# Localization in a one-dimensional Thue-Morse chain

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(Received 6 May 1992)

The mean resistance of a one-dimensional wire is calculated with the use of Landauer formula for three types of arrangements: the random, Thue-Morse, and Fibonacci chains for which the positions of the atoms and the scattering strengths are modulated according to the prescribed rules. Comparison of the obtained numerical results shows that for the position modulation, a Thue-Morse chain is more localized than a Fibonacci chain, while for the scattering strength modulation it is less localized. It is shown that the Thue-Morse chain can be switched from being localized to being extended when the ratio of the strength modulation to the position modulation is increased. A similar change occurs in the generalized Thue-Morse chain.

## I. INTRODUCTION

Technical advances in submicron physics have enabled experimentalists to fabricate nearly ideal one-dimensional wires (see, for example, Ref. 1 and references given therein). In these systems, both the magneto-optical absorption and the transport properties have been studied recently.[1,2] The relationship between the electrical conductance at zero temperature and the transmission coefficient, given by the well-known Landauer formula,[3] indicate that some experimentally measurable quantities can be adequately explained when regarding a one-dimensional (1D) wire as an infinite 1D array of potentials.

The discovery of quasicrystals[4] has stimulated interest in exploring the physical nature of quasiperiodic sequences[5] as well as commensurate-incommensurate systems.[6] Work on aperiodic sequences, including the Thue-Morse, incommensurate, and others, has also included studies of direct experimental relevance to semiconductor multilayers (superlattices) (Ref. 7) and superconducting networks.[8] It is interesting to note the contradiction between some calculated thermodynamic and spectral properties and the results from the structure factor in a Thue-Morse (TM) sequence as to whether the TM chain is more random or more "periodic" than the Fibonacci chain (see Kolář *et al.* in Ref. 5). In this paper, we will concentrate on comparing the *mean resistance* for the TM and the Fibonacci chains with position or scattering strength modulation. Our calculations of the *localization length* as a function of the chain length have shown that the TM chain is more localized for position modulation but less localized for the strength modulation.

The TM sequence composed of two symbols, "0" and "1," can be generated by the following recursion formula:

$$\epsilon_0 = 0 \ , \quad \epsilon_{2n} = \epsilon_n \ , \quad \epsilon_{2n+1} = 1 - \epsilon_n \ , \tag{1}$$

where $\epsilon_n$ denotes the $n$th element of the sequence. The resulting infinite string of digits never repeats itself. In spite of this *aperiodicity*, the TM sequence is still *self-similar* in nature. Both the TM and Fibonacci sequences can be generated by the simple substitution rules: $0 \to 01$, $1 \to 10$ for the TM sequence, and $0 \to 1$, $1 \to 10$ for the Fibonacci one. The finite chain obtained from a single "0" by $n$ times of applications of the respective substitution rule is called the $n$th generation chain. It has $2^n$ elements for the TM case, and $F_n$ elements for the Fibonacci case, where the Fibonacci numbers $\{F_n\}$ are recursively defined by $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$. For the TM sequence, the ratio of the number of 0's to the number of 1's is equal to one in any generation. However, this is not true for the Fibonacci sequence. For the infinite Fibonacci sequence, this ratio has the limiting value $\tau = (\sqrt{5}+1)/2$ (golden mean), an irrational number. We can choose two fundamental lengths $a$ and $b$, and assign them to the numbers "0" and "1" in the sequence, respectively. Then the successive TM chains are $\{ab, abba, abbabaab, \ldots\}$, and the Fibonacci ones are $\{b, ba, bab, babba, \ldots\}$. The $n$th generation of these sequences is obtained in a straightforward way. For the TM case, we have $W_{n+1} = W_n W_n'$, where $W_0 = a$ and $W_n'$ is obtained from $W_n$ by exchanging $a$ and $b$. For the Fibonacci sequence, we obtain $S_{n+1} = S_n S_{n-1}$ with $S_0 = a$, $S_1 = b$. The lengths $a$ and $b$ are the separations between neighboring scatterers and the whole sequence determines where the scatterers are. For example, the third-generation Fibonacci chain ($bab$) has scatterers at $\{x_n\}_{n=1,\ldots,F_3} = b$, $(b+a)$, and $2b+a$. The length of the $n$th generation ($n > 0$) is $L_{\rm TM} = 2^{n-1}(a+b)$ and $L_F = (F_{n-1}b + F_{n-2}a)$ for the TM and Fibonacci sequences, respectively.

The rest of the paper is organized as follows. In Sec. II, we describe our model and present the theory for calculating the resistance. This includes a brief review of the transfer-matrix and optical methods, along with some related numerical results. In Sec. III, we calculate the mean resistance for the random, TM, and Fibonacci chains, and compare the localization length as a function of the chain length for these three systems. Section IV is devoted to a discussion on the generalized TM and Fibonacci chains. A short summary of the key results is also presented at the end of Sec. IV.

