# Quantum dynamics of ultrasmall tunnel junctions: Real-time analysis

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We present a real-time path-integral analysis of the quantum dynamics of an ultrasmall tunnel junction interacting with an *arbitrary* external impedance. For a normal junction, we derive a quasiclassical Langevin equation for the phase variable and calculate the *I-V* curve *beyond* perturbation theory for the junction conductance. In the superconducting case, we develop a nonperturbative calculation of the time-dependent expectation value of the voltage operator and voltage-voltage correlation functions. Provided that dissipation is small enough, both of these quantities show damped oscillations and a power-law decay in the low-temperature limit. We also analyze the effect of resonant voltage steps on the *I-V* curve of an ac-driven tunnel junction and evaluate the linewidth of Bloch oscillations in the quantum limit.

## I. INTRODUCTION

It was understood quite early on that the phase difference $\varphi$ and the charge $Q$ of a Josephson tunnel junction are to be treated as macroscopic quantum conjugate variables with the associated uncertainty relation $\delta Q\,\delta\varphi \gtrsim e$.[1] For small capacitance junctions at low temperature, the Coulomb interaction effectively suppresses the charge fluctuations $\delta Q$ and accordingly leads to strong quantum fluctuations of the phase $\varphi$. Under these conditions, a discrete charge-transfer mechanism shows up: The junction charge can be changed only in units of $e$ due to single-electron tunneling (SET) or in units of $2e$ due to Cooper pair tunneling (CPT). For small external bias $V_x$, both SET and CPT are energetically forbidden and thus no charge transfer across the junction takes place (Coulomb blockade of tunneling). For larger values of $V_x$, the junction is periodically charged and discharged due to SET and CPT events. As a result, coherent voltage oscillations with fundamental frequencies $I/e$ and $I/2e$ (so-called SET and Bloch oscillations) occur, where $I$ is the current across the junction. For a review, we refer the reader to Refs. 2 and 3.

Modern lithographic techniques allow reliable fabrication of tunnel junctions with capacitances in the range $C \lesssim 10^{-15}$–$10^{-16}$ F, thus opening the possibility to investigate the above-mentioned effects at temperatures below $T \sim 1$ K. Coulomb-blockade effects have been clearly demonstrated in experiment[4–6] (see also references cited in Refs. 2 and 3). SET oscillations of the voltage have been also observed in chains of Josephson junctions.[7] Very recently, experimental evidence of Bloch oscillations has been reported.[8,9] These experimental findings confirm the main theoretical predictions made in Refs. 2 and 3.

In any real experimental situation, a tunnel junction always interacts with an external electromagnetic environment. Though not very important for many-junction systems, this interaction can play a crucial role in experiments with single junctions (e.g., Refs. 8–10). This problem has already been discussed in a number of papers[2,3,11–18] for both normal and superconducting junctions and, in some limiting cases, a coherent theoretical picture emerged. Nevertheless several important issues still remain to be clarified. For example, of fundamental importance is the question about the influence of an arbitrary dissipative environment on the Bloch oscillations. Do these oscillations survive or are they destroyed in the weak-dissipation limit (as was suggested in Ref. 19)? What is the role of quantum fluctuations of the charge? Is it possible to study these problems perturbatively (e.g., as in Refs. 2 and 20) or does perturbation theory fail even for small dissipation?[19] These and some other problems will be investigated in this paper.

In Sec. II, we briefly outline a general path-integral formalism which we use to describe the quantum dynamics of both the phase and charge variables of a tunnel junction shunted by an *arbitrary* external impedance $Z_S(\omega)$. In Sec. III we derive a quasiclassical-Langevin equation *for the phase* in the presence of two types of noise: junction shot noise and quantum noise of an external circuit. Using this approach, we calculate the current-voltage characteristic of a normal tunnel junction with arbitrary tunneling resistance $R_t$ and arbitrary external impedance $Z_s(\omega)$. Section IV is devoted to a detailed analysis of the charge dynamics of a superconducting junction in the Bloch-oscillation regime. Eliminating the phase variable, we derive a quasiclassical-Langevin equation *for the charge* and evaluate the current-voltage characteristic as well as the linewidth of Bloch oscillations (in both classical and quantum limits) for arbitrary $Z_s(\omega)$. For large Josephson coupling energy, it is possible to go beyond the quasiclassical approximation and solve the problem exactly. In this limit, we present a direct calculation of the time-dependent expectation value of the voltage operator in the Bloch-oscillations regime and evaluate the voltage-voltage correlation functions. We also describe the effect of resonances (voltage steps) on the *I-V* curve in the presence of an external ac signal. The results of this section are of particular importance in view of recent experimental findings.[8,9] In Sec. V we briefly discuss the main results of this paper and give a comparison with available experimental data. Some technical details of our theory are presented in the Appendix.

## II. GENERAL FORMALISM

We shall consider a Josephson tunnel junction shunted by an external impedance $Z_s(\omega)$. Depending on the experimental conditions, either the voltage source $V_x$ [Fig. 1(a)] or the current source $I_x$ [Fig. 1(b)] configuration might be relevant. We shall describe below both configurations within the same formalism. As in Ref. 3, we start from the microscopic Hamiltonian for the "junction+environment" system. After taking a trace over the electron degrees of freedom (see Ref. 3 for details), we arrive at an expression for the reduced density matrix in the phase representation, $\rho(\varphi_1,\varphi_2;t)$. Within the Feynman-Vernon path-integral formalism,[21] we introduce the phase variables $\varphi_1$ and $\varphi_2$ related, respectively, to forward and backward time contours (it is necessary to introduce two $\varphi$ variables, instead of one, because we are dealing with the density matrix rather than the amplitude) and get for the configuration of Fig. 1(b) (see e.g., Ref. 3),

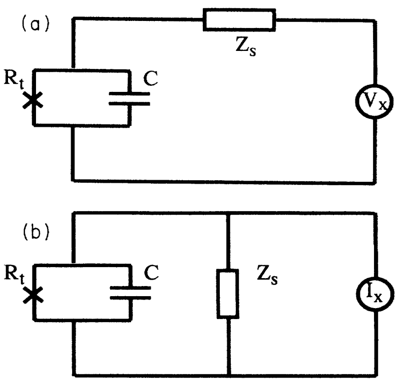


FIG. 1. Voltage source (a) and current source (b) configurations.

