Theory of high-frequency linear response of isotropic type-II superconductors in the mixed state

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A self-consistent approach to vortex dynamics, including the effects of nonlocal vortex interaction, pinning, and creep, is further clarified and unified by its restatement in terms of an initial-boundary value problem. We derive and solve a single vector partial differential equation describing the linear response of a type-II superconductor in the mixed state at frequencies well below the gap frequency. The solution of this equation, presented here for several sample geometries, provides the phenomenological superconductor dispersion relation, accompanying complex penetration depths, and complex response functions. The theory is expected to have applicability to a wide range of experiments involving vortex dynamics.

INTRODUCTION

In this paper we present a general theoretical description of the high-frequency linear response of isotropic type-II superconductors in the mixed state. The approach used here unifies several of the elements of our theory of rf vortex dynamics[1–6] wherein the alteration of electromagnetic fields produced by vortex motion is included self-consistently with the rest of the dynamics. By using a complex-valued dynamic vortex mobility, our theory is able simultaneously to take into account effects of vortex pinning, flux flow, and flux creep. The phenomenological formulation of the theory allows for the ready refinement of portions of it. For instance, the results of a microscopic treatment of pinning could be incorporated in the vortex-lattice dynamics.

The present approach leads to the formulation of an initial-boundary-value problem for one of the total (or net) rf fields or densities **a**, **b**, **j**, **E**, or **u**. Here **a** is the vector potential, **b** the magnetic field, **j** the current density, **E** the electric field, and **u** the vortex-displacement field. Once one of these quantities has been found, the others follow from various electrodynamic relations such as Maxwell's equations. The boundary-value problem to be solved in the superconductor geometry is usually of combined elliptic and parabolic type. In general, a coupled elliptic boundary-value problem outside the superconductor also needs to be solved for any rf fields. In the following section, we derive a single vector partial differential equation which governs the total type-II superconductor response in the linear regime, showing the generalization of many previous treatments. Example geometries and solutions of this equation are then considered.

We are concerned with calculating electrodynamic-response functions for a type-II superconductor subjected to a combination of large dc field and small rf field. When vortex pinning is significant, we assume that the vortex displacements are small in comparison with the intervortex spacing. (The pinning forces produce a net linear restoring force on a typical vortex, as in a parabolic pinning-potential well.) Since the amplitude of the rf field is much less than that of the dc field, linear-response theory can be applied. The calculated response functions will then be independent of the rf-field amplitude.

On the other hand, if vortex pinning is significant and the displacements large, the critical state is fully built up near the sample's surface. Theory then requires a model of the critical state, such as the Bean model.[7] The response functions will depend strongly on the rf-field amplitude, reflecting hysteretic effects.

Our theory is expected to be applicable to a wide range of phenomena involving vortex dynamics. Corresponding experiments include surface impedance, rf permeability, vibrating reed, torque, and (torsional) oscillators.[8–10] Measurements of ultrasonic attenuation and dispersion also involve vortex dynamics.[11] In the first class of experiments, the interaction of vortices with either a perturbing applied field or current occurs, while in the second class vortices interact with the acoustic waves of the crystal lattice. The present form of our theory should be immediately applicable to experiments of both classes when small vortex amplitudes are involved. With appropriate extensions, such as use of a critical-state model, modeling of hysteretic effects should also be possible in the theory.

The present theory is applicable over a wide range of frequencies, including the radio- and microwave-frequency range (below superconducting-gap frequencies), subject to certain provisions on the effective periodic pinning-potential height U, as we now discuss. We assume a single height $U(B,T)$ currently lacking dependence on angular frequency ω and current density **J**. It is likely that a more detailed treatment should include some type of statistical distribution of pinning heights which has these additional parameter dependences. At higher frequencies it is likely that only the pinning wells with lower heights are involved in the vortex dynamics. The pinning-potential height is an average or effective one not particularly tied to any one microscopic creep model unless the averaging method is spelled out. Note that in the case of dislocation-mediated creep, the relevant activation barrier is not the pinning potential itself, but its variance.[12]

At this stage of the theory as regards pinning and flux creep, we are content to in effect make the approximation that the average over pinning-potential barriers of a given linear-response function is close to that same linear-response function evaluated with an effective pinning-potential barrier. Symbolically, using the surface impedance as an example, this is the statement $Z_s(\langle U\rangle)\approx\langle Z_s(U)\rangle$, where we suppress all the other parameter dependences of Z_s.

We employ a Brownian-motion model for flux creep in which the vortices are assumed to be uncorrelated while diffusing. This is a limitation in the current theory, which would require modification in order to apply it in a detailed study of phase transitions involving vortices.

Through the wide variety of experiments mentioned above, we anticipate that our theory and its extensions can provide detailed information on the anisotropic electrodynamic response of type-II superconductors. Given material constants, our theory does not contain adjustable parameters. Conversely, from experimental data one could apply the theory to obtain, e.g., an effective linear pinning constant or effective pinning-potential height. As an example, the latter approach has recently been applied[13] to obtain an effective vortex-activation energy.

For the very small driving fields considered here, the vortex displacements have very small amplitude, usually less than 1 Å.[14] Since the range of the pinning potential is typically of the size of the coherence length,[12] the oscillatory part of the vortex motion stays very close to the bottom of the potential well. As mentioned above, we take the form of the pinning potential to be independent of the magnitude of the driving-current density.

Our approach self-consistently includes vortex interactions by accounting for the coupling of current density and vortex displacements. The results for the response functions are given in terms of one or more complex penetration depths which characterize the spatial variation of the electrodynamic fields and densities. In this paper a single governing vector partial differential equation (PDE) is derived from a generalized diffusion London equation with vortex-density term and a vortex equation of motion. (The latter includes a random or Langevin force[15] when flux creep is modeled.) The PDE represents a linearization about zero vortex velocity and constant vortex density. As before, nonlocal vortex interactions are included in our theory. Note that in Refs. 2, 4, 5, and 16 numerical results for linear-response functions have been given over a broad range of static field magnitude and of frequency, for temperatures from absolute zero through the transition temperature, and for different angles of both the applied static field with respect to the superconductor surface and microwave field. The emphasis in this paper is on the unified derivation of analytical results for the complex linear-response functions from the solution of certain boundary-value problems.