## II. MODEL AND THEORETICAL DESCRIPTION

The scaled ($\hbar = 2m = 1$) one-dimensional (1D) Schrödinger equation in the presence of potential scatterers is given by

$$-\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)\ . \tag{2}$$

The short-range scattering potential for $N$ scatterers can be expressed as

$$V(x) = \sum_{n=1}^{N} q_n \delta(x - x_n)\ . \tag{3}$$

Reading and Sigel[9] obtained an exact solution of Eq. (2) using the potential in Eq. (3) with finite $N$ and arbitrary strengths and positions. Gumbs[10] later solved the 1D Dirac equation for arbitrary $\delta$-function scatterers. For simplicity, in this paper, we let $q_n$ have two values to simulate the randomness in the scattering strength. That is, we take $q_n = q_1,\ q_2\ (q_1 \neq q_2)$. The positions $\{x_n\}$ of the scatterers are taken as being given by the Thue-Morse (TM) sequence. Specifically, if we know the positions of the scatterers for the $n$th generation TM chain, say $\{y_1, y_2, \ldots, y_{2^n}\}$, then the positions of the scatterers for the next generation $\{x_1, x_2, \ldots, x_{2^{n+1}}\}$ are given by

$$x_i = \begin{cases} y_i & \text{for } i = 1, \ldots,\ 2^n, \\ y_{2^n} + [(b+a)(i-2^n) - y_{i-2^n}] & \text{for } i = (2^n+1), \ldots,\ 2^{n+1}\ , \end{cases} \tag{4}$$

with the known first-generation scatterers located at $\{a, b+a\}$. To define the notation and for comparison, we also list the results for the Fibonacci chain. In this case, if the positions of the scatterers for the $(n-1)$th and $n$th generations are $\{y_1, y_2, \ldots, y_{F_{n-1}}\}$ and $\{z_1, z_2, \ldots, z_{F_n}\}$, respectively, we can obtain the positions of the scatterers for the $(n+1)$th generation at $\{x_1, x_2, \ldots, x_{F_{n+1}}\}$ from

$$x_i = \begin{cases} z_i & \text{for } i = 1, \ldots,\ F_n, \\ z_{F_n} + y_{i-F_n} & \text{for } i = (F_n+1), \ldots,\ F_{n+1}\ , \end{cases} \tag{5}$$

with the first- and second-generation scatterers located at $\{b\}$ and $\{b, b+a\}$.

With the use of the 1D single-channel Landauer formula, we can calculate the dimensionless conductance $g$ of the system from

$$g = \left[\frac{G}{2e^2/h}\right]\left[\frac{T}{1-T}\right]\ , \tag{6}$$

where $T$ is the transmission coefficient through the whole quasicrystal chain. For the calculation of transmission coefficient $T$, we shall use two methods, i.e., the transfer-matrix and optical methods which are described in detail in the following subsections.

### A. Transfer-matrix method

Making use of the transfer matrix, we can obtain the relationship between the wave functions $\psi_n^{(L,R)}(x)$ on the left and right sides of the $n$th scattering barrier. A straightforward calculation yields

$$\underline{\psi}_n^{(R)}(x) = \underline{T}_n \underline{\psi}_n^{(L)}(x)\ , \tag{7}$$

where $\psi_n^{(R)}(x)$ and $\psi_n^{(L)}(x)$ are represented by the spinors of the coefficients from the plane-wave expansion of their wave functions. The *transfer matrix* $\underline{T}_n$ is given explicitly by

$$\underline{T}_n = \begin{bmatrix} 1 - iq_n/2k & -iq_n/2k \\ iq_n/2k & 1 + iq_n/2k \end{bmatrix}\ , \tag{8}$$

which is position dependent due to the two different kinds of scatterers in the system. Here, $k = \sqrt{E}$ is the wave vector along the chain direction and $E$ is the incident energy of an electron. The propagation of the right-going plane wave between the $n$th and $(n+1)$th barriers is related by

$$\underline{\psi}_{n+1}^{(L)}(x) = \underline{D}_n \underline{\psi}_n^{(R)}(x)\ , \tag{9}$$

where $\underline{D}_n$ is the *displacement matrix*, given by

$$\underline{D}_n = \begin{bmatrix} e^{iky_n} & 0 \\ 0 & e^{-iky_n} \end{bmatrix}\ , \tag{10}$$

and $y_n = (x_n - x_{n-1})$. The scattering strengths $\{q_n\}$ are distributed according to

$$q_n = \begin{cases} q_1 \text{ for } y_{n-1} = a, \\ q_2 \text{ for } y_{n-1} = b, \end{cases} \tag{11}$$

with $y_1 = x_1$. The *total transfer matrix* $\underline{M}$ for this chain with $N$ scatterers is obtained from

$$\begin{aligned} \underline{M} &= (\underline{T}_N) \otimes (\underline{D}_{N-1}\underline{T}_{N-1}) \otimes \cdots \otimes (\underline{D}_1\underline{T}_1) \\ &= \begin{bmatrix} 1/t^* & -r^*/t^* \\ -r/t & 1/t \end{bmatrix}\ . \end{aligned} \tag{12}$$

The quasiperiodic, periodic, and random systems are simulated in our model by taking the scattering sites $\{x_n\}$ to be distributed quasiperiodically, periodically, and randomly, respectively. The total transmission coefficient $T$ and the reflection coefficient $R$ can be calculated with the use of

$$T = 1/|M_{2,2}|^2 \ , \quad R = |M_{2,1}/M_{2,2}|^2 \ , \tag{13}$$

where $|t|^2 = T$ is the transmission coefficient, $|r|^2 = R = (1-T)$ is the reflection coefficient, and $M_{i,j}$ is the component of the total transfer matrix $\underline{M}$. Therefore, when the component $M_{2,1}$ of the total transfer matrix is known, we can obtain the conductance $g$ or the resistance $\rho = 1/g$ of the system. By using this transfer-matrix theory, Das Sarma and Xie[11] and Avishai and Berend[11] calculated the conductance in the Fibonacci chain with position modulation.