$$\rho(t,\varphi_f)=\int d\varphi_i J(t,\varphi_f,\varphi_i)\rho(0,\varphi_i),\quad \varphi_f=(\varphi_{1f},\varphi_{2f}),\varphi_i=(\varphi_{1i},\varphi_{2i})\,, \tag{1}$$

$$J(t,\varphi_f,\varphi_i)=\int D\varphi_1 D\varphi_2 \exp(i\{S_0[\varphi_1]-S_0[\varphi_2]+\tilde{W}[\varphi_1,\varphi_2]+W[\cos(\varphi_1/2),\cos(\varphi_2/2)]+W[\sin(\varphi_1/2),\sin(\varphi_2/2)]\})\,, \tag{2}$$

where

$$S_0[\varphi]=\int_0^t\left[\frac{C}{2}\left[\frac{\dot\varphi}{2e}\right]^2+E_J\cos\varphi+\frac{I_x}{2e}\varphi\right]d\tau\,, \tag{3}$$

$$W[\chi_1,\chi_2]=-\frac{1}{R_te^2}\int_0^t\dot\chi^+\dot\chi^-d\tau+\frac{i}{2R_te^2}\int_0^t d\tau\int_0^t ds\,\chi^-(\tau)G(\tau-s)\chi^-(s)\,, \tag{4}$$

$$G(t)=\int\omega\coth\left[\frac{\omega}{2T}\right]\exp(-i\omega t)\frac{d\omega}{2\pi}=\left[\frac{\pi Tt}{\sinh(\pi Tt)}\right]^2\frac{1d}{\pi\,dt}\mathrm{P}\frac{1}{t}\,,$$

$$\tilde{W}[\varphi_1,\varphi_2]=-\int_0^t d\tau\int_0^t ds\frac{\varphi^{-1}(\tau)}{2e}\hat{Z}_S^{-1}(\tau-s)\frac{\dot\varphi^+(s)}{2e}+\frac{i}{2}\int_0^t d\tau\int_0^t ds\frac{\varphi^-(\tau)}{2e}\tilde{G}(\tau-s)\frac{\varphi^-(s)}{2e}\,, \tag{5}$$

$$\hat{Z}_s^{-1}(t)=\int\frac{d\omega}{2\pi}\frac{1}{Z_s(\omega)}\exp(-i\omega t),\quad \tilde{G}(t)=\int\frac{d\omega}{2\pi}\mathrm{Re}\left[\frac{\omega}{Z_S(\omega)}\right]\coth\left[\frac{\omega}{2T}\right]\exp(-i\omega t)\,.$$

Here we set $t=1$, introduce $\chi^+=(\chi_1+\chi_2)/2$, $\chi^-=\chi_1-\chi_2$ and an implied analogous definition for any other "mixed" variable here and below. The term $S_0[\varphi]$ [Eq. (3)] represents the contribution from a Josephson junction that interacts with a current source $I_x$. Due to the presence of an external circuit, the junction charge changes continuously and thus the phase $\varphi$ has to be treated as an extended variable.[3,22] This variable is linked to the voltage $V(t)$ across the junction by the usual relation $\dot\varphi=2eV(t)$, $C$ and $E_J$ are, respectively, the capacitance and the Josephson coupling energy of the junction. The nonlocal terms $W$ [Eq. (4)] and $\tilde{W}$ [Eq. (5)] represent contributions, respectively, from single-electron tunneling across the junction and continuous charge flow in the external circuit. Both these terms come from integration over the electron degrees of freedom either localized in the vicinity of a junction (in which case, the term $W[\cos\varphi_1/2,\cos\varphi_2/2]+W[\sin\varphi_1/2,\sin\varphi_2/2]$ follows) or belonging to the external leads (yielding the term $\tilde{W}[\varphi_1,\varphi_2]$), both describe dissipation and noise (shot noise for $W$ and Gaussian noise for $\tilde{W}$). For a normal junction, the quantity $R_t$ is equal to the normal-state tunneling resistance. For a superconducting junction at frequencies well below the superconducting gap and at low temperature, $R_t$ coincides with the junction subgap resistance.

To decouple the phase variables $\varphi_1$ and $\varphi_2$ on forward and backward time contours, we introduce additional path integrals over new variables, $q$, $v$, and $w$ and rewrite (2) in the form

$$J(t,\varphi_f,\varphi_i)=\int dq_f dq_i dv_f dv_i dw_f dw_i \int D\mathbf{q}\, D\mathbf{v}\, D\mathbf{w}\, D\boldsymbol{\varphi} \exp(iA[\boldsymbol{\varphi},\mathbf{q},\mathbf{v},\mathbf{w}]) , \tag{6}$$

$$A[\boldsymbol{\varphi},\mathbf{q},\mathbf{v},\mathbf{w}]=S[\varphi_1,q_1,v_1,w_1]-S[\varphi_2,q_2,v_2,w_2]+\int_0^t V_x q^-(\tau)d\tau+W_q[q_1,q_2]+R_t W[v_1,v_2]+R_t W[w_1,w_2] , \tag{7}$$

$$W_q[q_1,q_2]=\int_0^t d\tau \int_0^t ds[-q^-(\tau)\hat{Z}_s(\tau-s)\dot{q}^{+}(s)+\frac{i}{2}q^-(\tau)G_q(\tau-s)q^-(s)] ,$$

$$\hat{Z}_s(t)=\int Z_s(\omega)\exp(-i\omega t)\frac{d\omega}{2\pi}, \quad G_q(t)=\int \omega\coth\frac{\omega}{2T}\mathrm{Re}[Z_s(\omega)]\exp(-i\omega t)\frac{d\omega}{2\pi} , \tag{8}$$