In obtaining the complex penetration depth(s), we are in effect finding the dispersion relation for the superconductor. We show that the rf surface impedance $Z_s=R_s-iX_s$, defined by

$$Z_s=\frac{E_t}{h_t}\ , \tag{1}$$

is given by the complex self-consistently determined phase velocity of radiation in the superconductor. Here E_t and h_t are, respectively, the tangential components of the rf electric and magnetic fields at the superconductor surface, and they vary with time as $e^{-i\omega t}$. After presenting the coupled boundary-value problem, we next illustrate its solution for several sample geometries. We then discuss more involved problems where large demagnetization effects arise, emphasizing the vector nature of the governing PDE. We compare our approach of determining complex penetration depths with other models, including the theory of thermally assisted flux flow, and note various complex diffusion constants which appear in these theories.

THEORY OF LINEAR RESPONSE

We first recall some background on our theory[1–5] necessary for the derivation of the generalized flux-diffusion partial differential equation. The electrodynamics of the superconductor are described by Maxwell's curl equations, the two-fluid equation $\mathbf{J}=\mathbf{J}_n+\mathbf{J}_s$, the constitutive relation $\mathbf{J}_n=\sigma_{\rm NF}\mathbf{E}$, and two equations which describe the source of the total-current density $\mathbf{J}$. [Here $\mathbf{J}_n$ is the normal-current density, $\mathbf{J}_s$ is the supercurrent density, and $\sigma_{\rm NF}(B,T)=1/\rho_{\rm NF}(B,T)$ is the local electrical conductivity of the normal fluid.] The supercurrent-source equation is

$$\nabla\times\mathbf{J}_s=-\frac{1}{\mu_0\lambda^2}(\mathbf{B}-\phi_0 n\hat{\mathbf{B}}_0)\ , \tag{2}$$

where $n(\mathbf{x},t)$ is a local area density of vortices and ϕ_0 is the flux quantum. In Eq. (2) the local direction of the vortices is given by the unit vector $\hat{\mathbf{B}}_0=\mathbf{B}_0/B_0$ and $\mathbf{B}_0$ is the internal magnetic induction generated by the vortex array, which is assumed to be uniform. For more generality an equilibrium magnetization curve or equivalent relation may be included for the magnetic behavior.

Equation (2) accounts for nonlocal vortex interaction by way of the continuum density $n(\mathbf{x},t)$. We recall that the nonlocality of the elastic response of the vortex lattice means that the elastic moduli depend on the length scale of the elastic strain.[17] In this paper we restrict attention to isotropic, high-κ superconductors for which $\lambda(B,T)$ in Eq. (2) is independent of the direction of current flow. In this situation the wave-vector-dependent elastic moduli[17,18] satisfy the approximate relations $c_{11}(k)\approx c_{44}(k)$ between the compressional and tilt moduli and $c_{66}\approx 0$ for the shear modulus. These relations hold as long as the field is not too high, $B\lesssim 0.2B_{c2}$, and k stays away from the boundary $k_{\rm BZ}$ of the first Brillouin zone.[17,18] Hence we consider nonlocal effects, but do not include the additional geometric dispersion coming from the vortex lattice itself when $k\approx k_{\rm BZ}$. For greater generality the vortex equation of motion below can be modified to include higher-order elastic effects.

The vortex equation of motion we typically use includes a random (or Langevin) force[15] $\mathbf{F}(\mathbf{x},t)$ per unit length on the right-hand side to represent thermally generated forces:

$$\eta\dot{\mathbf{u}}(\mathbf{x},t)+\kappa_p\mathbf{u}(\mathbf{x},t)=\mathbf{J}(\mathbf{x},t)\times\phi_0\hat{\mathbf{B}}_0+\mathbf{F}(\mathbf{x},t)\ . \tag{3}$$

The Langevin force in Eq. (3) is assumed to be Gaussian white noise, with a δ-function autocorrelation function and zero time and ensemble average.[15,19,20] Equation (3) represents the force balance between a rf Lorentz force per unit length, $\mathbf{f}(\mathbf{x},t)=\mathbf{J}(\mathbf{x},t)\times\phi_0\hat{\mathbf{B}}_0$, a viscous drag force per unit length, and a restoring pinning force per unit length in addition to the effect of flux creep as modeled by the Langevin term. In Eq. (3), $\mathbf{u}$ is the vortex-displacement field, as measured from an equilibrium pinning site, η is the viscous-drag coefficient (e.g., Ref. 21), κ_p is the restoring-force constant (Labusch parameter[7]) of a pinning-potential well, and we ignore a possible vortex-mass term.[22] The periodic pinning potential is taken to be sinusoidally varying.[4,5]

When higher-order elastic effects are included, the left-hand side of Eq. (3) is modified. It is of note that for short tilt wavelengths the effective range of the vortex interaction is reduced to a length $\approx 1/k$. Subsequently, each vortex experiences interaction with fewer vortices; for large values of k, the expression for c_{44} should be modified as in Ref. 17.

The random force in Eq. (3) can be thought of as due to the interaction of the flux lines (in particular their core regions) with phonons of the crystal lattice, but need not depend on this particular mechanism. The random force per unit length makes the vortex equation of motion analogous to that of a particle undergoing Brownian motion in a periodic potential $U(u)$. [The pinning-force term in Eq. (3) is the linearization of $\partial U/\partial u$ per unit length of vortex about an equilibrium pinning site.]

The equations apart from the vortex equation of motion lead to a generalized diffusion London equation for $\mathbf{B}(\mathbf{x},t)$:

$$\nabla^2\mathbf{B}=\frac{\mu_0}{\rho_{\mathrm{NF}}}\dot{\mathbf{B}}+\frac{1}{\lambda^2}(\mathbf{B}-\phi_0 n\hat{\mathbf{B}}_0)\ . \tag{4}$$

Equations similar to Eq. (4) without the vortex-density term have appeared, e.g., in Refs. 12 and 23.

