi\alpha\nu] + 1 , \tag{3.10}$$

$$n^*(m) = \mathrm{Int}[(2\pi\alpha m \nu/c_1)^{1/(1-\nu)}] + 1 , \tag{3.11}$$

and

$$M = \mathrm{Int}[c_1 (N-1)^{1-\nu}/2\pi\alpha\nu] , \tag{3.12}$$

where $\mathrm{Int}[x]$ denotes the integer part of x, namely the largest interger that is smaller than x.

The sum over n in (3.8) can be approximated by an integral, namely

$$\Delta S_{n^*,N}(\alpha m) = S_N^\nu(\alpha m) - S_{n^*}^\nu(\alpha m) = \sum_{n=n^*(m)}^{N-1} \cos 2\pi \alpha m n^\nu \approx \int_{n^*}^{N-1} dn \cos 2\pi m \alpha n^\nu . \tag{3.13}$$

Introducing the variable $x=n^\nu$, one obtains

$$\Delta S_{n^*,N}^\nu(\alpha m) \approx \frac{1}{\nu} \int_{x^*}^{X} dx\, x^{(1/\nu)-1} \cos 2\pi \alpha m x , \tag{3.14}$$

where

$$X = (N-1)^\nu \tag{3.15}$$

and

$$x^* = n^{*\nu} . \tag{3.16}$$

Integrating by parts one finds for large N

$$\begin{aligned}\Delta S_{n^*,N}^\nu(\alpha m) &\sim \frac{X^{(1/\nu)-1}}{2\pi\alpha m \nu} \sin 2\pi\alpha m X - \frac{x^{*(1/\nu)-1}}{2\pi\alpha m\nu} \sin 2\pi \alpha m x^* + R_{N,m} \\ &= \frac{N^{1-\nu}}{2\pi\alpha m\nu} \sin 2\pi\alpha m N^\nu - \frac{n^{*1-\nu}}{2\pi\alpha m \nu} \sin 2\pi \alpha m n^{*\nu} + R_{N,m} ,\end{aligned} \tag{3.17}$$

where the remainder term is

$$R_{N,m} = \mathcal{O}\left[\frac{N^{1-2\nu}}{m^2}\right] + \mathcal{O}\left[\frac{n^{*(1-2\nu)}}{m^2}\right] . \tag{3.18}$$

The sums $\Delta S_{n^*,N}(\alpha m)$ should be substituted in (3.8) in order to calculate B_N^ν. We will show first that the contribution of the second term in (3.17) is negligible for large N. Its contribution to B_N^ν is

$$\begin{aligned}R_B &= -\sum_{m=1}^{M} \frac{(-1)^m}{m^2} \frac{(n^*)^{1-\nu}}{2\pi\alpha\nu} \sin(2\pi\alpha m n^{*\nu}) \\ &\approx -\frac{1}{c_1} \sum_{m=1}^{M} \frac{(-1)^m}{m} \sin(2\pi\alpha m n^{*\nu}) ,\end{aligned} \tag{3.19}$$

where (3.11) was used.

This sum is clearly bounded by a term that grows as $\ln M$, which is proportional to $\ln N$ for large N due to (3.12). The estimate of R_B can be improved since for

$0<\nu<1$, $1<1/(1-\nu)<\infty$ and the phase of the sine in (3.19) varies rapidly with m [see (3.11)]. The contribution of a group of terms in the vicinity of some m is of the order of $1/m^2$; consequently the sum is convergent in the limit $M\to\infty$ and therefore it behaves as a constant for large N. The contribution of the remainder term $R_{N,m}$ of (3.18) is of the order of $N^{1-2\nu}$ since the sum $\sum_{m=1}^{\infty}[(-1)^m/m^2]$ is convergent. Therefore, substitution of (3.17) in (3.8) yields

$$B_N^\nu=\frac{N^{1-\nu}}{2\pi\alpha\nu}\sum_{m=1}^{M}\frac{(-1)^m}{m^2}\sin 2\pi\alpha m N^\nu+R_N\ , \qquad (3.20)$$

where R_N is a remainder term of the order $N^{1-2\nu}$. The sum over M is convergent (in the limit $M\to\infty$), therefore it is dominated by the small m terms. The remainder term R_N is negligible for sufficiently large N.