$$S[\varphi,q,v,w]=\int_0^t \left[ \frac{C}{2}\left[\frac{\dot{\varphi}}{2e}\right]^2+E_J\cos\varphi+\frac{\dot{q}}{2e}\varphi+\frac{\dot{v}}{e}\cos\left[\frac{\varphi}{2}\right]+\frac{\dot{w}}{e}\sin\left[\frac{\varphi}{2}\right]\right]d\tau ,$$

where we define $V_x(t)=\int_0^t\hat{Z}_s(t-s)I_x(s)ds$, set $q^+(0)=q^-(t)=0$, $q^+(t)=q_f$, and $q^-(0)=q_i$, and adopt analogous boundary conditions for the variables $v$ and $w$. Then evaluation of the path integrals over the phase variables can be fulfilled separately for $\varphi_1$ and $\varphi_2$. Furthermore, bearing in mind a representation for the kernel of the evolution operator,

$$\int_{\varphi_i}^{\varphi_f} D\varphi \exp(iS[\varphi,q,v,w]) = \left\langle \varphi_f \left| \hat{T}\exp\left[-i\int_0^t \hat{H}[q,v,w]d\tau\right]\right| \varphi_i \right\rangle , \tag{9}$$

we can reduce each of these integrations to a quantum-mechanical problem with the Hamiltonian

$$\hat{H}[q,v,w]=\hat{H}_0-\frac{\dot{v}}{e}\cos\left[\frac{\varphi}{2}\right]-\frac{\dot{w}}{e}\sin\left[\frac{\varphi}{2}\right]-\frac{\dot{q}(\tau)\varphi}{2e} , \tag{10}$$

where

$$\hat{H}_0=-\frac{1}{2C}\frac{\partial^2}{\partial(\varphi/2e)^2}-E_J\cos\varphi \tag{11}$$

is the Hamiltonian of a quantum-mechanical particle $\varphi$ in a $2\pi$-periodic potential, $-E_J\cos\varphi$. It leads to the usual picture of Bloch states $\psi_q(\varphi)=u_q(\varphi)\exp(iq\varphi/2\pi)$, $u_q(\varphi+2\pi)=u_q(\varphi)$, and $2e$-periodic bands. The terms $-(\dot{v}/e)\cos(\varphi/2)$ and $-(\dot{w}/e)\sin(\varphi/2)$ represent the contribution to the potential energy from the SET process, where $v$ and $w$ are stochastic collective variables describing this process (the corresponding correlation functions will be defined below). These terms are $4\pi$ periodic in $\varphi$ and thus the corresponding Brillouin zones are $e$ periodic. The last term, $-\dot{q}\varphi/2e$, describes the interaction of the phase $\varphi$ with a current $\dot{q}$ in the external circuit. It violates the translational invariance $\varphi\to\varphi+2\pi$ of the Hamiltonian (11).

Note that Eqs. (6)–(11) describe both configurations of Figs. 1(a) and 2(b). The variable $\dot{q}$ describing the current across the external impedance $Z_S(\omega)$ of Fig. 1(a) is linked to the analogous variable $\dot{q}\,'$ for the configuration of Fig. 1(b) by an obvious relation $\dot{q}=I_x+\dot{q}\,'$. Below, we shall switch freely our analysis between these two configurations.

Reformulation of our problem in terms of the Schrödinger equation with the Hamiltonian (10) and (11) is particularly useful in the case of a superconducting junction, provided that we can neglect both Zener tunneling and thermal activation to higher zones and confine our description to the lowest Brillouin zone. In this case, we can integrate out the $\varphi$ variables and reduce the problem to that of the quantum dynamics of charge variables $q$, $v$, and $w$. This analysis is presented in the Appendix.

## III. NORMAL JUNCTION: QUANTUM LANGEVIN EQUATION FOR THE PHASE

To describe the quantum dynamics of a normal tunnel junction it is convenient for us to start from Eqs. (6)–(8), where we set $E_J=0$. Provided that the voltage, $V=\dot{\varphi}/2e$, across the junction is large enough, the phase dynamics is nearly classical. In this case, we can evaluate the path integral over $\varphi$ within the classical approximation. Proceeding in much the same way as in Ref. 23, we assume fluctuations of $\varphi^-=\varphi_1-\varphi_2$ to be small. Then we get ($\varphi^+\equiv\varphi$)

$$\frac{\delta A}{\delta\varphi}=-C\frac{\ddot{\varphi}}{2e}+\dot{q}-\dot{v}\sin\left[\frac{\varphi}{2}\right]+\dot{w}\cos\left[\frac{\varphi}{2}\right]=0 . \tag{12}$$

This equation regulates the balance of currents in our circuit. Integrals over the charge variables are Gaussian and can be handled without problem. As a result, we obtain

$$\frac{\delta A}{\delta q^-}=\int\frac{d\omega}{2\pi}\exp(-i\omega t)\left[i\omega Z_s(\omega)q(\omega)+i\omega\frac{\varphi(\omega)}{2e}\right] =\xi_q(t) , \tag{13a}$$

$$\frac{\delta A}{\delta v^-}=-R_t\dot{v}+\frac{\dot{\varphi}}{2e}\sin^2\left[\frac{\varphi}{2}\right]=\xi_v(t) , \tag{13b}$$

$$\frac{\delta A}{\delta w^-}=-R_t\dot{w}-\frac{\dot{\varphi}}{2e}\cos\left[\frac{\varphi}{2}\right]=\xi_w(t) , \tag{13c}$$

where $\xi_v$, $\xi_w$, and $\xi_q$ are Gaussian stochastic variables, with

$$\langle\xi_v(t_1)\xi_v(t_2)\rangle=\langle\xi_w(t_1)\xi_w(t_2)\rangle=R_tG(t_1-t_2) , \tag{14a}$$

$$\langle\xi_q(t_1)\xi_q(t_2)\rangle =\int\frac{d\omega}{2\pi}\omega\coth\left[\frac{\omega}{2T}\right]\mathrm{Re}[Z_s(\omega)]\exp[-i\omega(t_1-t_2)] . \tag{14b}$$