By making use of the complex dynamic mobility $\tilde{\mu}_v$,[2,4] the generalized diffusion London equation (4) may be combined with a vortex equation of motion such as Eq. (3) including flux creep. This allows all the equations governing the linear response to be combined in a single equation, which can then be solved subject to suitable initial and boundary conditions. Using the relation

$$\mathbf{v}(\mathbf{x},t)=\tilde{\mu}_v(\omega,B_0,T)\mathbf{f}(\mathbf{x},t) \tag{5}$$

between the vortex velocity and driving (Lorentz) force, we have

$$\mathbf{u}(\mathbf{x},t)=\frac{i}{\omega}\tilde{\mu}_v\mathbf{f}(\mathbf{x},t) \tag{6}$$

for the vortex displacement. For the vortex-density term $\mathbf{B}_v(\mathbf{x},t)\equiv n(\mathbf{x},t)\phi_0\hat{\mathbf{B}}_0(\mathbf{x})$ in Eq. (4), we write

$$\mathbf{B}_v=\mathbf{B}_0+\mathbf{b}_v=\mathbf{B}_0-\nabla\times(\mathbf{B}_0\times\mathbf{u})\ . \tag{7}$$

Equation (7) follows from integrating the vortex continuity equation with respect to time.[5] Upon using Eqs. (4), (6), and (7) and setting the normal-fluid diffusion coefficient $D_{\mathrm{NF}}(B,T)=\rho_{\mathrm{NF}}/\mu_0$, we have

$$\nabla^2\mathbf{B}=\frac{1}{D_{\mathrm{NF}}}\dot{\mathbf{B}}+\frac{1}{\lambda^2}\left\{\mathbf{B}-\mathbf{B}_0+\frac{i}{2}\mu_0\tilde{\delta}_{vc}^{\,2}\nabla\times[\hat{\mathbf{B}}_0\times(\mathbf{J}\times\hat{\mathbf{B}}_0)]\right\}\ . \tag{8}$$

This equation is of course first order in the vortex velocity and the deviation of the vortex density from its equilibrium value of $n_0\approx B_0/\phi_0$. We recall that the complex effective skin depth $\tilde{\delta}_{vc}=(2\tilde{\rho}_v/\mu_0\omega)^{1/2}$ simultaneously includes the effects of pinning, flux flow, and flux creep. Here the complex-valued effective resistivity $\tilde{\rho}_v(\omega,B_0,T)$ associated with the vortex motion is related to the dynamic mobility[2,4] $\tilde{\mu}_v$ by $\tilde{\rho}_v(\omega,B_0,T)=B_0\phi_0\tilde{\mu}_v(\omega,B_0,T)$.

We can use Ampere's law to obtain a single partial differential equation for the rf magnetic field $\mathbf{b}(\mathbf{x},t)=\mathbf{B}(\mathbf{x},t)-\mathbf{B}_0$,

$$\nabla^2\mathbf{b}=\frac{1}{D_{\mathrm{NF}}}\dot{\mathbf{b}}+\frac{1}{\lambda^2}\left\{\mathbf{b}+\frac{i}{2}\tilde{\delta}_{vc}^{\,2}\nabla\times(\hat{\mathbf{B}}_0\times[(\nabla\times\mathbf{b})\times\hat{\mathbf{B}}_0])\right\}\ , \tag{9}$$

which contains a fourfold vector cross product. Equation (9) is the single vector PDE that our self-consistent approach, as presented in Refs. 1–5, in effect solved. Solutions in the linear regime obtained by some authors[12,24] are solutions of approximations to this equation. By again making use of the dynamic vortex mobility, it is possible to derive generalized diffusion London equations similar to Eq. (9) for the other fields and densities. In the Appendix it is shown how this can be done for the electric field.

In Ref. 25 a linear PDE similar to Eq. (9) has been obtained in the pinning-dominated limit of vortex motion and a similar nonlinear PDE has been obtained in the flux-flow-dominated limit. However, corresponding analytic results in Ref. 25 were illustrated only with a simple diffusion equation. In our theory an extension to nonlinear response has been made in Ref. 26, where bilinear field nonlinearity has been retained in the vortex continuity equation. The possibility of nth-order harmonic generation has been discussed there.

An important special case of Eq. (9) is when $\hat{\mathbf{B}}_0$ is a constant vector. The cross products in Eq. (9) can be expanded using vector identities, using $\nabla\cdot\hat{\mathbf{B}}_0=0$, which always holds, and taking the gradients and curl of $\hat{\mathbf{B}}_0$ to be zero. We have

$$\nabla^2\mathbf{b}=\frac{1}{D_{\mathrm{NF}}}\dot{\mathbf{b}}+\frac{1}{\lambda^2}\left\{\mathbf{b}+\frac{i}{2}\tilde{\delta}_{vc}^{\,2}[(\hat{\mathbf{B}}_0\cdot\nabla)-\hat{\mathbf{B}}_0\nabla\cdot][\hat{\mathbf{B}}_0\times(\nabla\times\mathbf{b})]\right\}\ , \tag{10}$$

where the PDE for $\mathbf{b}$ is written in operator form. In this case we have reduced Eq. (9) to an equation containing two twofold vector cross products. Recall that the last factor on the right-hand side of Eq. (10) represents the Lorentz force on the vortices. This factor shows that the components of the rf-current density transverse and parallel to $\mathbf{B}_0$ separately enter the equation, with the former providing a nonzero contribution.

With $\hat{\mathbf{B}}_0$ constant, $\mathbf{b}$ is the single vector unknown in Eq. (10). In general, $\mathbf{b}$ can only be found by solution in both the superconductor and outside and matching components at the surface. In some special instances, usually involving a parallel applied rf field, the solution can be found from boundary data only on the superconductor surface. However, when demagnetization effects are significant, $\mathbf{b}$ is not a constant vector outside the superconductor. The form of $\mathbf{b}$ is usually found there by solving the vector Laplace equation together with boundary conditions at infinity. These topics are further discussed after we solve Eq. (10) for two sample geometries.