We turn now to analyze the behavior of D_N^ν of (3.9). The terms in the sum vary rapidly with m. If one groups several terms in the vicinity of m, their sum is of the order $1/m^2$. Consequently the sum over m in (3.9) is approximately

$$\tilde{S}_n^\nu=\sum_{m=m^*(n)}^{\infty}\frac{(-1)^m}{m}\cos 2\pi\alpha m n^\nu\sim\frac{\eta_1(n)}{m^*(n)}\ , \qquad (3.21)$$

where $\eta_1(n)$ is a function of n with a bounded amplitude. It is determined mainly by the terms in the vicinity of m^* and the function Int$[x]$ of (3.10). In the derivation we used the fact that up to a constant $\sum_{m=m^*}^{\infty}(1/m^2)\sim 1/m^*$. With the help of (3.10) one finds that for large N the sum (3.9) is approximately

$$D_N^\nu\sim\frac{2\pi\alpha\nu}{c_1}\sum_{n=0}^{N}\frac{\eta_1(n)}{n^{1-\nu}}\sim\frac{2\pi\alpha\nu}{c_1}\int_{\bar{n}}^{N}dn\frac{\eta_1(n)}{n^{1-\nu}}\ , \qquad (3.22)$$

where $\bar{n}$ is some lower cutoff that is not important since the integral diverges in the limit $N\to\infty$ and is therefore dominated by the vicinity of N. The function $\eta_1(x)$ is determined mainly by the terms in the vicinity of m^* in the sum (3.21), where m^* is determined by (3.10). One can see from direct examination of these that the values of n where function $\eta_1(n)$ changes sign become sparser as n increases. This function is expected to exhibit randomlike behaviors on very long scales due to the nature of the functions Int$[x]$ and $\cos x$ and the slow variation of their argument with n. Because of the slowness of the variation of $\eta_1(n)$ for large N, the asymptotic behavior of the integral (3.22) is

$$D_N^\nu\sim\frac{2\pi\alpha\nu}{c_1}\eta_1(N)\int_{\bar{n}_1}^{N}\frac{dn}{n^{1-\nu}}\sim\eta(N)N^\nu\ , \qquad (3.23)$$

where $|\eta(N)|\approx 2\pi\alpha/c_1$ and $\bar{n}_1$ is some lower cutoff that has no significance.

The asymptotic form of the sum (3.3) is

$$A_N^\nu\sim-\frac{N^{1-\nu}}{2\pi\alpha\nu}\sum_{m=1}^{M}\frac{(-1)^m}{m^2}\sin 2\pi\alpha m N^\nu-\eta(N)N^\nu\ , \qquad (3.24)$$