Combining (12) and (13), we arrive at the quasiclassical-Langevin equation for the phase

$$C\frac{\ddot{\varphi}}{2e}+\frac{1}{R_t}\frac{\dot{\varphi}}{2e}+\int \hat{Z}_s^{-1}(t-\tau)\frac{\dot{\varphi}(\tau)}{2e}d\tau-\frac{V_x}{Z_s(0)} =\frac{\xi_v(t)}{R_t}\sin\left[\frac{\varphi}{2}\right]+\frac{\xi_w(t)}{R_t}\cos\left[\frac{\varphi}{2}\right]+\xi_s(t)\ , \quad (15)$$

$$\langle \xi_S(t_1)\xi_S(t_2)\rangle =\int\frac{d\omega}{2\pi}\coth\left[\frac{\omega}{2T}\right]\mathrm{Re}\left[\frac{\omega}{Z_s(\omega)}\right]\exp[-i\omega(t_1-t_2)]\ .$$

The first two terms in the right-hand side of (15) describe the shot noise of a tunnel junction (see also Ref. 24), while the last term describes the Gaussian noise of an external circuit. Averaging Eq. (15), over all possible realizations of $\xi_v(t)$, $\xi_w(t)$, and $\xi_s(t)$ we arrive at the current-voltage characteristic

$$I=\frac{1}{R_t}\{V-\langle[\xi_v\sin(\varphi/2)+\xi_w\cos(\varphi/2)]\rangle\}\ , \quad (16)$$

where $I$ and $V$ are, respectively, the current and the voltage across the junction.

The average in (16) can be easily calculated, provided that we can treat the noise terms perturbatively. This can be done if the phase diffusion is slow enough, i.e., if the average deviation of the phase from its deterministic value for the period of order $\delta t\sim(eV)^{-1}$ is much smaller than 1. Accordingly, the total impedance $Z(\omega)$ given by

$$\frac{1}{Z(\omega)}=\frac{1}{R_t}-i\omega C+\frac{1}{Z_S(\omega)}$$

should obey the condition

$$\int\coth\left[\frac{\omega}{2T}\right]\mathrm{Re}[Z(\omega)]\frac{1-\cos(\omega t)}{\omega}\frac{d\omega}{2\pi}\ll R_q \quad (17)$$

for all values $t\lesssim 2\pi/eV$, $R_q=\pi/2e^2\simeq 6.5$ k$\Omega$. Combining Eq. (16) with inequality (17), one can easily derive the $I$-$V$ characteristic of a normal tunnel junction shunted by an arbitrary external impedance $Z_s(\omega)$,

$$I=\frac{V}{R_t}-\frac{e}{\pi R_t}\int_0^\infty K(t)[\pi T/\sinh(\pi T t)]^2\sin(eVt)dt\ , \quad (18)$$

where

$$K(t)=i\int\frac{Z(\omega)}{\omega+i0}\exp(-i\omega t)\frac{d\omega}{2\pi}\ . \quad (19)$$

Calculating the second derivative $d^2I/dV^2$ at $T=0$ we can rewrite the result (18) and (19) in a particularly simple form

$$\frac{d^2I}{dV^2}=\frac{e^2}{\pi R_t V}\mathrm{Re}[Z(eV)]\ . \quad (20)$$

In the limit of large voltages

$$\int_{eV}^\infty\frac{d\omega}{\omega}\mathrm{Re}[Z(\omega)]\ll R_q\ ,$$

we obtain the following asymptotic form of the $I$-$V$ curve

$$I=\frac{V}{R_t}-\frac{e}{2R_t}\left[\mathrm{P}\int Z(\omega)\frac{d\omega}{2\pi}\right]=\frac{1}{R_t}\left[V-\frac{e}{2C_{\mathrm{eff}}}\right]\ , \quad (21)$$

where we denote $C_{\mathrm{eff}}=C+\lim_{\omega\to\infty}[i/\omega Z_s(\omega)]$. Equation (21) shows that for any $R_t$ and $Z_s(\omega)$ [provided that $1/Z_s(\omega)\propto\omega^\nu$, $\nu\le 1$] there is a universal offset (Coulomb gap) $\Delta V=e/2C_{\mathrm{eff}}$ on the $I$-$V$ curve for large $V$. This result can be used for a reliable experimental estimation of the effective capacitance $C_{\mathrm{eff}}$.

Let us briefly discuss several special choices for the external impedance $Z_s(\omega)$. For a purely inductive impedance $Z_s(\omega)=-i\omega L$ at $T=0$ and $R_tC>\sqrt{LC/2}$, we get, from (21),

$$I=\frac{V}{R_t}-\frac{e}{\pi R_t C y}\left[\frac{y+eV}{2}\arctan\left[\frac{eV+y}{x}\right]+\frac{y-eV}{2}\arctan\left[\frac{eV-y}{x}\right]+\frac{x}{4}\ln\left[\frac{(eV-y)^2+x^2}{(eV+y)^2+x^2}\right]\right]\ , \quad (22)$$

where we define $x=1/2R_tC$, $y=\sqrt{\omega_1^2-x^2}$. Criterion (17) for result (22) reads $E_c\ll\omega_1$. In the limit $R_t\gg R_q$, Eq. (22) reduces to $IR_t=V-e\kappa/2C$, where $\kappa=eV/y$ for $0<eV<y$ and $\kappa=1$ for $eV>y$ (cf., Ref. 14)]. The physical reason for the sharp crossover at $eV=y$ is transparent: For $eV<y$, an environmental mode $\omega_1$ cannot be excited by a tunneling electron and therefore only elastic SET takes place, while for $eV>y$, this electron can create a quantum $\omega_1$ and SET process becomes inelastic. For smaller values of $R_t\lesssim R_q$, this crossover is smoothened (see Fig. 2) because of intensive virtual electron tunneling across the junction.

