Glauber dynamics of the kinetic Ising model

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In this work we study the Glauber dynamics of the one-dimensional Ising model with nearest-neighbor and next-nearest-neighbor interactions, for which an approximate solution of the magnetization per site is obtained. When the dynamical critical exponent z is investigated following the treatment of Cordery, Sarker, and Tobochnik [Phys. Rev. B **24**, 5402 (1981)], our observation shows that its upper-bound value is the same as the known value, thus implying that z is independent of the range of the interaction. We also suggest a high-temperature expansion approximation which is then used to solve the two-dimensional Glauber dynamics governed by a master equation; this solution is compared with that of the decoupling method. The time-delayed correlation function is also calculated.

I. INTRODUCTION

Since the time when Glauber discussed the time-dependent statistics of the Ising model and gave a one-dimensional exact solution,[1] many developments in critical dynamics have occurred and extensions have been made. For the one-dimensional kinetic Ising model (of the Glauber type) the time-dependent magnetization per site and spin-spin correlation function can be obtained by an exact solution of the master equation[1] and the dynamic critical exponent z is obtained by using renormalization-group methods[2] and physical arguments.[3] Subsequently, the dynamical critical exponent was found to be nonuniversal.[3-7] In these studies only nearest-neighbor-interaction Ising systems have been considered.

In the two dimensions, much work has been done on such systems by solving the master equation approximately or by using renormalization-group methods and computer simulations. The multispin correlation function and dynamical critical exponent have been obtained and discussed[8-20] The master equation for systems with $d \geq 2$ is known to be very complicated, for which an exact solution, in general, cannot be obtained; therefore a decoupling approximation is often made.

In this work we will investigate the $d=1$ kinetic Ising model with not only nearest-neighbor but also next-nearest-neighbor interactions in the absence of a magnetic field. In the equilibrium case, it is well known that this model is equivalent to a model with only nearest-neighbor interactions in the presence of an effective "external" field and the static critical behavior is independent of the range of the interaction. In the dynamical case, however, our observation shows that this equivalence is never attained. We also find an approximate expression for the time-dependent magnetization per site. In addition, by using the same physical arguments as presented by Cordery, Sarker, and Tobochnik (CST),[3] we obtain an upper-bound value of 2 for the dynamical critical exponent z, implying that z is independent of the range of the interaction.

To find an exact solution of the two-dimensional kinetic Ising model appears to be a much more difficult task. In fact, in such a system we will be faced with a set of coupled equations in various n-spin correlation functions. However, we note that if the transition probability is appropriately chosen, a compact solution of the master equation can be obtained. We also note that at a critical point Onsager's result,[23] $k_B T_C = 2.269J$ (where k_B is the Boltzmann constant; T_c, the critical temperature; and J the exchange integral) can be used, and then an approximation that corresponds to a high-temperature expansion can be applied. We will solve the master equation under the above approximation and find n-spin correlation functions ($n=1,2$). Our result is compared with that of decoupling method. We will also find the time-delayed correlation function.

This paper is organized as follows. In Sec. II, we study one-dimensional Ising model with nearest-neighbor (NN) and next-nearest-neighbor (NNN) interactions in the absence of a field. We argue that it is not equivalent to the model with only NN interaction in the existence of an effective external field in dynamics. Then solving the master equation and employing a decoupling approximation, we get an expression for the magnetization per site. Finally, we use the arguments of CST to find the dynamical critical exponent z. In Sec. III, we investigate the solution of the master equation for an anisotropic Ising model on a two-dimensional lattice under the high-temperature expansion and using the decoupling approach; n-spin correlation functions and the time-delayed correlation function are found. Finally, in Sec. IV conclusions are given.