In Figs. 1(a) and 1(b), we plot the resistance as a function of the incident energy $E$ of an electron for the TM sequence. Two different values of the ratios of the position and strength modulation are used in these calculations. When the position-modulation ratio increases, the energy scale of the spectrum is *contracted*, but the self-similarity of the spectrum is preserved (compare the left-arrow indicated ones). On the other hand, when the strength-modulation ratio increases, the energy scale of the spectrum *expands*, and the resistance peak is broadened. The height of the resistance peak is greatly increased and becomes quite *sensitive* to the ratio of the strength modulation (see the right-arrow indicated ones). In Figs. 2(a) and 2(b), we present the resistance as a function of $E$ for the TM sequence with different ratios of the position modulation to the strength modulation. The *ratio effect* can be clearly seen in these four figures.

We emphasize that although there are several studies on the transmission coefficient for the Fibonacci sequence with position modulation,[11–13] there is comparatively little discussion of the effect of the strength modulation or the combined effect of the two modulations. Therefore, we will briefly discuss the strength modulation in the Fibonacci sequence and compare it to the TM case. The ratio effect for the Fibonacci sequence is similar to that in the TM sequence. However, for the Fibonacci sequence the magnitude of the resistance peak is *more sensitive* to the ratio of the position modulation. As the ratio of the strength modulation increases, the peak value is only *slightly reduced*. Consequently, there might be a partial cancellation of the effects of the two modulations in the case where both occur simultaneously.

### B. Optical method

The optical method has been used for calculating the transmission coefficient $T$ for a periodic arrangement of scatterers.[14] In this section, we generalize this method to 1D quasiperiodic lattice and then compare the results of our calculations for the periodic and quasiperiodic lattices with the results obtained when the transfer-matrix method is used. We begin by considering the transmission through the $n$th barrier which includes multiple electron reflections between the $(n-1)$th and $n$th barriers. After a straightforward calculation, we obtain the following recursion formula:

$$t_n = \begin{cases} t^{(1)} t_{n-1}/(1-\gamma^{(1)}\gamma_{n-1}e^{i\theta_{n-1}}) & \text{for } y_{n-1}=a \ , \\ t^{(2)} t_{n-1}/(1-\gamma^{(2)}\gamma_{n-1}e^{i\theta_{n-1}}) & \text{for } y_{n-1}=b \ , \end{cases} \tag{14}$$

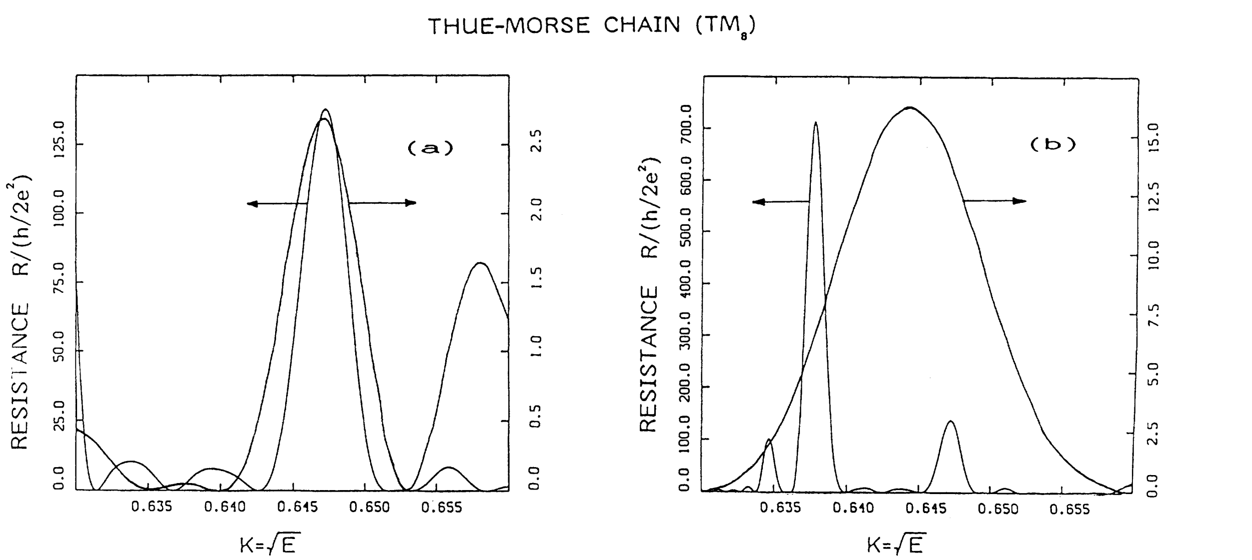


FIG. 1. Calculated resistance $R$ as a function of the incident energy $E$ of an electron for the eighth-generation Thue-Morse chain with position and scattering strength modulations, respectively. (a) $q_1 = q_2 = 0.5$ and $a = 1$, $b = \tau$ (scale on left) and $q_1 = \sqrt{2}/8$, $q_2 = 0.25$ and $a = b = 1$ (scale on right); (b) $q_1 = q_2 = 0.5$ and $a = 1$, $b = 4\tau$ (scale on left) and $q_1 = \sqrt{2}/32$, $q_2 = 0.25$ and $a = b = 1$ (scale on right).