For a purely Ohmic impedance, $Z_s(\omega)=R_s$, we get, from (18) and (19),

$$K(t)=R_0(1-\exp(-t/R_0C)),\quad 1/R_0=1/R_t+1/R_s. \quad (23)$$

After a proper translation between current and voltage source configurations, this result practically coincides with that derived by Odintsov[12] within the framework of a different technique. However, in some cases the criteria for (18) and (23) differ from those obtained in Ref. 12. To derive these conditions, one has to combine inequality

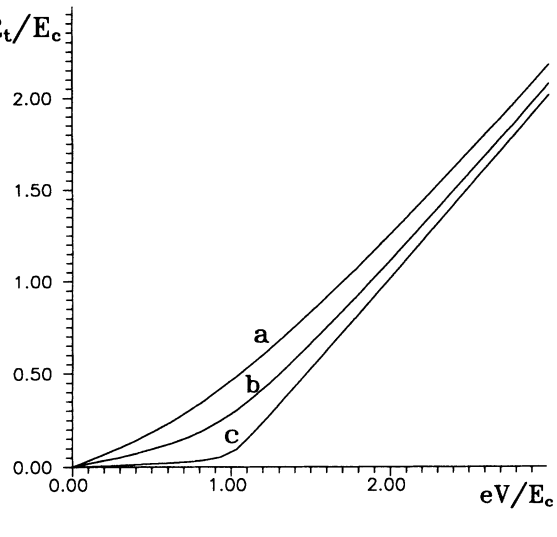


FIG. 2. The $I$-$V$ curves for a normal tunnel junction coupled to an inductive external impedance $Z_s(\omega)=-i\omega L$. The curves correspond to $\alpha_t \simeq 1.3$ (a), $\alpha_t \simeq 0.5$ (b), and $\alpha_t \simeq 0.1$ (c).

(17) with the criterion for the quasiclassical-Langevin-equation approach developed here. We analyze the latter condition, making use of the results of Ref. 18. Let us consider the case $T=0$ in which (23) reduces to[12]

$$I=\frac{V}{R_t}\left\{1-\frac{1}{4(\alpha_s+\alpha_{t1})}\ln\left[1+\left[\frac{1}{eVR_0C}\right]^2\right]-\frac{2E_c}{\pi eV}\arctan(eVR_0C)\right\}, \tag{24}$$

where we have defined $\alpha_s=R_q/R_s$ and $\alpha_{t1}=\pi^2\alpha_t/4=R_q/R_t$. For $\alpha_{t1}\lesssim 1$, both criteria coincide and yield $eV \gg E_c=e^2/2C$ for $\alpha_s \lesssim 1$ and $eV\gg\alpha_s E_c\exp(-2\alpha_s)$ for $\alpha_s \gg 1$. For lower voltages, the $I$-$V$ curve deviates from (24) and (at least for $\alpha_s\lesssim 1$) shows the Coulomb-blockade feature (see Ref. 14 to recover the complete $I$-$V$ curve in the limit $\alpha_{t1}\ll 1$). For $\alpha_t \gg 1$, the criterion for the Langevin-equation approach is more stringent than that defined by (17). By corresponding nonperturbative analysis that has been made in Ref. 18, there is a metal-insulator phase transition in our system that takes place at $\alpha_s=\alpha_{sc}=1/[4-1/a_{t1})]$. For $\alpha_s>\alpha_{sc}\simeq 1/4$ and small voltages, the junction behaves like an Ohmic resistance, which (in the main approximation in $1/\alpha_{t1}$) is equal to $R_t$:

$$I\simeq V/R_t\ . \tag{25}$$

In this regime, Coulomb blockade is completely destroyed by charge fluctuations in the Ohmic shunt. Equation (25) holds for $V$ satisfying

$$eV \lesssim (\alpha_{t1}+\alpha_s)E_c\exp[-2(\alpha_{t1}+\alpha_s)]$$

while result (24) is valid for larger values of $V$. For $\alpha_s<\alpha_{sc}$ the junction behavior is entirely different. As has been shown in Ref. 18 for $eV\lesssim\Delta_t\sim\alpha_t^2E_c\exp(-2\alpha_{t1})$, Coulomb blockade takes place. It is only partially destroyed by quantum fluctuations of the charge and the junction effective resistance is very high in this regime. For $\alpha_s\ll 1$, there is an additional parameter region, $\Delta_t\lesssim eV\ll\Delta_t/\alpha_s$, in which the system shows SET oscillations. The corresponding part of the $I$-$V$ curve has been derived in Ref. 18. For higher voltages, $\Delta_t/\alpha_s\lesssim eV\lesssim E_c$, the renormalization-group analysis of Ref. 18 yields, with logarithmic accuracy,

$$I=\frac{V}{R_t}\left[1+\frac{1-4\alpha_0}{2\alpha_{t1}}\ln\left[\frac{eV}{E_c}\right]\right],\quad \alpha_0=\frac{R_q}{R_s+R_t}\ . \tag{26}$$

The analysis presented here clearly illustrates the qualitatively different roles of Ohmic and tunneling conductances in our problem. Indeed, in the limit $\alpha_s\gg 1$ quantum fluctuations of $\varphi$ are strongly suppressed and the $I$-$V$ curve is described by the quasiclassical result (24) for

$$eV\gg(\alpha_{t1}+\alpha_s)E_c\exp[-2(\alpha_{t1}+\alpha_s)]$$

and by Eq. (25) for smaller values of $V$. On the other hand, for $\alpha_s\lesssim 1$ the quasiclassical approach (15) is adequate only for large voltages $eV\gtrsim E_c$, *however large $a_{t1}$ may be.* For $eV\ll E_c$, a more sophisticated technique[18] is needed to describe nontrivial effects of quantum fluctuations of the phase.

## IV. SUPERCONDUCTING JUNCTION: THE QUANTUM DYNAMICS OF THE CHARGE

In the preceding section we analyzed the quantum dynamics of a normal tunnel junction within the quasiclassical approximation for the phase and derived the corresponding Langevin equation for $\varphi$. In the case of a superconducting junction, we are mostly interested in the regime of strong quantum fluctuations of $\varphi$. Therefore we make use of a different approximation and consider the quantum dynamics of the charge restricted to the lowest Brillouin zone. Provided that we neglect both Zener tunneling and thermal activation to higher zones, it becomes possible to integrate out the phase variable and reformulate the problem in terms of charges (see Appendix for details).