II. ONE-DIMENSIONAL KINETIC ISING MODEL WITH NN AND NNN INTERACTIONS

We study an Ising model with nearest-neighbor and next-nearest-neighbor interactions whose Hamiltonian is

$$-\beta H - k_1 \sum_i \sigma_i \sigma_{i+1} + k_2 \sum_i \sigma_i \sigma_{i+2} \, , \tag{1}$$

where k_1 and k_2 are the interaction parameters between nearest and next-nearest-neighbors, respectively, $\beta=1/k_BT$ and $\sigma=\pm 1$. In the equilibrium case, one customarily introduces a change in variable, $\mu_i=\sigma_i\sigma_{i+1}=\pm 1$, so that Eq. (1) may be rewritten as

$$-\beta H=k_1\sum_i \mu_i+k_2\sum_i \mu_i\mu_{i+1} , \quad (2)$$

which is just the Hamiltonian of a Ising model with only nearest-neighbor interaction k_2 in the existence of an effective external field k_1. The equilibrium thermodynamical behavior is determined by the partition function

$$Z=\sum_{\{\sigma\}} e^{-\beta H}=\sum_{\{\sigma\}}\exp\left[k_1\sum_i \sigma_i\sigma_{i+1}+k_2\sum_i \sigma_i\sigma_{i+2}\right]$$
$$=2\sum_{\{\mu\}}\exp\left[k_1\sum_i \mu_i+k_2\sum_i \mu_i\mu_{i+1}\right] , \quad (3)$$

which is easy to calculate by using, for example, the well-known transfer-matrix method. The equivalence between model (1) and (2) is then very obvious.

For dynamics, the situation becomes much more complicated. Our analysis shows that the above equivalence is not carried over. To see that, we start from the master equation, which describes the time-dependent behavior of the spin system under study. Let $P(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots;t)$ be the probability of the occurrence of the configuration $(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_n)$ at time t. The evolution of $P(\sigma_1,\sigma_2,\ldots,t)$ obeys the master equation[1]

$$\frac{d}{dt}P(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,t)=-\sum_i W_i(\sigma_i)P(\ldots,\sigma_i,\ldots;t)+\sum_i W_i(-\sigma_i)P(\ldots,-\sigma_i,\ldots;t) , \quad (4)$$

where $W_i(\sigma_i)$ is the probability, per unit time, of a transition from configuration $(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots)$ to $(\sigma_1,\sigma_1,\ldots,-\sigma_i,\ldots)$. In Glauber dynamics, only one single spin flip is allowed each time. According to statistical physics, the magnetization per site and the spin-spin correlation function are given, respectively, by

$$q_i(t)=\langle\sigma_i\rangle=\sum_{\{\sigma\}}\sigma_i P(\ldots,\sigma_i,\ldots;t) , \quad (5)$$

$$r_{i\cdot k}(t)=\langle\sigma_i\sigma_k\rangle=\sum_{\{\sigma\}}\sigma_i\sigma_k P(\ldots,\sigma_i,\ldots,\sigma_k,\ldots;t) . \quad (6)$$

These functions obey the following equations:

$$\frac{d}{dt}q_i(t)=-2\langle\sigma_i(t)W_i(\sigma_i)\rangle , \quad (7)$$

$$\frac{d}{dt}r_{i,k}(t)=-2\langle\sigma_i(t)\sigma_k(t)[W_i(\sigma_i)+W_k(\sigma_k)]\rangle . \quad (8)$$

As usual, $W_i(\sigma_i)$ is phenomenologically given by means of the condition of detailed balance,[1,2] and we propose

$$W_i(\sigma_i)=\tfrac{1}{2}\alpha\{1-\tfrac{1}{2}\delta_1\sigma_i(\sigma_{i-1}+\sigma_{i+1})\}\{1-\tfrac{1}{2}\delta_2\sigma_i(\sigma_{i-2}+\sigma_{i+2})\} , \quad (9)$$

with $\delta_1=\tanh(2k_1)$, $\delta_2=\tanh(2k_2)$. Substituting (9) into (7) and (8), we arrive at

$$\frac{d}{d(\alpha t)}q_i(t)=-q_i(t)+\tfrac{1}{2}\delta_1(q_{i-1}+q_{i+1})+\tfrac{1}{2}\delta_2(q_{i-2}+q_{i+2})$$
$$-\tfrac{1}{4}\delta_1\delta_2(\langle\sigma_i\sigma_{i-1}\sigma_{i-2}\rangle+\langle\sigma_i\sigma_{i+1}\sigma_{i-2}\rangle+\langle\sigma_i\sigma_{i-1}\sigma_{i+2}\rangle+\langle\sigma_i\sigma_{i+1}\sigma_{i+2}\rangle) . \quad (10)$$