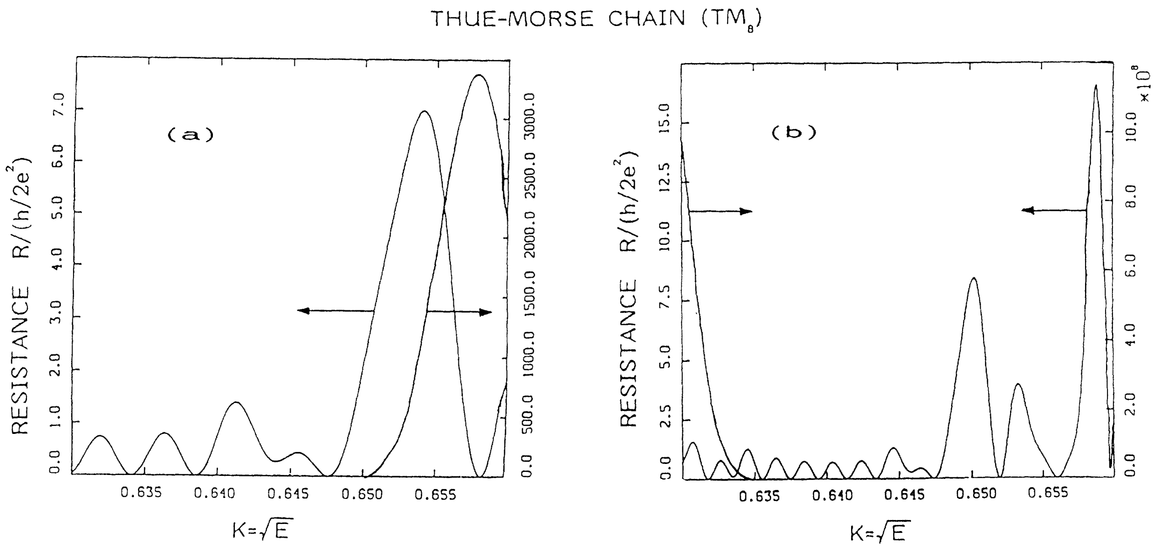


FIG. 2. Calculated resistance $R$ as a function of the incident energy $E$ of an electron for the eighth-generation Thue-Morse chain with both position and scattering strength modulations. (a) $q_1 = \sqrt{2}/4$, $q_2 = 0.5$ and $a = 1$, $b = \tau$ (scale on left) and $q_1 = \sqrt{2}/16$, $q_2 = 0.5$ and $a = 1$, $b = \tau$ (scale on right); (b) $q_1 = \sqrt{2}/4$, $q_2 = 0.5$ and $a = 1$, $b = 4\tau$ (scale on left) and $q_1 = \sqrt{2}/16$, $q_2 = 0.5$ and $a = 1$, $b = 4\tau$ (scale on right).

where $t_1 = t^{(1,2)}$ and $\gamma_1 = \gamma^{(1,2)}$. Also, $\theta_{n-1} = 2ky_{n-1}$ is the phase angle acquired by an electron after two successive reflections at the $(n-1)$th and $n$th barriers. The amplitudes for transmission and reflection $(t,\ \gamma)$ through a single $\delta$ barrier are given by

$$t^{(1,2)} = \frac{2k}{2k + iq_{1,2}} ,$$
$$\gamma^{(1,2)} = \frac{-iq_{1,2}}{2k + iq_{1,2}} . \tag{15}$$

The sequence $\{q_n\}$ is the same as in Eq. (11). Current conservation gives two additional constraints

$$t_n t_n^* + \gamma_n \gamma_n^* = 1 ,$$
$$t_n \gamma_n^* + \gamma_n t_n^* = 0 . \tag{16}$$

If the values of $(t_{n-1},\ \gamma_{n-1})$ are known, we can calculate $(t_n,\ \gamma_n)$ from

$$t_n = \frac{t^{(1,2)} t_{n-1}}{1 - \gamma^{(1,2)} \gamma_{n-1} e^{i\theta_n}} ,$$
$$\gamma_n = \left( \frac{-it_n}{|t_n|} \right) \left[ 1 - |t_n|^2 \right]^{1/2} . \tag{17}$$

The total transmission coefficient through $N$ barriers is obtained by $T = t_N t_N^*$.

In Figs. 3(a) and 3(b), we compare the results of our calculations for the resistance of a periodic chain by using the transfer-matrix and optical methods, respectively. These calculations show that there is a good agreement of the values of the height and position of each peak obtained using two methods. If more scatterers are added to the chain, less of the fine structure will be retained from the optical calculation. There is also less agreement between the results of the two calculations. In Figs. 3(c) and 3(d), we plot the calculated results of the resistance of the Fibonacci chain with the use of transfer-matrix and optical methods, respectively. Clearly, there is still some agreement both in the height and in the position of the peaks obtained with these two methods. If several more scatterers are added to the chain, the fine structure will be gradually eliminated from the optical calculations. However, the overall agreement between the results is not as good as in a periodic chain since there is more of the fine structure in a quasiperiodic chain. We conclude that the optical method could be used in both periodic and quasiperiodic chains if the number of scatterers is not very large.

## III. LOCALIZATION EFFECT

In this section, we address the main question of our investigation, consisting of which of the two chains, *Thue-Morse* or *Fibonacci*, is more localized. We answer this question by comparing the mean resistance as a function of the chain length. This will show which chain has the shorter localization length for the same chain length. We have computed the mean resistance by averaging the resistance over an interval of $k \in [k_0,\ k_0 + N\Delta k]$ with the step $\Delta k$ for different lengths $L$ or different generations:

$$\overline{\rho} = \frac{1}{N+1} \sum_{i=0}^{N} \rho_i(k_0 + i\Delta k) . \tag{18}$$