as one finds from substitution of (3.20) and (3.23) into

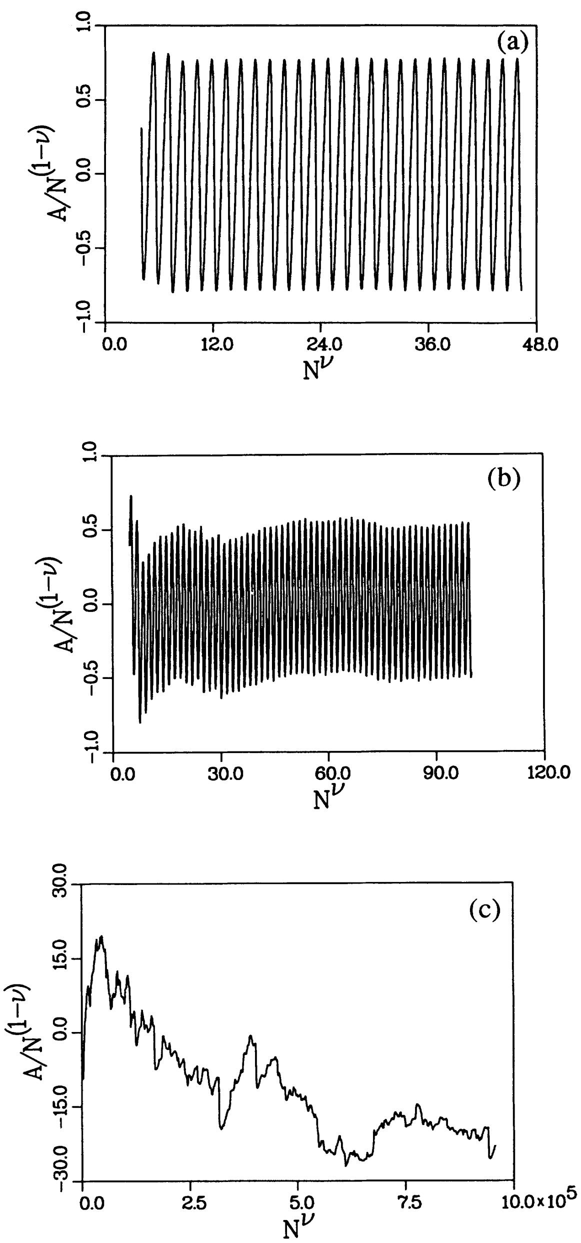

FIG. 2. The sums A_N^ν as a function of $x=N^\nu$ for $\alpha=\frac{1}{2}(\sqrt{5}-1)$ and (a) $\nu=1/3$, (b) $\nu=1/2$, and (c) $\nu=0.8$.

(3.7). The first term is an oscillating function of $X=N^\nu$ with the period $1/\alpha$, while the second is a slowly varying "randomlike" term. The ratio of their magnitudes is of the order

$$\lim_{N\to\infty}N^{1-2\nu}=\begin{cases}\infty & \text{for } \nu<\frac{1}{2}\\ 0 & \text{for } \nu>\frac{1}{2}\ .\end{cases} \qquad (3.25)$$

Therefore the first term dominates for $\nu<1/2$ while the second dominates for $\nu>\frac{1}{2}$. In order to investigate how well the approximation (3.24) works we plotted, in Fig. 2, the A_N^ν of (3.1) for several values of ν. Actually, we plotted

$$F_\nu(x)=\frac{A_N^\nu}{N^{1-\nu}} \tag{3.26}$$

as a function of $X=N^\nu$. In Fig. 3 the Fourier transform $\hat{F}_\nu(Q)$ of $F_\nu(x)$ is presented. It is clear that the period is indeed $1/\alpha$. For $\nu \le \frac{1}{2}$ the sum is purely periodic in X and the weight of the high harmonics, corresponding to large values of m in the sum (3.24), is small. For $\nu > \frac{1}{2}$ the erratic behavior resulting from the nature of the function $\eta(N)$ is obvious. Since it is slow it results in a peak at $Q=0$ of the Fourier transform. The oscillating behavior resulting from the first term is observed as well. For $\nu=1$ the periodic structure that is found for $\nu<1$ disappears, as is clear from Fig. 4.