$$\frac{d}{d(\alpha t)}r_{i,i+1}(t)=-2r_{i,i+1}+\delta_1+\tfrac{1}{2}\delta_1(r_{i-1,i+1}+r_{i,i+2})$$
$$+\tfrac{1}{2}\delta_2(r_{i+1,i-2}+r_{i+1,i+2}+r_{i,i-1}+r_{i,i+2})$$
$$-\tfrac{1}{4}\delta_1\delta_2(\langle\sigma_i\sigma_{i-2}\rangle+\langle\sigma_i\sigma_{i+2}\rangle+\langle\sigma_{i+1}\sigma_{i-1}\rangle+\langle\sigma_{i+1}\sigma_{i+3}\rangle+\langle\sigma_i\sigma_{i+1}\sigma_{i-1}\sigma_{i-2}\rangle$$
$$+\langle\sigma_i\sigma_{i+1}\sigma_{i-1}\sigma_{i+2}\rangle+\langle\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i-1}\rangle+\langle\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}\rangle) . \quad (11)$$

Equation (11) gives the time-dependent behavior of the nearest-neighbor spin-correlation function.

Similarly, for model (2), from the master equation with $P(\mu_1,\mu_2,\ldots,\mu_i,\ldots;t)$ and $W_i(\mu_i)$, we obtain

$$\frac{d}{d(\alpha t)}Q_i(t)=-Q_i(t)+\delta_1'+\frac{\delta_2}{2}[Q_{i-1}(t)+Q_{i+1}(t)]$$
$$-\frac{\delta_2'\delta_2}{2}[R_{i-1,i}(t)+R_{i,i+1}(t)] , \quad (12)$$

where

$$W_i(\mu_i)=\tfrac{1}{2}\alpha[1-\tfrac{1}{2}\delta_2\mu_i(\mu_{i-1}+\mu_{i+1})][1-\mu_i\delta_1'] ,$$
$$Q_i(t)=\langle \mu_i\rangle , \tag{13}$$
$$R_{i,i+1}(t)=\langle \mu_i\mu_{i+1}\rangle ,$$

and $\delta_1'=\tanh k_1$, $\delta_2=\tanh(2k_2)$. Comparing Eq. (11) with Eq. (12) and noting that $\mu_i=\sigma_i\sigma_{i+1}$, we find that $r_{i,i+1}$ and Q_i obey completely different equations, and therefore, in general, they have different solutions. This implies that in dynamics we cannot identify model (1) with model (2).

We now turn our attention to Eqs. (10) and (11). Because the 3- and 4-spin-correlation functions appear on the right-hand side of these equations, we have to work with equations including higher-order correlation functions. As a result, we will have an infinite set of equations. In order to solve them, we use a decoupling technique as usual. Here we take the following prescription:

$$\begin{aligned}
\langle \sigma_i\sigma_{i-1}\sigma_{i-2}\rangle &\approx \langle \sigma_i\sigma_{i-1}\rangle\langle \sigma_{i-2}\rangle = r_{i,i-1}q_{i-2} ,\\
\langle \sigma_i\sigma_{i+1}\sigma_{i-2}\rangle &\approx \langle \sigma_i\sigma_{i+1}\rangle\langle \sigma_{i-2}\rangle = r_{i,i+1}q_{i-2} ,\\
\langle \sigma_i\sigma_{i-1}\sigma_{i+2}\rangle &\approx \langle \sigma_i\sigma_{i-1}\rangle\langle \sigma_{i+2}\rangle = r_{i,i-1}q_{i+2} ,\\
\langle \sigma_i\sigma_{i+1}\sigma_{i+2}\rangle &\approx \langle \sigma_i\sigma_{i+1}\rangle\langle \sigma_{i+2}\rangle = r_{i,i+1}q_{i+2} .
\end{aligned} \tag{14}$$

In the zeroth-order approximation, we assume that $r_{i,i+1}=r^e_{i,i+1}$, where $r^e_{i,i+1}$ is the equilibrium value, and

$$r^e_{i,i+1}=r^e_{i,i-1}=\theta(k_1k_2)=\theta_0 ,$$

we thus write Eq. (10) as

$$\frac{d}{d(\alpha t)}q_i(t)=-q_i(t)+\frac{\delta_1}{2}[q_{i-1}(t)+q_{i+1}(t)] +\frac{\delta_2'}{2}[q_{i-2}(t)+q_{i+2}(t)] , \tag{15}$$

where

$$\delta_2'=\delta_2-\theta_0\delta_1\delta_2=\delta_2(1-\theta_0\delta_1) . \tag{16}$$