From this calculation, we are able to compare the $L$ de-

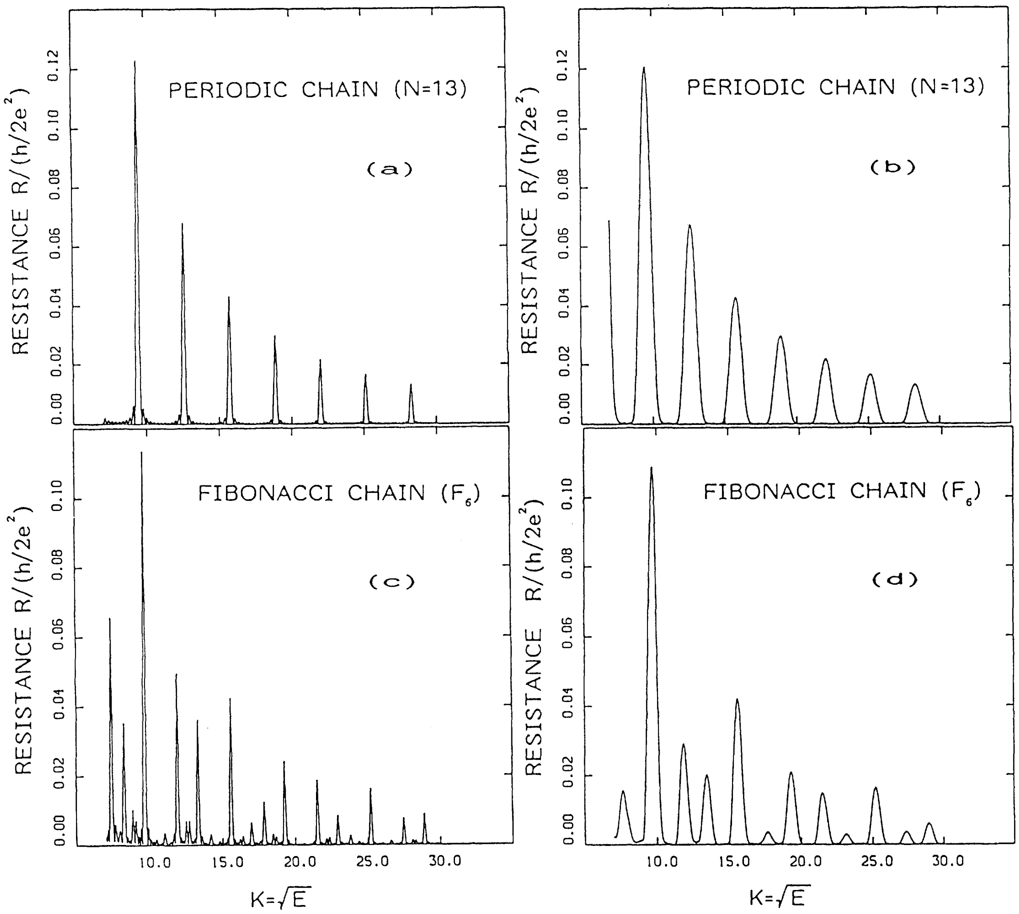


FIG. 3. Calculated resistance $R$ as a function of the incident energy $E$ of an electron for the periodic chain with 13 scatterers and the sixth-generation Fibonacci chain with position modulation by using the transfer-matrix and optical methods, respectively. (a) and (b) correspond to $q_1 = q_2 = 0.5$ and $a = b = 1$ for transfer-matrix and optical methods, respectively; (c) and (d) correspond to $q_1 = q_2 = 0.5$ and $a = 1$, $b = \tau$ for transfer-matrix and optical methods, respectively.

pendence of $\overline{\rho}$ for different systems. The so-called $\alpha$ *parameter*[15] can also be obtained from this $L$ dependence:

$$\alpha = \frac{1}{L} \ln [1 + \overline{\rho}(L)] \; . \tag{19}$$

The inverse of $\alpha$ defines the localization length $\xi$. In our numerical calculations, we take the interval to be [0.9, 1.0] and $\Delta k = 10^{-3}$.

Figures 4(a)–4(c) show plots of the mean resistance as a function of the chain length $L$ for the random, TM, and Fibonacci chains with position modulation. These calculations show that the threshold value of $L$ (indicated by a vertical arrow in the figures) is the largest for the Fibonacci chain. For the random chain, the mean resistance increases exponentially with $L$. However, for the periodic and quasiperiodic chains, this increase is according to a power law. This is clearer from Figs. 5(a)–5(c), where the localization length $\xi$ is plotted as a function of $L$. In a random chain, there are oscillations of $\xi$ only for a small range of values of $L$. As $L$ increases, there is Anderson localization[16] where $\xi$ is independent of $L$, i.e., the mean resistance increases exponentially with $L$. For the Fibonacci and TM chains, there are substantial oscillations in the entire range of values of $L$. This means that there is a power-law increase of the mean resistance with $L$. The mean localization length in the TM chain is smaller than in the Fibonacci chain. This implies that in the case of position modulation the TM chain is more localized than the Fibonacci one.

For the strength modulation in the random, TM, and Fibonacci chains, from the $L$ dependence of the mean resistance we find that the threshold values of $L$ do not mutually differ very much in all three arrangements. The main difference is that the rate of increase of the mean resistance with $L$ is the largest for the random and Fibonacci chains. The absolute magnitude of the mean resistance is much larger in the Fibonacci chain. In Figs. 6(a)–6(c), the $L$ dependence of the localization length is presented. In the random and Fibonacci chains, there are oscillations of the localization length in the short chain regime only. As $L$ increases, there is localization for which $\xi$ is independent of $L$. For the TM chain, on the other hand, there are large oscillations of the localization length which is attributed to a power-law increase