We remark that it is possible to include the displacement-current term in Ampere's law in the derivation of the generalized flux-diffusion equation. The result is to add the term $\mu_0\epsilon\partial^2\mathbf{b}/\partial t^2$ to Eq. (9) or (10), where ϵ is the dielectric constant of the superconductor. This term is potentially important for propagation-dominated problems[16] at frequencies approaching the gap frequency. This term is ignored in the determination of the complex dispersion relations to be derived in the following sections. Its presence would introduce a new characteristic length v/ω, where $v=1/\sqrt{\mu_0\epsilon}$ is the speed of light in the superconductor.

CYLINDER IN PARALLEL APPLIED STATIC MAGNETIC FIELD

We first illustrate the solution of Eq. (10) for the case of an infinitely long right-circular cylinder of radius a in a parallel applied static magnetic field. With this orientation of $\hat{\mathbf{B}}_0$, the more general Eq. (9) need not be solved. We take the axis of the cylinder to be along the $\hat{\mathbf{z}}$ direction and employ cylindrical coordinates (ρ,ϕ,z) centered on this axis. We assume that the rf field satisfies $\mathbf{b}(\rho=a,t)=\hat{\mathbf{z}}b_0e^{-i\omega t}$ at the surface. We require the general conditions that $\nabla\cdot\mathbf{b}=0$ and $\mathbf{J}\cdot\hat{\mathbf{n}}|_{\rho=a}=J_\rho|_{\rho=a}=0$ ($\hat{\mathbf{n}}$ is the unit outward normal vector). A suitable form for $\mathbf{b}$ is therefore $\mathbf{b}(\mathbf{x},t)=\mathbf{b}(\mathbf{x})e^{-i\omega t}$, where $\mathbf{b}(\mathbf{x})=\hat{\mathbf{z}}f(\rho)$.

Substituting this form of $\mathbf{b}(\rho)$ into Eq. (10) gives

$$\left[\lambda^2+\frac{i}{2}\tilde{\delta}^2_{vc}\right]\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]+\left[2i\frac{\lambda^2}{\delta^2_{\rm NF}}-1\right]f=0\ , \quad (11)$$

where the square of the normal-fluid skin depth is $\delta^2_{\rm NF}=2D_{\rm NF}/\omega$. We write the complex quantity

$$\tilde{\lambda}^2(\omega,B_0,T)=\frac{\lambda^2(B_0,T)+(i/2)\tilde{\delta}^2_{vc}(B_0,T,\omega)}{1-2i\lambda^2(B_0,T)/\delta^2_{\rm NF}(B_0,T,\omega)}\ . \quad (12)$$

Then Eq. (11) becomes

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{df}{d\rho}\right]-\frac{1}{\tilde{\lambda}^2}f=0\ . \quad (13)$$

We want the solution of Eq. (13) which is finite at the origin and satisfies the boundary condition $f(\rho=a)=b_0$. The form of Eq. (13) shows that the solution is a function of the variable $\rho/\tilde{\lambda}$. Therefore the interpretation of $\tilde{\lambda}$ as a complex-valued penetration depth is obtained as before. Note, however, that this fact can be seen directly from the differential equation for the rf magnetic field and prior to finding the explicit solution. This is expected to be an important point when other geometries and boundary conditions are considered. The solution of Eq. (13) can be written in terms of a modified Bessel function[4]

$$f(\rho)=b_0\frac{I_0(\rho/\tilde{\lambda})}{I_0(a/\tilde{\lambda})}\ . \quad (14)$$

As an example of finding the other fields and densities, the rf electric field is found from Eq. (14) and the use of Faraday's law. For a discussion of the dimensionless rf magnetic permeability $\tilde{\mu}$ in this geometry, see Ref. 4. When the radius of the cylinder is much larger than the modulus of the complex penetration depth, we can conveniently define the surface impedance from Eq. (1), in which case we have $Z_s=i\omega\mu_0\tilde{\lambda}$. When the condition $a\gg|\tilde{\lambda}|$ holds, all the rf fields and densities, which decay approximately exponentially with distance from the surface, become negligible around the cylinder axis and we have the simple relation $Z_s=ia\mu_0\omega\tilde{\mu}/2$.

The quantity $i\omega\tilde{\lambda}$ appearing in Z_s is the complex-valued phase velocity $v_{\rm ph}$ of the superconductor, as obtained from Eq. (10), appropriate for a problem where attenuation of the rf field dominates. Equation (12) is the corresponding dispersion relation. The two-fluid model is a special case of the above method. When the vortex density term $n=0$ in Eq. (4), simple plane-wave solutions immediately give the two-fluid model results for $\tilde{\lambda}$, $v_{\rm ph}$, and Z_s. The implicit relation $\tilde{\lambda}=\tilde{\lambda}(\omega)$ contains a great deal of information. In principle, this dispersion relation contains the group velocity $v_{\rm gr}$, from which a phenomenological density of states might be obtained.

The problem of an infinitely long type-II superconducting cylinder of elliptical cross section in a parallel static applied magnetic field can be solved by means analogous to this section. The explicit solution for the rf magnetic field involves a product of an "angular" and "radial" Mathieu function. The solution reduces to that of the cylinder of circular cross section discussed above as the ellipse's semimajor and semiminor axes become equal, but the details are not presented here.