Immediately we obtain the solution of Eq. (15) as follows:

$$q_i(t)=e^{-\alpha t}\sum_{n,m=-\infty}q_{i-n-2m}(0)I_n(\alpha\delta_1 t)I_m(\alpha\delta_2' t) , \tag{17}$$

where $I_n(x)$ is the modified Bessel function and $q_k(0)$ is the initial value of the kth spin. In the limit $\delta_2\to 0$, corresponding to the vanishing of the next-nearest-neighbor interaction, Eq. (17) reduces to

$$q_i(t)=e^{-\alpha t}\sum_{n=-\infty}q_{i-n}(0)I_n(\alpha\delta_1 t) , \tag{18}$$

which reproduces precisely Glauber's result.[1]

In a higher-order approximation, we use the nonequilibrium value of $r_{i,i+1}(t)$ instead of $r^e_{i,i+1}$, so that

$$\delta_2'(t)=\delta_2-\theta(t)\delta_1\delta_2=\delta_2[1-\theta(t)\delta_1] , \tag{19}$$

and find the solution of (15) as

$$q_i(t)=e^{-\alpha t}\sum_{n,m=-\infty}^{\infty}q_{i-n-2m}(0)I_n(\alpha\delta_1 t)I_m(\alpha G(t)), \tag{20}$$

where

$$G(t)=\int_0^t\delta_2'(t')dt' . \tag{21}$$

Comparing (20) with (17), we find that the only difference between them is that the linear function of time, $\alpha\delta_2' t$, in the second modified Bessel function is replaced by the nonlinear function of time, $\alpha G(t)$.

Finally, we are interested in the dynamical critical exponent z, which is defined as

$$\tau=\xi^z \tag{22}$$

where τ is the relaxation time and ξ is the correlation length at the critical temperature. For a determination of z, using simple physical arguments relating to movement of domain walls proposed by CST (Ref. 3), the behavior of the relaxation time near the transition point is determined by the time that a domain wall takes to move through a distance ξ. Suppose that the rate at which a domain wall moves by one step is W. Then according to random-walk theory, the wall must make, on average, $N\sim\xi^2$ steps to move through a distance ξ (if ξ is large enough); it follows that τ behaves like ξ^2/W. In the arguments of CST, the fastest mechanism of the motion of the wall is chosen, and so the resulting value of z should be an upper bound on the exact one. From (9), we have the highest transition rate $W_i(\sigma_i)=\alpha/2$, so that we obtain $z=2$.

We find that, as a reasonable extension, the above analysis can be applied to the one-dimensional Ising system including further finite- and long-range interaction, the transition rate can be written as

$$\begin{aligned}
W_i(\sigma_i)=\tfrac{1}{2}\alpha\{1-\delta_1\sigma_i(\sigma_{i-1}+\sigma_{i+1})\}\\
\times\tfrac{1}{2}\{1-\delta_2\sigma_i(\sigma_{i-2}+\sigma_{i+2})\}\\
\times\cdots\{1-\tfrac{1}{2}\delta_k\sigma_i(\sigma_{i-k}\sigma_{i+k})\} ,
\end{aligned} \tag{23}$$

which corresponds to the existence of nearest-, next-nearest-, and the kth-neighbor interactions. Obviously here we obtain $z=2$ as well.

Our result leads to the following statement: the dynamical critical exponent z is independent of the range of the interaction for one-dimensional Ising system, which is similar to that in equilibrium case. As we have already mentioned, we discuss in this paper the relationship to universality in dynamics of the further- and long-range interactions.