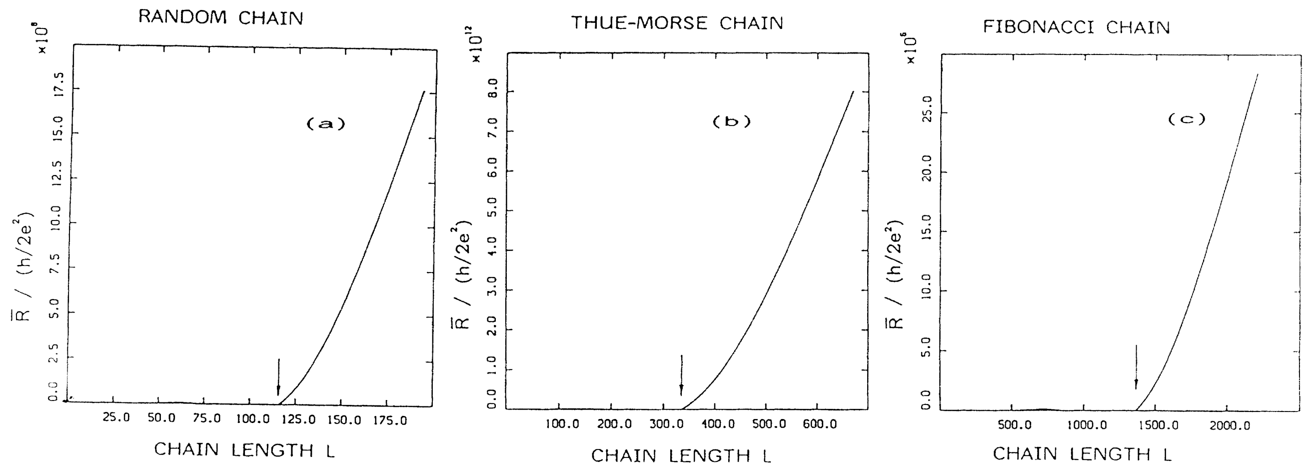


FIG. 4. Calculated mean resistance $\overline{R}$ as a function of the chain length $L$ for the random, Thue-Morse and Fibonacci chains with position modulation, respectively. (a) Random chain: $q_1 = q_2 = 0.5$ and $0 < y_i < \tau$; (b) TM chain: $q_1 = q_2 = 0.5$ and $a = 1,\ b = \tau$; (c) Fibonacci chain: $q_1 = q_2 = 0.5$ and $a = 1,\ b = \tau$.

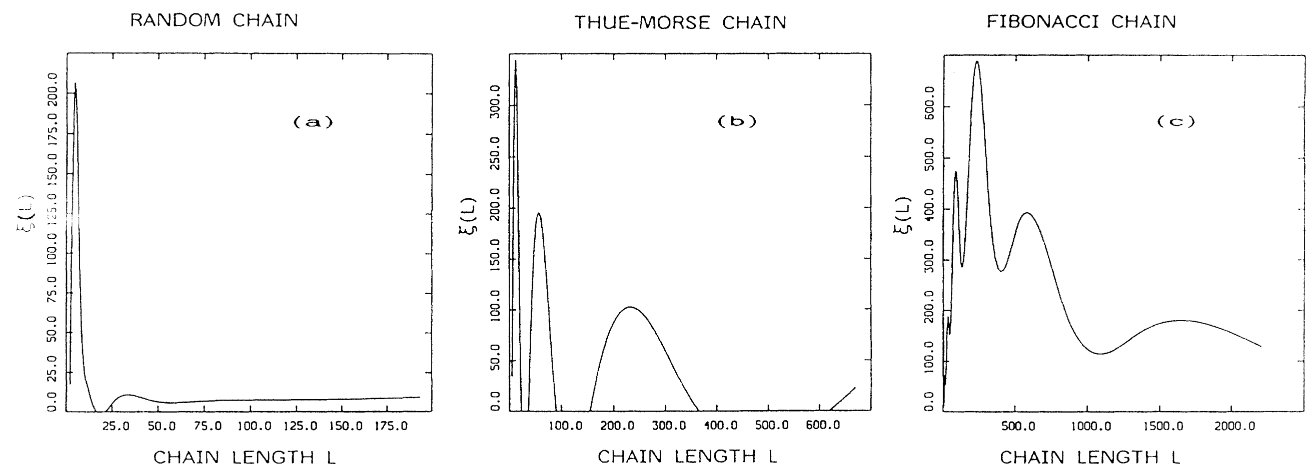


FIG. 5. Calculated localization length $\xi$ as a function of the chain length $L$ for the random, Thue-Morse, and Fibonacci chains with position modulation, respectively. (a) Random chain: $q_1 = q_2 = 0.5$ and $0 < y_i < \tau$; (b) TM chain: $q_1 = q_2 = 0.5$ and $a = 1,\ b = \tau$; (c) Fibonacci chain: $q_1 = q_2 = 0.5$ and $a = 1,\ b = \tau$.