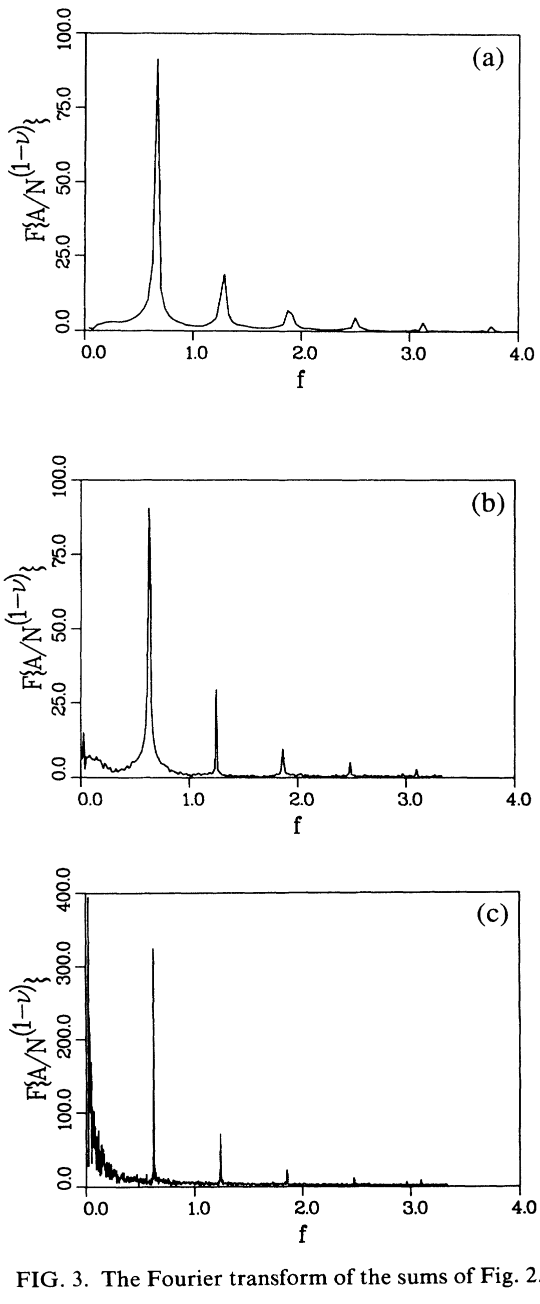

FIG. 3. The Fourier transform of the sums of Fig. 2.

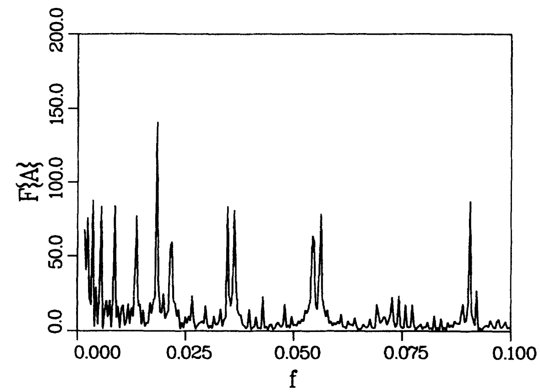

FIG. 4. The Fourier transform of the sequence of Fig. 1(a).

B. The regime $1<\nu<2$

In this regime the sums are dominated by terms in the vicinity of n that satisfy the condition[8,9]

$$2\pi\alpha\nu n^{\nu-1} \approx 2l\pi\ , \tag{3.27}$$

where l is an arbitrary integer. For large n the distance between two consecutive values of n that satisfy (3.27) is

$$\Delta n = n^{2-\nu}/\alpha\nu(\nu-1)\ . \tag{3.28}$$

Note that $\Delta n \to \infty$ in the limit $n \to \infty$. For large N the typical size of $S_N^\nu(\alpha)$ in this regime is $\sqrt{N}$, as if ϕ_n^ν would be random as demonstrated in Fig. 5. Note that the number of terms in the sums of this figure is of the order of 10^4. Although the typical size is similar to the one found for a random phase, the variations differ strongly from those found for random phases. The variations are large and take place for values of n that satisfy (3.27), and the resulting separation between them is (3.28). The growth takes place in extremely narrow regions and is most pronounced for small m where the separation between these regions is the largest. [For $m \neq 1$, α in (3.28) should be replaced by $m\alpha$.] The sum (3.3) is dominated

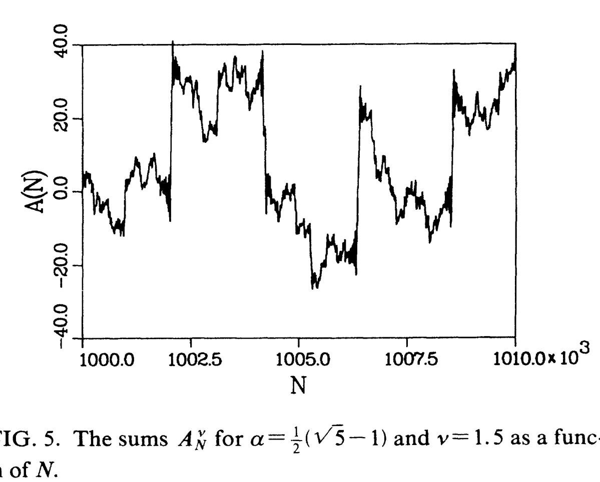

FIG. 5. The sums A_N^ν for $\alpha=\frac{1}{2}(\sqrt{5}-1)$ and $\nu=1.5$ as a function of N.

by terms with small m, therefore the separation between regions of the largest variation is given approximately by (3.28). This is demonstrated in Fig. 5, where the variation in a relatively small region is plotted. The prediction of (3.28) for separation between the regions of large variation in Fig. 5 is $\Delta n = 2.15 \times 10^3$ in good agreement with the numerical results.