PLANAR GEOMETRY WITH OBLIQUE APPLIED STATIC MAGNETIC FIELD

The problem considered here is that of planar geometry with an applied static magnetic field arbitrarily oriented with respect to both the superconductor surface and microwave field. This geometry is expected to have many applications. Our previous derivation by different means was presented succinctly in Ref. 3. Figure 1 shows the geometry wherein the rf field is chosen to lie along $\hat{\mathbf{z}}$ and x measures the distance into the superconductor. At the surface $x=0$, we have $\mathbf{J}\cdot\hat{\mathbf{x}}=0$ and $\mathbf{b}(\mathbf{x})=b_0\hat{\mathbf{z}}$. Requiring $\mathbf{b}$ to be divergenceless, a suitable form is $\mathbf{b}(\mathbf{x},t)=\mathbf{b}(\mathbf{x})e^{-i\omega t}$, where

$$\mathbf{b}(\mathbf{x})=\hat{\mathbf{y}}f_1(x)+\hat{\mathbf{z}}f_2(x)\ . \quad (15)$$

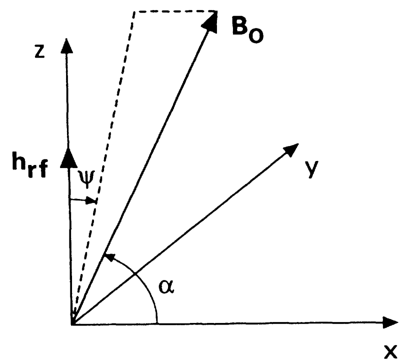

FIG. 1. Geometry of the semi-infinite superconductor with an oblique applied static magnetic field considered in the text. The superconductor occupies the region $x>0$, and the magnetic-flux density $\mathbf{B}_0$ and rf magnetic field $\mathbf{h}_{rf}=\hat{\mathbf{z}}h_{rf}$ are indicated. Also shown are the angle α that $\mathbf{B}_0$ makes with the x axis and the angle ψ that its projection on the yz plane makes with the z axis.

The functions f_1 and f_2 satisfy the four boundary conditions

$$f_1(0)=0,\quad f_2(0)=b_0\ , \tag{16a}$$

$$\lim_{x\to\infty} f_i(x)=0,\quad i=1,2\ . \tag{16b}$$

Substitution of the form (15) into Eq. (10) gives coupled second-order ordinary differential equations for the field components f_1 and f_2. We find

$$af_1''+\beta f_1+cf_2''=0\ , \tag{17}$$

$$df_2''+\beta f_2+cf_1''=0\ , \tag{18}$$

where a prime denotes differentiation with respect to x and the (constant) parameter-dependent coefficients are given below. Equation (17) results from the y component of Eq. (10), Eq. (18) from the z component, and the x component is void. Since these equations are linear, the boundary conditions (16) provide a unique solution. The four distinct coefficients appearing in Eqs. (17) and (18) are given in terms of the components of $\hat{\mathbf{B}}_0$ and characteristic lengths as

$$a\equiv\lambda^2+\frac{i}{2}\tilde{\delta}_{vc}^{2}(\hat{B}_{0x}^2+\hat{B}_{0y}^2),\quad \beta\equiv 2i\lambda^2\delta_{\mathrm{NF}}^{-2}-1\ , \tag{19a}$$

$$c\equiv\frac{i}{2}\tilde{\delta}_{vc}^{2}\hat{B}_{0y}\hat{B}_{0z},\quad d\equiv\lambda^2+\frac{i}{2}\tilde{\delta}_{vc}^{2}(\hat{B}_{0x}^2+\hat{B}_{0z}^2)\ . \tag{19b}$$

The components of $\hat{\mathbf{B}}_0$ are (see Fig. 1) $\hat{B}_{0x}=\cos\alpha$, $\hat{B}_{0y}=\sin\alpha\sin\psi$, and $\hat{B}_{0z}=\sin\alpha\cos\psi$. Here α corresponds to the polar angle in spherical coordinates with the x axis the polar axis and ψ corresponds to the azimuthal angle. The coefficient c defined in Eq. (19b) represents the coupling of Eqs. (17) and (18). The single complex penetration depth which appears in the solution for the rf field is proportional to $(a/\beta)^{1/2}$ or $(d/\beta)^{1/2}$ when $\hat{B}_{0y}$ or $\hat{B}_{0z}$ is zero, respectively.

Because of the semi-infinite geometry, it is convenient to employ the Laplace transform with respect to x in order to solve the system (17) and (18).[27] We use the notation

$$\hat{f}(s)=\int_0^\infty e^{-sx}f(x)dx\ , \tag{20}$$

where s is the transform variable. We know the magnetic field components at the surface from Eq. (16a), but we must retain the current-density components there, proportional to $f_i'(0)$, as unknowns which are determined at the end by application of the other two boundary conditions (16b).

The respective Laplace transforms of f_1 and f_2 are

$$\hat{f}_1(s)=\frac{(a'd-cc')s^2+a'\beta}{(ad-c^2)s^4+\beta(a+d)s^2+\beta^2}\ , \tag{21a}$$

and

$$\hat{f}_2(s)=\frac{(ac'-a'c)s^2+\beta c'}{(ad-c^2)s^4+\beta(a+d)s^2+\beta^2}\ , \tag{21b}$$

where

$$a'(s)\equiv b_0cs+af_1'(0)+cf_2'(0)$$

and

$$c'(s)\equiv b_0ds+cf_1'(0)+df_2'(0)$$

come from the inhomogeneous terms in the system for the transforms. The Laplace transforms (21) satisfy the general conditions

$$\begin{gathered}\lim_{|s|\to\infty}[s^2\hat{f}_i(s)-sf_i(0)]=f_i'(0)\ ,\\ \lim_{|s|\to\infty}s\hat{f}_i(s)=f_i(0),\quad i=1,2\ ,\end{gathered} \tag{22}$$

as they should. The denominator of the transforms, quartic in s, can be factored and the transforms inverted by partial fraction decomposition. This leads to the following statement. The poles of an appropriate integral transform of the field provide the complex penetration depths. Since the governing PDE, i.e., Eq. (9) or (10), is linear, an integral transform can be developed. (In the previous section, the Hankel transform would be appropriate.) The reciprocals of the squares of the complex penetration depths are given by

$$\lambda_+^{-2}=-\frac{\beta}{\lambda^2+(i/2)\tilde{\delta}_{vc}^{2}\hat{B}_{0x}^2} \tag{23a}$$

and

$$\lambda_-^{-2}=-\frac{\beta}{\lambda^2+(i/2)\tilde{\delta}_{vc}^{2}}\ . \tag{23b}$$