### A. Quantum Langevin equation for the charge, $I$-$V$ curves and linewidth of Bloch oscillations

Let us first neglect SET effects and set $\alpha_t=0$. Then the $\varphi$-dependent part of the problem reduces to that of a quantum-mechanical particle $\varphi$ in a tilted $2\pi$-periodic potential $-E_J\cos\varphi-\dot{q}\varphi/2e$. The solution of the corresponding Schrödinger equation with the Hamiltonian (10) and (11) with $v=w=0$ becomes trivial and we get (see also the Appendix)

$$c_1(t)=\exp\left[-i\int_0^t E_0(q(\tau))d\tau\right],\quad c_2(t)=0\ .$$

Here $E_0(q)$ is the ground-state energy of the Hamiltonian (11) and $c_{1,2}$ are the amplitudes of the Bloch states $\psi_p^{(1)}(\varphi)=\psi_p(\varphi)$, $\psi_p^{(2)}(\varphi)=\psi_{p+e}(\varphi)$, $\psi_k(\varphi)=\exp(ik\varphi/2e)u_k(\varphi)$, $u_k(\varphi)=u_k(\varphi+2\pi)$, and $p(t)=k+q(t)-q(0)$. Then we immediately arrive at a simplified version of the equation for the voltage

$$\langle V(t)\rangle=\int dq\frac{dE_0}{dq}f(q,t)\ , \tag{27}$$

$$f(q,t)=\int dq_{i1}\,dq_{i2}\,\rho_0(q_{i1},q_{i2})\int_{q_{i1}}^{q}\int_{qi2}^{q}Dq_1\,Dq_2\exp(iS[q_1,q_2])\ , \tag{28}$$

$$S[q_1,q_2]=\int_0^t[-E(q_1)+E(q_2)+V_xq^-]d\tau+\int_0^t d\tau\int_0^t ds[-q^-(\tau)\hat{Z}_s(\tau-s)\dot{q}^+(s)+\tfrac{1}{2}iq^-(\tau)G_q(\tau-s)q^-(s)]\ . \tag{29}$$

Note that Eq. (29) again recovers the duality between the charge and the phase representations of the effective action.[25,17] This property of our problem becomes particularly clear if we set $Z_s(\omega)=R_s-i\omega L$.

To proceed further we shall treat quantum fluctuations of the charge $q$ within the quasiclassical approximation. Again after a slight generalization of the analysis[23] we get

$$\int_{-\infty}^{t}\hat{Z}_s(t-\tau)\dot{q}(\tau)d\tau+\frac{dE_0}{dq}=V_x+\xi(t),\quad \langle|\xi(\omega)|^2\rangle=\omega\mathrm{Re}[Z_s(\omega)]\coth\left[\frac{\omega}{2T}\right]\ . \tag{30}$$

This is a quantum Langevin equation for the charge [*not the phase*—cf., Eq. (15)] variable. In a special case $Z_s(\omega)=R_s-i\omega L$ Eq. (3) reads[17]

$$L\ddot{q}+R_s\dot{q}+\frac{dE_0}{dq}=V_x+\xi(t)\ . \tag{30a}$$

In this case the $I$-$V$ curves were analyzed in Refs. 20 (for $L=0$) and 17 (for $L\neq 0$). Here we derive the current-voltage characteristic directly from Eq. (30). For $V_x<V_c=(dE_0/dq)_{\max}$ Coulomb blockade of tunneling takes place and

$$I=\langle\dot{q}\rangle=0,\quad V=V_x\ . \tag{31}$$

For $V_x>V_c$, the charge $q$ moves in a tilted periodic potential $E_0(q)-V_x(q)$ and the junction voltage oscillates in time (Bloch-oscillation regime). In the limit $V_x\gg V_c$ and/or $T\gg eV_c$, one can treat the nonlinear term $dE_0/dq$ perturbatively. In doing so, we substitute the solution of Eq. (30)

$$\begin{gathered}q(t)=q_0+\frac{V_x}{Z_s(0)}t+2\frac{e^2}{\pi^2}\int_{-\infty}^{t}[k_s(t-\tau)-k_s(-\tau)]\left[\xi(\tau)-\frac{dE_0}{dq}\right]d\tau\ ,\\ k_s(t)=\frac{i\pi^2}{2e^2}\int\frac{\exp(-i\omega t)}{Z_s(\omega)(\omega+i0)}\frac{d\omega}{2\pi}\end{gathered} \tag{32}$$

into the expression for the $I$-$V$ curve

$$I=\langle\dot{q}\rangle=\frac{V_x}{Z_s(0)}-\frac{1}{Z_s(0)}\left\langle\frac{dE_0}{dq}\right\rangle \tag{33}$$

and expand it in powers of $dE_0/dq$. Making use of the equation

$$\left\langle\exp\left[2i\frac{en}{\pi}\int[k_s(t-x)-k_s(\tau-x)]\xi(x)dx\right]\right\rangle=\exp[-n^2f_s(t-\tau)]\ ,$$

where

$$f_s(t)=\left[\frac{\pi}{e}\right]^2\int\frac{d\omega}{2\pi}\mathrm{Re}\left[\frac{1}{Z_s(\omega)}\right]\coth\left[\frac{\omega}{2T}\right]\frac{1-\cos(\omega t)}{\omega}\ , \tag{34}$$

after straightforward algebraic manipulations, we get

$$I=\frac{V_x}{Z_s(0)}-\frac{1}{Z_s(0)}\sum_{m=1}^{\infty}\frac{\pi m^3a_m^2}{e}\int_0^\infty\exp[-m^2f_s(t)]k_s(t)\sin\left[\frac{\pi V_xm}{eZ_s(0)}t\right]dt \tag{35a}$$

for a voltage source configuration [Fig. 1(a)] and

$$V=\sum_{m=1}^{\infty}\frac{\pi m^3a_m^2}{e}\int_0^\infty\exp[-m^2f_s(t)]k_s(t)\sin\left[\frac{\pi I_xm}{e}\right]dt \tag{35b}$$