C. The regime $\nu > 2$

In this regime the sums $S_N^\nu(\alpha)$ are expected to behave as if the phases ϕ_n^ν were random.[8,13–16] Moreover, in this regime $(\phi_n^\nu)\text{mod}1$ is found to pass the χ^2 test for pseudorandomness.[16] It is expected that A_N^ν of (3.1) will behave as if the phases are random. This is demonstrated in Fig. 6, where the behavior of A_N^ν with $\nu=3$ is depicted. It is consistent with a $\sqrt{N}$ increase. This plot could not be distinguished from that obtained with "random" phases. (We recall, however, that the random number generator used to generate the random phases is also based on a quasiperiodic sequence.)

D. $\nu = 2$

This is a bordering case between regimes B and C. For $\nu=2$, $(\phi_n^\nu)\text{mod}1$ fails the χ^2 test for pseudorandomness, but not remarkably, and as tests of pseudorandomness are concerned it is quite close to a random sequence.[16] In particular S_N^ν exhibits fluctuations consistent with $\sqrt{N}$ but there are no regions of strong variations of the form

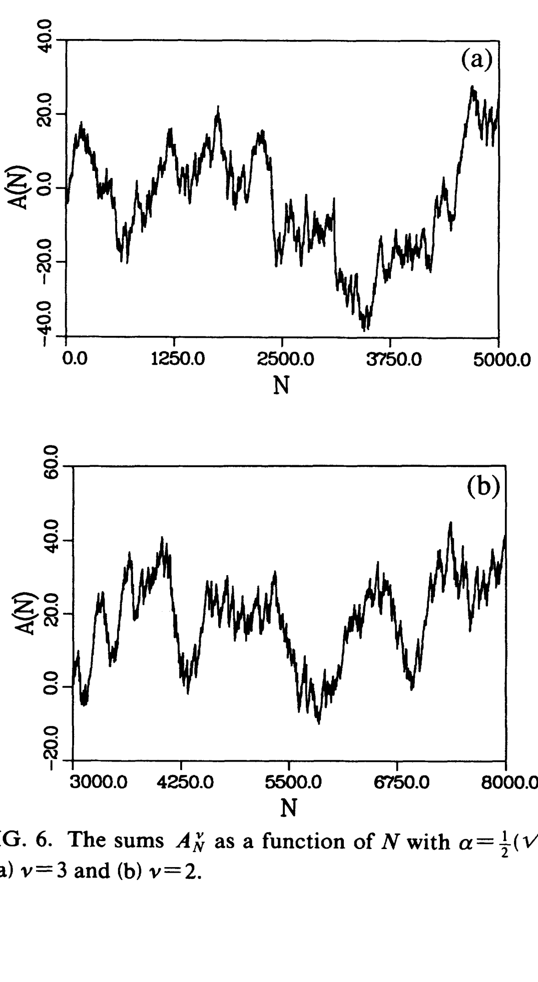

FIG. 6. The sums A_N^ν as a function of N with $\alpha = \frac{1}{2}(\sqrt{5}-1)$ for (a) $\nu=3$ and (b) $\nu=2$.

that were found for $1 < \nu < 2$. The sum A_N^ν is depicted in Fig. 6(b) (note its difference from that with $\nu = 1.5$ plotted in Fig. 5).

IV. CONCLUSIONS

To summarize, our findings on $A_\nu(t) = \ln|J_\nu(t)|$ are as follows:

(i) For $\nu=1$: The behavior depends on the number theoretic properties of the ratio $\alpha = \phi/\phi_0$. For algebraic irrational α, with bounded continued-fraction expansion, $|J^\nu(t)|$ increases algebraically as t^β. For a generic irrational number, whose continued fraction expansion consists of an unbound series of integers $|J^\nu(t)|$ increases faster than algebraically with analytical prediction of an increase such as $t^{\beta \ln t}$. It remains to understand the dependence of the exponents $\beta(\alpha)$ on α, which we leave for future studies. The same holds for the discontinuous jumps discussed at the end of Sec. II.