III. TWO-DIMENSIONAL GLAUBER DYNAMICS

We now start with an anisotropic Ising Hamiltonian on the square lattice

$$-\beta H=k_1\sum_{i,j}\sigma_{i,j}\sigma_{i+1,j}+k_2\sum_{i,j}\sigma_{i,j}\sigma_{i,j+1} , \tag{24}$$

in which the summations are taken over all nearest-neighbor-spin pairs along the X and Y axis, respectively. In Glauber dynamics, under the condition of detailed balance, we choose the transition rate as

$$W_{ij}(\sigma_{ij})=\tfrac{1}{2}\alpha\{1-\sigma_{ij}\tanh[k_1(\sigma_{i-1,j}+\sigma_{i+1,j})]\}\times\{1-\sigma_{ij}\tanh[k_2(\sigma_{i,j-1}+\sigma_{i,j+1})]\}\ . \qquad (25)$$

In the special case of $k_1=k_2=k$, (27) reduces to the isotropic model and the transition rate becomes

$$W_{ij}(\sigma_{ij})=\tfrac{1}{2}\alpha\{1-\sigma_{ij}\tanh[k(\sigma_{i-1,j}+\sigma_{i+1,j}+\sigma_{i,j-1}+\sigma_{i,j+1})]\}\ . \qquad (26)$$

To solve the master equation, we proceed to simplify the expression of (26). We make use of the following series expansion:

$$\tanh x=x-\frac{x^2}{3}+\frac{15}{2}x^5-\frac{17}{315}x^7+\cdots,\quad |x|<\frac{\pi}{2}\ . \qquad (27)$$

It is clear that the requirement of $|4k|<\pi/2$ will strictly be satisfied for $T>T_C$ (Onsager's result gives $k_C=J/k_BT_C=1/2.269$). We call it the high-temperature expansion. Retaining the leading term of (27), the transition rate is approximately written as

$$W_{ij}(\sigma_{ij})=\tfrac{1}{2}\alpha\{1-k\sigma_{ij}(\sigma_{i-1,j}+\sigma_{i+1,j}+\sigma_{i,j-1}+\sigma_{i,j+1})\}\ . \qquad (28)$$

As with Eqs. (7) and (8), we write

$$\frac{d}{dt}q_{ij}(t)=-2\langle\sigma_{ij}W_{ij}(\sigma_{ij})\rangle\ , \qquad (29)$$

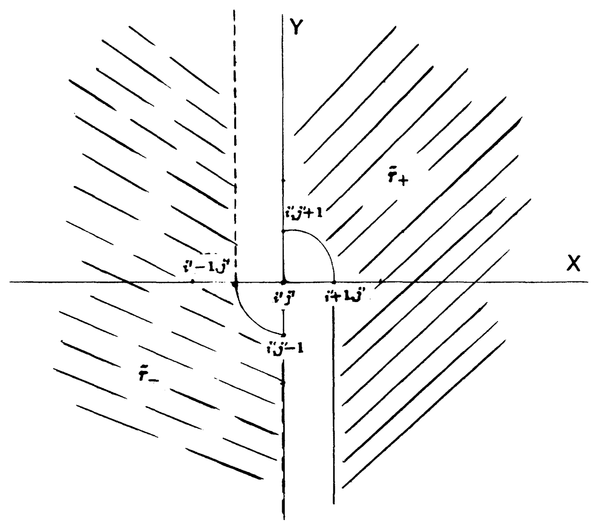

FIG. 1. The definition region of $\tilde{r}_+$ and $\tilde{r}_-$.

$$\frac{d}{dt}r_{ij,i'j'}=-2\langle\sigma_{ij}\sigma_{i'j'}[W_{ij}(\sigma_{ij})+W_{i'j'}(\sigma_{i'j'})]\rangle\ , \qquad (30)$$

where

$$q_{ij}(t)=\sum_{\{\sigma\}}\sigma_{ij}P(\ldots,\sigma_{ij},\ldots;t)\ ,$$

$$r_{ij;i'j'}(t)=\sum_{\{\sigma\}}\sigma_{ij}\sigma_{i'j'}P(\ldots,\sigma_{ij},\ldots,\sigma_{i'j'},\ldots;t)\ , \qquad (31)$$

in which q_{ij} is the magnetization at site (i,j) and $r_{ij;i'j'}(t)$ is the correlation function of spins σ_{ij} and $\sigma_{i'j'}$.