for a current source configuration [Fig. 1(b)]. Here $a_m$ are the Fourier coefficients:

$$E_0(q)=\sum_m a_m\cos\left[\frac{\pi m}{e}q\right]\ .$$

In the limit of a large external impedance and low temperature

$$f_s(t) \ll 1, \quad 1/t \gtrsim \omega_b = \frac{\pi I_x}{e} , \tag{36}$$

or roughly $\mathrm{Re} Z_s(\omega \lesssim \omega_b) \gg R_q$, $T \ll eV_c$, Eq. (35b) yields

$$V = \sum_{m=1}^{\infty} \frac{\pi^3 m^2 a_m^2}{2e^3 \omega_b} \mathrm{Re} \frac{1}{Z_s(m\omega_b)} , \tag{37}$$

with further simplifications for both small and large values of $E_J$

$$V = \frac{2e}{\pi C^2 \omega_b} \sum_{m=1}^{\infty} \frac{1}{m^2} \mathrm{Re} \frac{1}{Z_s(m\omega_b)} ,$$
$$E_J \ll E_c, \quad E_0(q) = \min_n \left\{ \frac{q-2en)^2}{2C} \right\} , \tag{37a}$$

$$V = \frac{\pi^3 \Delta^2}{2e^3 \omega_b} \mathrm{Re} \frac{1}{Z_s(\omega_b)}, \quad E_J \gg E_c, \quad E_0(q) = \Delta \cos \left[ \frac{\pi q}{e} \right] . \tag{37b}$$

Obviously, Eq. (37) can be also used to describe the intermediate case $E_J \sim E_c$, which is of particular experimental interest.[8,9] According to (37) the main contribution to the $I$-$V$ curve comes from an external impedance $Z_s(\omega)$ at a frequency equal to $\omega = \omega_b$ (and to several higher harmonics of $\omega_b$ in the limit $E_J \ll E_c$). This conclusion turns out to be important for a quantitative analysis of the experimental data (see below).

Now let us analyze the effect of quantum noise $\xi(t)$ and calculate the spectral function

$$S_v(\omega) = \int \exp(i\omega t)[\langle \overline{V(t_0+t)V(t_0)} \rangle - \langle V \rangle]^2 dt , \tag{38}$$

where the average over realizations of $\xi(t)$ and the time average are denoted, respectively, by angular brackets and an overbar. We then denote the solution of Eq. (30) as $q(t) = q_0(t) + \delta q(t)$, where $q_0(t)$ is the solution of (30) with $\xi(t) = 0$, which obeys the condition $\dot{q}_0(t + 2\pi/\omega_b) = \dot{q}_0(t)$. Proceeding perturbatively in $\xi(t)$ we obtain

$$V(t) = \langle V \rangle - Z_s(0) \sum_k b_k \exp\{-ik[\omega_b t + \delta q(t)]\} , \tag{39}$$

where $b_k$ are the Fourier coefficients for $q_0(t)$. Then substituting (39) into (38) we find

$$S_v(\omega) = S_0(\omega) + {\sum_k}' |b_k Z_s(0)|^2 S_k(\omega - k\omega_b) ,$$
$$S_k(\omega) = \int \exp(-k^2 g(t) + i\omega t) dt , \tag{40}$$

where the sum $\sum'$ is taken over all $k \neq 0$ and

$$g(t) = \frac{\pi^2}{2e^2} \int_0^t d\tau \int_0^t ds \langle \delta\dot{q}(\tau) \delta\dot{q}(s) \rangle .$$

Here $S_0(\omega)$ is the low-frequency part of $S_v(\omega)$ and $S_k(\omega)$ represents the contribution from the $k$th harmonics of Bloch oscillations. For large $\omega_b \gg eV_c$, we can drop the potential energy in (30) and, after straightforward calculation, we get

$$\langle |\delta\dot{q}_\omega|^2 \rangle \simeq \frac{\langle |\xi(\omega)|^2 \rangle}{|Z_s(\omega)|^2} .$$

Combining the last two equations, we find that $g(t)$ coincides with $f_s(t)$ Eq. (34): $g(t) = f_s(t)$.

It is reasonable to define the linewidth of Bloch oscillations $\Gamma$ by the equation $f_s(1/\Gamma) = 1$. Making use of Eq. (34) for $Z_s(\omega) = R_s$ in the limit $\omega_b \ll \omega_c$ we obtain

$$2\alpha_s \ln \left[ \frac{\omega_c}{\pi T} \sinh \left[ \frac{\pi T}{\Gamma} \right] \right] = 1 , \tag{41}$$

where $\omega_c$ is the effective cutoff frequency. For $E_J \gg E_c$ we put $\omega_c = \omega_0 = \sqrt{8E_J E_c}$. Further specification of the problem is needed to define $\omega_c$ in the opposite limit $E_J \ll E_c$. For example, for $Z_s(\omega) = R_s - i\omega L$ we have $\omega_c \sim R_s / L$.

For a high-temperature limit $T \gg T$ the function $S_1(\omega)$ is of a Lorentz form $S_1(\omega) = 2\Gamma/(\omega^2 + \Gamma^2)$, and we reproduce the well-known result[2,20] $\Gamma = 2\pi\alpha_s T$. In the opposite limit $T \ll \Gamma$ we get

$$\Gamma = \omega_c \exp \left[ -\frac{1}{2\alpha_s} \right], \quad S_1(\omega) = \frac{\pi}{\Gamma(2\alpha_s)} \frac{|\omega|^{2\alpha_s - 1}}{\omega_c^{2\alpha_s}} , \tag{42}$$

i.e., in accordance with Ref. 17, the linewidth of Bloch oscillations remains finite even at $T = 0$. Note, however, that the expression for $\Gamma$ Eq. (42), does not coincide with that for $\delta\omega_b / \omega_b$ evaluated in Ref. 17: $\delta\omega_b / \omega_b \propto \alpha_s \ln(\omega_c / \omega_b)$. This is a consequence of a "non-Lorentzian" form of $S_1(\omega)$ at $T \to 0$.