(ii) For $\nu \neq 1$ different regimes of behaviors were identified:

(1) For $0 < \nu < \frac{1}{2}$: The behavior of A_N^ν is essentially periodic in the variable $X = t^\nu$ when a finite number of harmonics dominates the whole sum.

(2) For $\frac{1}{2} < \nu < 1$: In this regime there appears a large-scale random component superimposed on the periodic behavior as exhibited in the Fourier spectrum by a noise band near the origin.

(3) For $1 < \nu < 2$: The behavior of the series is dominated by the special points where $\alpha \nu n^{\nu-1}$ is an approximate integer.

(4) For $\nu > 2$ the behavior of $J^\nu(t)$ is indistinguishable from that of a product of cosines with a random-phase uniformly distributed in $[0, 2\pi]$.

As for possible experimental realizations, for $\nu=1$ the best candidates are artificially fabricated superconducting (or Josephson junctions) networks.[21–23] The existence of surface states (localized solutions of the linearized Ginzburg-Landau equations) was already discussed in the context of numerical simulations of these networks.[24] Measurements of the exponential decay of the supercurrent transmission as a function of the magnetic field and/or the sample width may exhibit some of the properties analyzed here.

It will be interesting to go beyond the "directed path" approximation and see how the inclusion of returning loops modifies this behavior. Preliminary results indicate that the important features will be conserved as long as the energy is in the forbidden zone (and, hence, returning loops are finite).

Finally, crossing the "mobility edge" between localized and extended states would allow us to make the connection between our results and the previous studies of the Hofstadter bulk eigenstates.

ACKNOWLEDGMENTS

It is our pleasure to thank M. Kardar and E. Medina for useful discussions. This work was supported in part by the U.S.-Israel Binational Science Foundation (BSF).

Y.S. acknowledges the support of the Compton Foundation during his visits to the Technion and the hospitality of Ora Entin-Wohlman during his visit to Tel Aviv University. Support from the Minnesota University Supercomputer Institute is also acknowledged by X.R.W.

APPENDIX A: PARITY STATISTICS OF THE APPROXIMANTS' DENOMINATORS q_k

In this appendix we show that for a generic irrational number the q_k's are twice as likely to be odd than even in the $k \rightarrow \infty$ limit. We recall that the recursion relations for q_k and p_k in terms of the continued-fraction integers a_k are

$$q_k = a_k q_{k-1} + q_{k-2} \quad \text{with} \quad q_0 = 1, q_1 = a_1 \ , \tag{A1}$$

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{with} \quad p_0 = q_0, p_1 = a_0 a_1 + 1 \ . \tag{A2}$$

We begin by considering the parities $\pi_k = \pi(q_k)$, $\pi_k = 0(1)$ if q_k is even (odd).

Let us define the two component vectors $\mathbf{v}^{(k)} = \binom{\pi_{k-2}}{\pi_{k-2}}$ over the Z_2 field $\{0,1\}$. The whole vector field consists of four elements:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \text{and } \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The recursion relation (A1) for q_k defines a linear map between $v^{(k)}$ and $v^{(k+1)}$ which only depends on $\pi(a_k)$. If we denote by $\omega_i^{(k)}$ the probability for $v^{(k)} = \mathbf{e}_i$, $i = 1,2,3,4$, the probability distribution for $\pi(a_k)$ will thus determine a relation between $\omega_i^{(k+1)}$ and $\omega_i^{(k)}$.

For a generic irrational number, the probability for a_k to be equal to m is $1/m(m+1)$. Hence the probability for a_k to be odd is ln2. The matrix relating $\omega_i^{(k)}$ for consecutive k's is found to be

$$\begin{bmatrix} 1-\ln 2 & 0 & \ln 2 & 0 \\ \ln 2 & 0 & 1-\ln 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{A3}$$

Obviously $\omega_4 = 0$ because $q_0 = 1$.