Substituting (30) into (31) and (32), we obtain

$$\frac{d}{d(\alpha t)}q_{ij}(t)=-q_{ij}(t)+k(q_{i-1,j}+q_{i+1,j}+q_{i,j-1}+q_{i,j+1})\ , \qquad (32)$$

$$\frac{d}{d(\alpha t)}r_{ij;i'j'}(t)=-2r_{ij;i'j'}(t)+k(r_{i'j';i-1,j}+r_{i'j';i+1,j}+r_{ij;i'-1,j'}+r_{ij;i'+1,j'}+r_{i'j';i,j-1}+r_{i'j';i,j+1}+r_{ij;i',j'-1}+r_{ij;i'j'+1})\ . \qquad (33)$$

Immediately we can write the exact solution of (32) as follows:

$$q_{mn}(t)=e^{-\alpha t}\sum_{m',n'=-\infty}^{\infty}q_{m',n'}(0)I_{m-m'}(2k\alpha t)I_{n-n'}(2k\alpha t)\ . \qquad (34)$$

For Eq. (33), we note that it is not applicable to $r_{ij,ij}=\langle\sigma_{ij}\sigma_{ij}\rangle$ because the latter is a constant, always equal to 1. To account for this exceptional case, we introduce the following definition:[21]

$$\tilde{r}_{ij;i'j'}=\begin{cases}r_{ij;i'j'}(t)-r^e_{ij;i'j'}\equiv\tilde{r}_+ & \text{for } i>i' \text{ or } i=i',\ j>j'\\ -r_{ij;i'j'}(t)+r^e_{ij;i'j'}\equiv\tilde{r}_- & \text{for } i<i' \text{ or } i=i',\ j<j'\\ 0\equiv\tilde{r}_0 & \text{for } i=i',j=j'\ .\end{cases} \qquad (35)$$

The definition regions of $\tilde{r}_+$ and $\tilde{r}_-$ are shown in Fig. 1. Here $r^e_{ij,i'j'}$ represents the equilibrium spin-spin correlation function. With the above consideration, a similar equation is obtained:

$$\frac{d}{d(\alpha t)}\tilde{r}_{ij;i'j'}(t)=-2\tilde{r}_{ij;i'j'}(t)+k(\tilde{r}_{i'j';i-1,j}\,\tilde{+}\,r_{i'j';i+1,j}+\tilde{r}_{ij;i'-1,j'}+\tilde{r}_{ij;i'+1,j'}+\tilde{r}_{i'j';i,j-1}+\tilde{r}_{i'j';i,j+1}+\tilde{r}_{ij;i',j'-1}+\tilde{r}_{ij;i',j'+1})\ . \qquad (36)$$

In terms of a generating function, we finally get the solution

$$r_{mn,m'n'}(t)=r^{e}_{mn,m'n'}+e^{-2\alpha t}\sum_{i,j;i',j'}(r_{ij;i'j'}(0)-r^{e}_{ij;i'j'})\times[I_{m-i}(2k\alpha t)I_{n-j}(2k\alpha t)I_{m'-i'}(2k\alpha t)I_{n'-j'}(2k\alpha t)-I_{m-i'}(2k\alpha t)I_{n-j'}(2k\alpha t)I_{m'-i}(2k\alpha t)I_{n'-j}(2k\alpha t)]\ ,$$

$$\text{for } m>m' \text{ and } m=m',\ n>n'\ , \tag{37}$$

where $r_{ij,i'j'}(0)$ is the initial value of $r_{ij,i'j'}$.

We now return to the anisotropic Ising model (25) on the square lattice and study its dynamical behavior governed by a master equation. As before, we rewrite the transition probability as

$$W_{ij}(\sigma_{ij})=\tfrac{1}{2}\alpha\{1-\tfrac{1}{2}\delta_1\sigma_{ij}(\sigma_{i-1,j}+\sigma_{i+1,j})\}\{1-\tfrac{1}{2}\delta_2\sigma_{ij}(\sigma_{i,j-1}+\sigma_{i,j+1})\}\ , \tag{38}$$

where $\delta_1=\tanh(2k_1)$, $\delta_2=\tanh(2k_2)$. The first- and second-order spin correlation function obey Eqs. (29) and (30), respectively. On the right-hand side of Eqs. (29) and (30) there appear higher-order correlation functions. We again use a decoupling procedure as follows:

$$\begin{aligned}
\langle\sigma_{ij}\sigma_{i-1,j}\sigma_{i,j+1}\rangle&\approx\langle\sigma_{ij}\sigma_{i,j+1}\rangle\langle\sigma_{i-1,j}\rangle=r_{ij;i,j+1}q_{i-1,j}\ ,\\
\langle\sigma_{ij}\sigma_{i+1,j}\sigma_{i,j-1}\rangle&\approx\langle\sigma_{ij}\sigma_{i,j-1}\rangle\langle\sigma_{i+1,j}\rangle=r_{ij;i,j-1}q_{i+1,j}\ ,\\
\langle\sigma_{ij}\sigma_{i+1,j}\sigma_{i,j+1}\rangle&\approx\langle\sigma_{ij}\sigma_{i+1,j}\rangle\langle\sigma_{i,j+1}\rangle=r_{ij;i+1,j}q_{i,j+1}\ ,\\
\langle\sigma_{ij}\sigma_{i-1,j}\sigma_{i,j-1}\rangle&\approx\langle\sigma_{ij}\sigma_{i-1,j}\rangle\langle\sigma_{i,j-1}\rangle=r_{ij;i-1,j}q_{i,j-1}\ .
\end{aligned} \tag{39}$$

We then get

$$\frac{dq_{ij}(t)}{d(\alpha t)}=-q_{ij}(t)+\left[\frac{\delta_1}{2}-\frac{\delta_1\delta_2}{4}\theta_1\right](q_{i-1,j}+q_{i+1,j})+\left[\frac{\delta_2}{2}-\frac{\delta_1\delta_2}{4}\theta_2\right](q_{i,j-1}+q_{i,j+1})\ , \tag{40}$$

$$\frac{dr_{ij;i'j'}(t)}{d(\alpha t)}=-2r_{ij;i'j'}(t)+\left[\frac{\delta_1}{2}-\frac{\delta_1\delta_2}{4}\theta_1\right](r_{i'j';i-1,j}+r_{i'j';i+1,j}+r_{ij;i'-1,j'}+r_{ij;i'+1,j'})+\left[\frac{\delta_2}{2}-\frac{\delta_1\delta_2}{4}\theta_2\right](r_{i'j';i,j-1}+r_{i'j';i,j+1}+r_{ij;i',j'-1}+r_{ij;i',j'+1})\ , \tag{41}$$

where we have assumed that $r_{ij;i,j-1}\approx r_{ij;i,j+1}\equiv\theta_1(t)$ and $r_{ij;i-1,j}\approx r_{ij;i+1,j}=\theta_2(t)$. We now take the equilibrium values of $\theta_1(t)$ and $\theta_2(t)$ as a zeroth-order approximation, so that[22]

$$\begin{aligned}
\theta_{10}&=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\left[\frac{(1-\alpha_1e^{i\varphi})(1-\alpha_2e^{-i\varphi})}{(1-\alpha_1e^{-i\varphi})(1-\alpha_2e^{i\varphi})}\right]^{1/2}\ ,\\
\theta_{20}&=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\left[\frac{(1-\alpha_2e^{i\varphi})(1-\alpha_1e^{-i\varphi})}{(1-\alpha_2e^{-i\varphi})(1-\alpha_1e^{i\varphi})}\right]^{1/2}\ ,
\end{aligned} \tag{42}$$

in which

$$\begin{cases}\alpha_1=\dfrac{Z_1(1-|Z_2|)}{1+|Z_2|}\\[2ex] \alpha_2=\dfrac{Z_1^{-1}(1-|Z_2|)}{1+|Z_2|}\end{cases}\quad\begin{cases}Z_1=\tanh k_1\\ Z_2=\tanh k_2\end{cases}\ . \tag{43}$$

It is immediately found that Eqs. (40) and (41) are precisely the same as Eqs. (32) and (33) if one replaces $(\frac{1}{2}\delta_1-\frac{1}{4}\delta_1\delta_2/\theta_1)$ and $(\frac{1}{2}\delta_2-\frac{1}{4}\delta_1\delta_2/\theta_1)$ with k. Therefore, their solution can be written as

$$q_{mn}(t)=e^{-\alpha t}\sum_{m',n'}q_{m'n'}(0)I_{m-m'}(2a\alpha t)I_{n-n'}(2b\alpha t)\ , \tag{44}$$

$$r_{mn;m'n'}(t)=r^{e}_{mn;m'n'}+e^{-2\alpha t}\sum_{iji'j'}(r_{ij;i'j'}(0)-r^{e}_{ij;i'j'})[I_{m-i}(2a\alpha t)I_{n-j}(2b\alpha t)I_{m'-i'}(2a\alpha t)I_{n'-j'}(2b\alpha t)-I_{m-i'}(2a\alpha t)I_{n-j'}(2b\alpha t)I_{m'-i}(2a\alpha t)I_{n'-j}(2b\alpha t)] \tag{45}$$

for $m>m'$ and $m=m'$, $n>n'$,

where

$$a \equiv (\tfrac{1}{2}\delta_1 - \tfrac{1}{4}\delta_1\delta_2\theta_{10}) , \quad b \equiv (\tfrac{1}{2}\delta_2 - \tfrac{1}{4}\delta_1\delta_2\theta_{20}) . \tag{46}$$