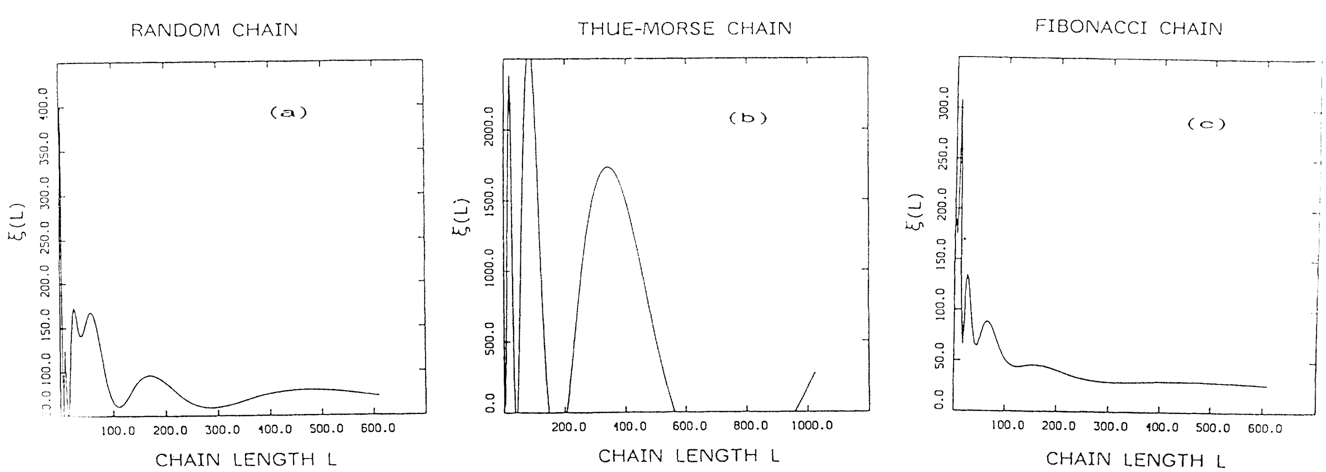


FIG. 6. Calculated localization length $\xi$ as a function of the chain length $L$ for the random, Thue-Morse, and Fibonacci chains with scattering strength modulation, respectively. (a) Random chain: $0 < q_i < 0.5$ and $a = b = 1$; (b) TM chain: $q_1 = \sqrt{2}/8,\ q_2 = 0.5$ and $a = b = 1$; (c) Fibonacci chain: $q_1 = \sqrt{2}/8,\ q_2 = 0.5$ and $a = b = 1$.

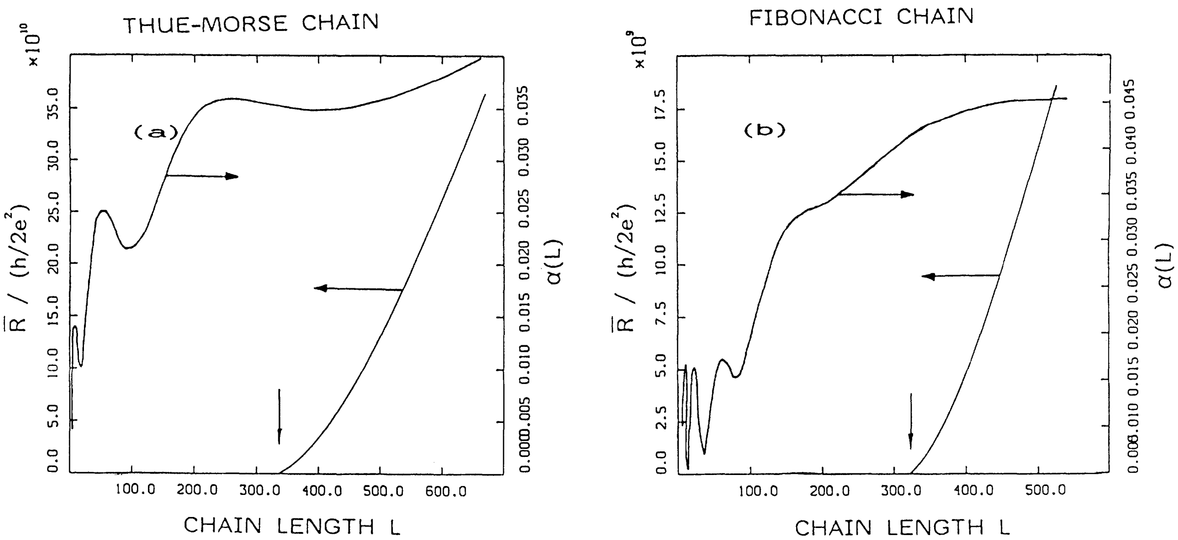


FIG. 7. Calculated mean resistance $\overline{R}$ (scale on left) and the $\alpha$ parameter (scale on right) as a function of the chain length $L$ for the Thue-Morse and Fibonacci chains with both position and scattering strength modulation. (a) TM chain: $q_1 = \sqrt{2}/8$, $q_2 = 0.5$ and $a = 1$, $b = \tau$; (b) Fibonacci chain: $q_1 = \sqrt{2}/8$, $q_2 = 0.5$ and $a = 1$, $b = \tau$.

of the mean resistance with $L$. The localization length in the Fibonacci chain is comparable to that in a random chain. This indicates that the Fibonacci chain with position modulation is almost localized. We conclude that in the case of strength modulation, the TM chain is much more extended than the Fibonacci one.

From the results presented above, we expect some cancellation of the effects of the position and strength modulation in the TM and Fibonacci chains. This is confirmed in Figs. 7(a) and 7(b), where we have shown the $L$ dependences of the mean resistance (see the left-arrow indicated ones) and the $\alpha$ parameter (see the right-arrow indicated ones) for the TM and Fibonacci chains with both position and strength modulation. The threshold values of $L$ and the rate of the resistance increase now become comparable to each other. The behavior of $\alpha$ as a function of $L$ is quite similar in these two systems. This gives strong evidence for the large cancellation of the effects of the position and strength modulation in the TM and Fibonacci chains. Based on these studies, we predict that one can switch the TM sequence from an extended state to a localized state by increasing the ratio of amplitude of the position modulation to amplitude of the strength modulation.