To find the stationary distribution for $\omega_i^{(\infty)}$ $i = 1,2,3$ we note that (A3) defines a Markov process and the eigenvector with the largest eigenvalue (by virtue of the Perron-Frobeniüs theorem) has $\omega_1 = \omega_2 = \omega_3$ which (since $\omega_4 = 0$) is $\boldsymbol{\omega}^{(\infty)} = (1/3, 1/3, 1/3, 0)$. To assume that this distribution is reached we still need to assure that $\boldsymbol{\omega}^{(\text{in})}$ (the initial $\boldsymbol{\omega}$) is not orthogonal to $\boldsymbol{\omega}^{(\infty)}$. Indeed $\mathbf{v}^{(\text{in})} = \binom{1}{\pi(a_1)}$ and therefore $\boldsymbol{\omega}^{(\text{in})} = (1/2, 0, 1/2, 0)$ and $\boldsymbol{\omega}^{(\text{in})} \cdot \boldsymbol{\omega}^{(\infty)} = 1/3 > 0$.

Counting the numbers of times 1 and 0 appears in $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, we conclude that if they are uniformly distributed as $k \rightarrow \infty$ the probability for q_k to be odd is twice as large as that to be even.

[1] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[2] G. H. Wannier, Phys. Status Solidi B **88**, 757 (1978).

[3] M. Ya. Azbel, Zh. Eksp. Teor. Fiz. **46**, 929 (1964) [Sov. Phys. JETP **19**, 634 (1964)].

[4] For a recent review, see Y. Shapir and X. R. Wang, Mod. Phys. Lett. B **4**, 1301 (1990).

[5] E. Medina, M. Kardar, Y. Shapir, and X. R. Wang, Phys. Rev. Lett. **84**, 1916 (1990).

[6] Y. Shapir, X. R. Wang, E. Medina, and M. Kardar, in *Hopping and Related Phenomena,* edited by H. Fritzche and M. Pollak (World Scientific, Singapore, 1990) p. 20.

[7] V. L. Nguyen, B. Z. Spivak, and B. I. Shklovskii, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 35 (1985) [JETP Lett. **41**, 42 (1985)].

[8] M. Griniasty and S. Fishman, Phys. Rev. Lett. **60**, 1334 (1988).

[9] D. J. Thouless, Phys. Rev. Lett. **61**, 2144 (1988).

[10] S. Das-Sarma, S. He, and X. C. Xie, Phys. Rev. Lett. **61**, 2144 (1988); Phys. Rev. B **41**, 5544 (1990).

[11] S. Das-Sarma, A. Kobayashi, and R. E. Prange, Phys. Rev. Lett. **56**, 1280 (1986); Phys. Rev. B **34**, 5309 (1986).

[12] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982); D. R. Grempel, R. E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984).

[13] M. V. Berry and J. Goldberg, Nonlinearity **1**, 1 (1988).

[14] E. A. Coutsias and D. Kazarinoff, Physica **26D**, 295 (1987).

[15] G. H. Hardy and J. E. Littlewood, Acta Math. Acad. Sci. Hung. **37**, 25 (1914).

[16] N. Brenner and S. Fishman, Nonlinearity **5**, 211 (1992).

[17] H. J. Stöckmann (private communication).

[18] M. V. Berry, Physica **10D**, 369 (1984).

[19] A. Ya. Khinchin, *Continued Fractions* (University of Chicago Press, Chicago, 1964).

[20] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* 5th ed. (Oxford University Press, New York, 1979).

[21] P. G. de Gennes, C. R. Acad. Ser. B **292**, 9279 (1981); S. Alexander, Phys. Rev. B **27**, 1541 (1983).

[22] B. Pannetier, J. Chaussy, and R. Rammal, Jpn. J. Appl. Phys. Suppl. **26**, 1994 (1987).

[23] J. N. Gordon, A. M. Goldman, J. Maps, D. Costello, R. Tiberio, and B. Whitehead, Phys. Rev. Lett. **56**, 2280 (1986).

[24] J. Simonin, D. Rodriguez, and A. Lopez, Phys. Rev. Lett. **49**, 944 (1982); H. J. Fink, A. Lopez, and R. Maynard, Phys. Rev. B **26**, 5237 (1982).