It is interesting that (44) and (45) are found to be of the same structure with Eqs. (34) and (37), except for the difference of arguments in the modified Bessel functions. Thus it strongly suggests that the decoupling approximation may correspond to the high-temperature approximation. Physically speaking, the coupling between spins can be destroyed by thermal disturbances at high temperature. We find it to be a reasonable physical interpretation for the decoupling approximation.

Finally, we proceed to find the spin-spin correlation function at different times and different sites. Let us consider the time-delayed spin correlation function, which is defined as

$$\langle \sigma_{ij}(t)\sigma_{kl}(t+t')\rangle = \sum_{\{\sigma(t)\}}\sum_{\{\sigma(t')\}} \sigma_{ij}(t)P(\ldots,\sigma_{ij}(t),\ldots;t) \times P_c(\ldots,\sigma_{ij}(t)\ldots|\ldots,\sigma_{ij}(t'),\ldots;t')\sigma_{kl}(t') . \tag{47}$$

where $P_c(\ldots,\sigma_{ij}(t),\ldots|\ldots,\sigma_{ij}(t'),\ldots;t')$ is the condition probability of spin at site (ij) being in state $\sigma_{ij}(t)$ at time t and being in state $\sigma_{ij}(t')$ at time $t+t'$. Obviously, in the high-temperature approximation, we have

$$\sum_{\{\sigma(t')\}} P_c(\ldots,\sigma_{ij}(t),\ldots|\ldots,\sigma_{ij}(t'),\ldots;t')\sigma_{kl}(t') = q_{kl}(t+t') = e^{-\alpha t'}\sum_{k'',l''}\sigma_{k''l''}(t)I_{k-k''}(2k\alpha t')I_{l-l''}(2k\alpha t') . \tag{48}$$

Substituting (48) into (47), we get

$$\langle \sigma_{ij}(t)\sigma_{kl}(t+t')\rangle = \sum_{\{\sigma(t)\}}\sigma_{ij}(t)e^{-\alpha t'}\sum_{k'',l''}\sigma_{k''l''}(t)I_{k-k''}(2k\alpha t')I_{l-l''}(2k\alpha t')P(\ldots,\sigma_{ij}(t),\ldots;t)$$
$$= e^{-\alpha t'}\sum_{k'',l''} r_{ij;k''l''}(t)I_{k-k''}(2k\alpha t')I_{l-l''}(2k\alpha t') . \tag{49}$$

Similarly, in the decoupling approximation, we obtain

$$\langle \sigma_{ij}(t)\sigma_{kl}(t+t')\rangle = e^{-\alpha t'}\sum_{k'',l''} r_{ij;k''l''}(t)I_{k-k''}(2a\alpha t')I_{l-l''}(2b\alpha t') . \tag{50}$$

IV. CONCLUSIONS

In conclusion, we have studied the dynamics of the one-dimensional Ising model which NN interaction k_1 and NNN interaction k_2 in the absence of a field and showed that it is not equivalent to the Ising model with NN interaction k_2 in the presence of an effective external field k_1, in contrast with that in the equilibrium case. By solving a master equation, we obtained an approximate expression of the magnetization per site. Following the arguments of CST, we found an upper-bound value of 2 for the dynamical critical exponent z of our model which is the same value as that of the Ising model with only NN interactions. This knowledge increases our understanding of the concept of universality in the dynamic region. In the second part of this paper, we investigated the solution of the master equation in high-temperature expansion and using a decoupling treatment. We found that, in both cases, the results have the same structure, which strongly implies that the decoupling approximation corresponds to the high-temperature expansion, and decoupling approximation is then suitable for use under high-temperature conditions so we conclude that the reasonable physical considerations support this point of view.

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