## IV. GENERALIZED THUE-MORSE CHAIN

We now extend our discussion to the *generalized Thue-Morse* (GTM) chain. The GTM sequence is generated by the substitution $0 \to 0^m 1^n$, $1 \to 1^n 0^m$ (see Kolář *et al.* in Ref. 5). The $l$th generation chain has $m(m+n)^{l-1}$ digits "0" and $n(m+n)^{l-1}$ digits "1." The ratio of the number of 0's to the number of 1's is a rational number $m/n$, independent of the generation index. As an example, we choose two fundamental lengths $a$ and $b$, corresponding to the numbers "0" and "1," and generate the successive GTM chains $\{ab^3, ab^3(b^3a)^3, \ldots\}$ for $m = 1$ and $n = 3$. For the $l$th generation, the length of the string is $L = (m+n)^{l-1}(ma+nb)$. Procedures for the resistance calculations developed for the original TM chains ($m = n = 1$) can also be applied to this case. If the ratio $m/n$ is very large or very small, a GTM sequence will be very close to a periodic one.

Figures 8(a)–8(c) show plots of the resistance as a function of the incident energy $E$ of an electron for the periodic, GTM, and TM chains, with position modulation. When the ratio $m/n$ is small, say $m/n = \frac{1}{15}$, the behavior of the resistance in a GTM chain looks more like its periodic counterpart. In fact, we can consider a GTM chain with very large or very small ratios of $m/n$ as being periodic with a slight disorder in the positions of the scatterers. In this way, we can switch the TM sequence from a localized to an extended state by increasing or reducing this ratio from a value of one.

In a similar way, one can obtain the so-called *generalized Fibonacci* sequence.[13,17] The $l$th generation of this generalized sequence is given by the recursion relation $S_{l+1} = S_l^m S_{l-1}^n$, where $S_0 = a$, $S_1 = b$. The corresponding generalized Fibonacci numbers, defined by the recursion formula $F_{l+1} = mF_l + nF_{l-1}$ with $F_0 = F_1 = 1$, increase exponentially. For the *golden-*, *silver-*, and *copper-*mean sequences, we have $m = n = 1$; $m = 1$, $n = 2$; and $m = 2$, $n = 1$; respectively. It has been proved[13,17] that the matrix trace maps of the golden- and silver-mean recursion formulas belong to the same universality class of dynamical systems, while the map of the copper-mean sequence has a very different dynamical behavior: it is two dimensional, area nonpreserving, and noninvertable and has a fractal invariant.[18] For a generalized Fibonacci sequence, the ratio of the number of 0's to the number of 1's depends on the generation index. This is differ-

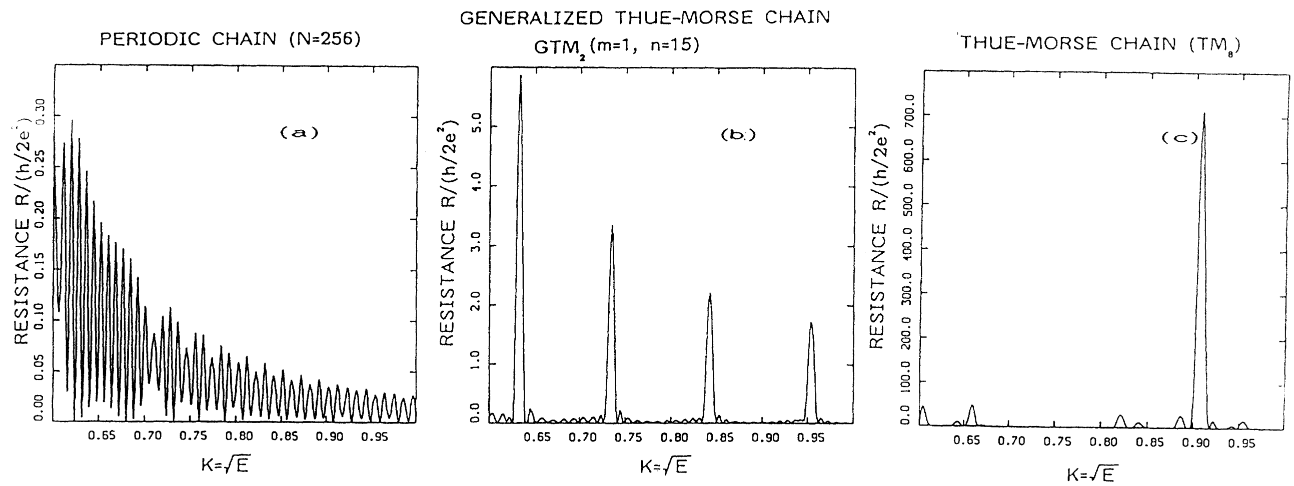


FIG. 8. Calculated resistance $R$ as a function of the incident energy $E$ of an electron for the periodic chain with 256 scatterers, the generalized Thue-Morse chain with $m/n = \frac{1}{15}$, and the eighth-generation Thue-Morse chain with position modulation, respectively. (a) Periodic chain: $q_1 = q_2 = 0.25$ and $a = b = 1$; (b) generalized TM chain: $q_1 = q_2 = 0.25$ and $a = 1,\ b = \tau$; (c) TM chain: $q_1 = q_2 = 0.25$ and $a = 1,\ b = \tau$.

ent from a GTM sequence. For the infinite generalized Fibonacci sequences, the limit of this ratio is equal to $(\sqrt{5}+1)/2$, $\sqrt{2}+1$, and 2 for the golden-, silver-, and copper-mean sequences, respectively.

In conclusion, compared to the Fibonacci chain, we have found that the TM chain is less localized for the strength modulation, and more localized for position modulation. We can switch it from an extended one to a localized one by increasing or reducing the ratio $m/n$ from one or by increasing the ratio of strength modulation to position modulation.

## ACKNOWLEDGMENTS

This work was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada and the City University of New York PSC-CUNY Research Award Program.

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