The solution is completed by imposing conditions (16b). We omit the detailed arguments and manipulations involved and present the final results:

$$f_1(x)=\frac{b_0\hat{B}_{0y}\hat{B}_{0z}}{(1-\hat{B}_{0x}^2)}(e^{-x/\lambda_+}-e^{-x/\lambda_-}) \tag{24}$$

and

$$f_2(x)=\frac{b_0}{(1-\hat{B}_{0x}^2)}(\hat{B}_{0y}^2e^{-x/\lambda_+}+\hat{B}_{0z}^2e^{-x/\lambda_-})\ . \tag{25}$$

By using the expressions for the components of the unit vector $\hat{\mathbf{B}}_0$ in terms of the angles α and ψ, the results given here are found to agree with those of Ref. 3. The complex penetration depths λ_+, λ_- correspond, respectively, to the penetration depths $\lambda_\gamma, \lambda_\beta$ developed from a geometrical approach. We stress that the complex dispersion relations (23) can be found without first explicitly solving for the rf field. Having found $\mathbf{b}(\mathbf{x},t)$, the other fields and densities can be found straightforwardly, yielding various linear-response functions. For a discussion of the surface impedance in this geometry, see Ref. 3. For numerical results on the magnetic permeability and their relevance to the irreversibility line, see Ref. 5.

GENERAL GEOMETRIES

We next discuss the procedure for solving Eq. (9) or (10) for general geometries. The point that must be stressed at the outset is that these PDE's are vector valued. First, consider the meaning of the Laplacian acting on $\mathbf{b}$. This operation becomes well defined in general coordinate systems through the expression

$$\nabla^2\mathbf{b} = -\nabla\times\nabla\times\mathbf{b} + \nabla(\nabla\cdot\mathbf{b}) \ . \tag{26}$$

In particular, the Laplacian acting on a vector field is not the same as the Laplacian acting on each component, unless rectangular components are taken. Loosely stated, the components "mix," so that even the relatively simple vector Laplace equation leads to a set of coupled PDE's.

In the present case, the vector Laplace equation is the one suitable for $\mathbf{b}$ in the source-free region outside the superconductor. This follows immediately from the vanishing curl and divergence of $\mathbf{b}$ and Eq. (26). To avoid the complication mentioned above, the usual method to solve the vector Laplace equation is to introduce a suitable scalar potential. If we take $\mathbf{b}_{\text{out}} = -\nabla\phi + \mathbf{b}_0$, then $\mathbf{b}$ satisfies the vector Laplace equation while the scalar potential ϕ satisfies the corresponding scalar equation. Assuming that $\mathbf{b}$ takes the constant value $\mathbf{b}_0$ at infinity, then the boundary condition on ϕ is that it vanish there. As a specific example, for a right-circular cylinder in a field perpendicular to its axis, the solution for ϕ is appropriately expressed in terms of cylindrical harmonics.

Now consider the solution of Eq. (10) within the superconductor for nonrectangular geometry. Direct solution by taking the coupled components seems formidable, and the question of suitable scalar potentials to reduce to scalar PDE's is an open one. Recall how the vector London equation [Eq. (10) without the vortex term] is solved.[28] If $\nabla^2\mathbf{A} = (1/\lambda^2)\mathbf{A}$, then one possibility is $\mathbf{A} = \text{const}\,\nabla\times(f\mathbf{B}_0)$, where the function f satisfies the scalar equation $\nabla^2 f = (1/\lambda^2)f$. Clearly, this prescription must be extended in order to apply to Eq. (10) because of the vector-valued vortex term there. Such an extension is beyond the scope of the present discussion. In summary, when large demagnetization effects are present, the analytical solution of Eq. (10) is hindered by the coupling of vector components due to the vortex response.

By using the well-developed techniques for solving the vector London equation,[28] we can present the complex penetration depth λ_ω in the absence of vortices. It is simply $\lambda_\omega^{-2} = \lambda^{-2} - 2i\delta_{\text{NF}}^{-2}$, which is the dispersion relation of the two-fluid model. Again, taking the example of a cylinder in transverse field, the form of the solution within the superconductor can be found explicitly in terms of modified Bessel functions and continuity can be enforced at the surface to provide the particular solution. (The other fields and linear-response functions then follow by standard means.)

COMPLEX DIFFUSION COEFFICIENTS

In this section we briefly mention various diffusion coefficients which arise in the dynamics of flux diffusion, providing an opportunity to compare with some other treatments. In particular, we stress that the various diffusion coefficients and corresponding resistivities are in general *not* simply additive. In our theory the complex diffusion coefficient $D_v = \tilde{\rho}_v/\mu_0 = (\omega/2)\tilde{\delta}_{vc}^2$ arises from the vortex-response term. In particular, at high temperature, $D_v \to D_f = \rho_f/\mu_0$, the flux-flow-diffusion coefficient, where $\rho_f = B_0\phi_0/\eta$ is the flux-flow resistivity. At low temperature we have $D_v \to (D_f^{-1} + D_p^{-1})^{-1}$, where $D_p = \rho_p/\mu_0 = -i\omega\lambda_C^2$ is the diffusion coefficient associated with pinning and $\lambda_C^2 = B_0\phi_0/\mu_0\kappa_p$ is the square of the pinning (or Campbell[29]) penetration depth. The coefficient D_p is imaginary, showing that it is associated with the oscillatory part of the vortex motion. In general, an imaginary diffusion coefficient may be interpreted as saying the diffusion only takes place in imaginary time. (Recall that the transformation $t \to it$ links the time-dependent Schrödinger equation to the diffusion or heat equation.) The other diffusion coefficients arising from Eq. (10) are illustrated by the two-fluid model, wherein $D_{\text{eff}} = (D_{\text{NF}}^{-1} + D_s^{-1})^{-1}$, where $D_s = -i\omega\lambda^2$ is pure imaginary, coming from the Meissner response. Each of the basic diffusion coefficients D_{NF}, D_p, D_f, and D_s is directly related to the square of a characteristic length.[5]

In the thermally assisted flux-flow (TAFF) theory,[12] the following diffusion equation in cylindrical geometry is employed:

$$\dot{\mathbf{B}} = \frac{\partial}{\partial r}\lambda_C^2\frac{\partial\dot{\mathbf{B}}}{\partial r} + \frac{\partial}{\partial r}D_0\frac{\partial\mathbf{B}}{\partial r} \ , \tag{27}$$

where $D_0 = \rho/\mu_0$ and ρ is the resistivity caused by TAFF. Aside from some differences in the treatment of thermal activation, Eq. (27) can be viewed as an approximation to Eq. (10). Equation (27) can be rewritten in the form $\dot{\mathbf{B}} = D_{\text{eff}}\partial^2\mathbf{B}/\partial r^2$, where the effective diffusion constant is $D_{\text{eff}} = D_p + D_0$. Note that in this approximate theory the diffusion constants add directly, in contrast to the behavior in our theory. Such direct addition usually reflects an approximate method, as in the addition of resistivities[30] or the use of an equivalent-circuit model.[31]

It is also possible to rewrite the complex ac penetration depth obtained on the basis of the TAFF model in Ref. 12 in terms of characteristic lengths and to compare it in detail with those derived here. However, this comparison is omitted for the sake of brevity.

SUMMARY

Our self-consistent approach to vortex dynamics, including the effect of nonlocal vortex interaction, has been further clarified and unified by its restatement in terms of an initial-boundary-value problem. We have derived and solved a single vector partial differential equation [Eq. (9)] describing the linear response of a type-II superconductor in the mixed state. The solution of this equation, as presented here for several sample geometries, provides the phenomenological dispersion relation for radiation principally attenuating in the superconductor and accompanying complex penetration depths. Once the complex penetration depths are known, the linear-response functions describing dissipation and screening can be found. An example is the rf surface impedance, which is determined by the phase velocity of radiation in the superconductor. The existence of more than one complex penetration depth in general has important physical implications, as discussed, for example, in Ref. 5 on the complex rf magnetic permeability.

The emergence of a second complex, self-consistently determined penetration depth was illustrated in this paper with the solution of the partial differential equation (9) governing the rf flux density $\mathbf{b}(x,t)$. This equation is a vector-generalized diffusion London equation valid in the linear-response regime. In mathematical terms the parabolic part of Eq. (9) or (A6) accounts for the diffusive nature of both the normal-fluid response and vortex-creep effects, while the elliptic part accounts for the finite-penetration effect (or Meissner response) of the superconductor. Of course, these effects are linked as dictated by the Maxwell and London equations for the electrodynamics. The vortex equation of motion provides the necessary ingredient to close the linearized equations. If the displacement current term is retained in Eq. (9) or (A6), these equations have an additional hyperbolic character which may be suitable for the modeling of propagation problems. The inclusion of all these features shows the mathematical and physical richness of this equation. By including the complex dynamic mobility for vortex motion, Eq. (9) encompasses simultaneously the superconductor Meissner response, the vortex response, and the response of the normal fluid. We expect our approach to find applications in many areas involving vortex dynamics, including the modeling of vibrating reed[8] and oscillator[9,10] experiments.

We compared our approach of determining self-consistent complex penetration depths with other models, including the theory of thermally assisted flux flow. Aside from differences in the handling of thermal activation, our theory generalizes the TAFF and other approaches, including the self-consistent coupling of vortex displacement and current density with nonlocal interaction and a fuller accounting of the vector-valued vortex response. We presented and discussed various complex diffusion constants which arise in the description of the coupled vortex-superconductor response. A typical combination of diffusion constants is seen in the two-fluid model, where they add by way of a harmonic sum. (The diffusion constant D_s intrinsic to the superconductor is pure imaginary, being associated with the Meissner response.)

We have presented the vector PDE's (9) and (A6) suitable for describing the response of isotropic superconductors. A more refined description may be expected to involve tensor PDE's. Specifically, Eq. (2) for the supercurrent density can be generalized in Ginzburg-Landau theory with the introduction of an effective-mass tensor. We expect that the vortex mobility $\tilde{\mu}_v$ in, e.g., Eqs. (5) and (6), will need to be treated as a tensor. As an example, at high temperature, the vortex mobility tensor can be expected to go over to the inverse of the viscosity tensor. The viscosity tensor in turn is related to the conductivity tensor (for the instance of a vortex oriented along a principal axis see Ref. 21). Such a theory, which would rely heavily on that for anisotropic superconductors, is worthy of separate consideration.

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APPENDIX: GOVERNING PARTIAL DIFFERENTIAL EQUATION FOR THE ELECTRIC FIELD

Here we show how a vector partial differential equation similar to Eq. (9) arises for the total electric field $\mathbf{E}$. Since the effects of vortex pinning, flux flow, and flux creep are included in a unified manner by way of the dynamic mobility $\tilde{\mu}_v$, this approach represents an improvement over the treatment of, e.g., Ref. 24.

By taking the curl of Faraday's law $\nabla\times\mathbf{E}=-\dot{\mathbf{B}}$, using Ampere's law $\nabla\times\mathbf{H}=\mathbf{J}$ and the two-fluid equation $\mathbf{J}=\mathbf{J}_n+\mathbf{J}_s$, together with $J_n=\sigma_{\mathrm{NF}}\mathbf{E}$, we have

$$\nabla\times(\nabla\times\mathbf{E})+\mu_0\sigma_{\mathrm{NF}}\dot{\mathbf{E}}=-\mu_0\dot{\mathbf{J}}_s\ . \tag{A1}$$

As usual, the overdot denotes time differentiation and $\sigma_{\mathrm{NF}}=1/\mu_0 D_{\mathrm{NF}}$ is the normal-fluid conductivity. From Eqs. (2) and (7), we have

$$\mathbf{J}_s=-\frac{1}{\mu_0\lambda^2}(\mathbf{A}+\mathbf{B}_0\times\mathbf{u})\ , \tag{A2}$$

where $\nabla\times\mathbf{A}-\mathbf{B}-\mathbf{B}_0=\mathbf{b}$, giving

$$\dot{\mathbf{J}}_s=-\frac{1}{\mu_0\lambda^2}(-\mathbf{E}+\nabla\phi+\mathbf{B}_0\times\mathbf{v})\ . \tag{A3}$$

In Eq. (A3), ϕ is a scalar potential. In linear response, as we are considering, $\dot{\mathbf{J}}_s=-i\omega\mathbf{J}_s$ and this equation becomes the generalization of the second London equation in the presence of moving vortices. Using Eq. (5) for the vortex velocity, we have

$$\dot{\mathbf{j}}=\frac{1}{\mu_0\lambda^2}\left[\mathbf{E}-\nabla\phi-\frac{\mu_0\omega}{2}\tilde{\delta}_{vc}^{\,2}\hat{\mathbf{B}}_0\times(\mathbf{J}\times\hat{\mathbf{B}}_0)\right], \tag{A4}$$

where the square of the complex effective skin depth due to vortex motion is $\tilde{\delta}_{vc}^{\,2}=2B_0\phi_0\tilde{\mu}_v/\mu_0\omega$. By using for the current density

$$\mathbf{J}=\frac{1}{\mu_0}\nabla\times\nabla\times\mathbf{A} \tag{A5}$$

and the relation for the vector potential $\mathbf{A}=-(i/\omega)(\mathbf{E}-\nabla\phi)$, we can combine Eqs. (A1) and (A4) to yield

$$\nabla\times(\nabla\times\mathbf{E})+\frac{1}{D_{\mathrm{NF}}}\dot{\mathbf{E}}=-\frac{1}{\lambda^2}\left\{\mathbf{E}-\nabla\phi+\frac{i}{2}\tilde{\delta}_{vc}^{\,2}\hat{\mathbf{B}}_0\times[(\nabla\times(\nabla\times\mathbf{E}))\times\hat{\mathbf{B}}_0]\right\}. \tag{A6}$$

Equation (A6) is the single vector PDE governing the behavior of the electric field $\mathbf{E}$ in linear response at frequencies well below the gap frequency.

[1] M. W. Coffey and J. R. Clem, IEEE Trans. Magn. **MAG-27**, 2136 (1991); **MAG-27**, 4396(E) (1991).
[2] M. W. Coffey and J. R. Clem, Phys. Rev. Lett. **67**, 386 (1991).
[3] J. R. Clem and M. W. Coffey, Physica C **185-189**, 1915 (1991).
[4] M. W. Coffey and J. R. Clem, Phys. Rev. B **45**, 9872 (1992).
[5] M. W. Coffey and J. R. Clem. Phys. Rev. B **45**, 10 527 (1992).
[6] J. R. Clem and M. W. Coffey, J. Supercond. **5**, 313 (1992).
[7] A. M. Campbell and J. E. Evetts, *Critical Currents in Superconductors* (Barnes & Noble, New York, 1972).
[8] A. Gupta *et al.*, Phys. Rev. Lett. **63**, 1869 (1989).
[9] D. E. Farrell *et al.*, Phys. Rev. Lett. **64**, 1573 (1990).
[10] F. de la Cruz and C. Durán, Supercond. Sci. Technol. **5**, 59 (1992).
[11] J. Pankert, Physica C **168**, 335 (1990).
[12] C. J. van der Beek and P. H. Kes, Phys. Rev. B **43**, 13 032 (1991).
[13] S. Sridhar *et al.*, Phys. Rev. Lett. **68**, 2220 (1992).
[14] W. Tomasch *et al.*, Phys. Rev. B **37**, 9874 (1988).
[15] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1989).
[16] M. W. Coffey and J. R. Clem (unpublished).
[17] A. Sudbø and E. H. Brandt, Phys. Rev. B **43**, 10 482 (1991).
[18] E. H. Brandt, Phys. Rev. Lett. **67**, 2219 (1991).
[19] P. Fulde *et al.*, Phys. Rev. Lett. **35**, 1776 (1975); **36**, 832(E) (1976).
[20] W. Dieterich, I. Peschel, and W. R. Schneider, Z. Phys. B **27**, 177 (1977).
[21] Z. Hao and J. R. Clem, IEEE Trans. Magn. **MAG-27**, 1086 (1991).
[22] M. W. Coffey and Z. Hao, Phys. Rev. B **44**, 5230 (1991); M. W. Coffey and J. R. Clem, *ibid.* **44**, 6903 (1991).
[23] F. London, *Superfluids* (Dover, New York, 1960).
[24] A. M. Campbell, in *Magnetic Susceptibility of Superconductors and Other Spin Systems,* edited by R. A. Hein, T. L. Francavilla, and D. H. Liebenberg (Plenum, New York, 1991), p. 129.
[25] E. H. Brandt, Z. Phys. B **80**, 167 (1990).
[26] M. W. Coffey, Phys. Rev. B **46**, 567 (1992).
[27] The corresponding eliminant equation for the field components is $(ad-c^2)f_i^{(4)}+\beta(a+d)f_i''+\beta^2 f_i=0$, $i=1,2$, which can also be solved by Laplace transformation.
[28] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chap. 13.
[29] A. M. Campbell, J. Phys. C **2**, 1492 (1969); **4**, 3186 (1971).
[30] H. Blackstead *et al.*, Physica C **174**, 394 (1991); **175**, 534 (1991).
[31] A. M. Portis and M. Hein (unpublished); S. K. Remillard (unpublished); A. M. Portis and D. W. Cooke, Supercond. Sci. Technol. **5**, 5395 (1992).