As we already pointed out, the analysis presented here is adequate, provided that both Zener tunneling and thermal activation to higher Brillouin zones can be neglected. For $E_J \ll E_c$ and $Z_s(\omega) = R_s$, the corresponding limitations are[2,3,26]

$$\alpha_s \ll (E_J/E_c)^2, \quad T \ll E_c, \quad I \lesssim I_{cr}^s = \begin{cases} e(2\alpha_s^2 E_J^2 T/\pi)^{1/3} , & T \gg \alpha_s E_c \\ (2e\alpha_s/\pi)[E_J^2 E_c \ln(\omega_c \tau_s)]^{1/3}, & T \ll \alpha_s E_c , \end{cases}$$

while for $E_J \gg E_c$, we have $\max(I/e, T) \ll \omega_0$. Note that the above expression for the current $I_{cr}^s$ (Ref. 26), at which the crossover between charge dynamics in the lowest Brillouin zone and its dissipative motion in higher zones takes place, differs significantly from the analogous expression $I_{\mathrm{cr}}^t \sim e\sqrt{\alpha_t} E_J$ derived in Ref. 27 for the case of quasiparticle tunneling (see Ref. 26 for details). This fact might be important for the analysis of experimental data.[8,9] Another validity condition for our analysis $\Gamma \ll \omega_b$ yields $2\alpha_s \ll 1/\ln(\omega_c/\omega_b)$ and $T \ll \omega_b/\alpha_s$. Similar conditions can be formulated for a non-Ohmic case.

### B. Bloch oscillations in the small bandwidth limit: Charge dynamics and correlation functions

In the limit of large $E_J \gg E_c$, our analysis allows a complete description of the charge dynamics for arbitrary values of $\alpha_t$ and external impedance $Z_s(\omega)$. Making use of the equation

$$\langle \psi_q | [\dot{v}\cos(\varphi/2) + \dot{w}\sin(\varphi/2)] | \psi_{q+e} \rangle = \dot{v} \ , \tag{43}$$

one can drop the variable $w$ from Eqs. (A10) and rewrite these equations in the form

$$\begin{aligned} i\dot{c}_1 &= E_0(q(t))c_1 - (\dot{v}(t)/e)c_2 \ , \\ i\dot{c}_2 &= -(\dot{v}(t)/e)c_1 + E_0(q(t))c_2 \ , \end{aligned} \tag{44}$$

where we put $E_0(q) = \Delta\cos(\pi q/e)$. Substituting the solution of (44) into the expression for the expectation value of the voltage operator (A11) (see Appendix) and expanding in powers of $\Delta$, after straightforward algebraic manipulations we get, for $t \gg \omega_0^{-1}$ and not very small $\omega_b$,

$$\begin{aligned} \langle V(t) \rangle &= -\frac{\pi\Delta}{e}\exp(-f(t))A(t) + \frac{\pi\Delta^2}{e}B\int_0^t \exp(-f(t))\sin(k(\tau))\sin(\omega_b\tau)d\tau \\ &\quad + \frac{\pi\Delta^2}{e}\int_0^t \exp[-f_t(\tau) - 2f_s(t) - 2f_s(t-\tau) + f_s(\tau)]\sin[k_t(\tau) - k_s(\tau)]C(2t-\tau)d\tau \ , \\ f_t(t) &= \frac{4}{e^2R_t}\int \coth\left[\frac{\omega}{2T}\right]\frac{1-\cos(\omega t)}{\omega}\frac{d\omega}{2\pi}, \quad k_t(t) = \frac{2i}{e^2R_t}\int \frac{\exp(-i\omega t)}{\omega + i0}\frac{d\omega}{2\pi} \ , \end{aligned} \tag{45}$$

where integration is taken over the frequencies within the interval $|\omega| \lesssim \omega_0$ and we define $f(t) = f_s(t) + f_t(t)$, $k(t) = k_s(t) + k_t(t)$. As is shown in the Appendix, the quantities $A(t)$ and $C(t)$ depend on the initial density matrix. For a particular case, $\rho(0,k_1,k_2) = \delta[(k_1+k_2)/2]$, the Eqs. (A12) and (A14) yield $A(t) = C(t) = \sin(\omega_b t)$. The parameter $B$ coincides with the trace of the density matrix (A13), i.e., here we have $B = 1$. Nevertheless, we keep this parameter for further calculation of the voltage-voltage correlation function (see below).

Equations (45) describe the voltage dynamics for an arbitrary external impedance $Z_s(\omega)$. Calculating the time average of (45) $\langle \bar{V} \rangle$, we arrive at the current-voltage characteristic

$$\langle \bar{V}(I_x) \rangle = \frac{\pi\Delta^2}{e}\int_0^\infty \exp(-f(t))\sin(k(t))\sin(\omega_b t)dt \ , \tag{46}$$

which coincides with our previous result (37) in the limit (36). For a special case of an Ohmic shunt $Z_s(\omega) = R_s$ at $T = 0$ and large $t$, Eq. (45) yields

$$\begin{aligned} \langle V(t) \rangle &= \frac{\pi^2\Delta^2}{2e\Gamma(2\alpha_s + 2\alpha_t)}\frac{\omega_b^{2\alpha_s+2\alpha_t-1}}{\omega_c^{2\alpha_s+2\alpha_t}} - \frac{\pi\Delta}{e}(\omega_0 t)^{-2\alpha_t - 2\alpha_s}\sin(\omega_b t) + \frac{\pi\Delta^2}{e}\sin(\pi\alpha_t + \pi\alpha_s) \\ &\quad \times \int_t^\infty \frac{\sin(\omega_b\tau)}{(\omega_0\tau)^{2\alpha_t+2\alpha_s}}d\tau + \frac{\pi\Delta^2}{e}\frac{\sin(\pi\alpha_t - \pi\alpha_s)}{\omega_0^{2\alpha_t+2\alpha_s}}t^{-4\alpha_s}\int_0^t \frac{\sin(2\omega_b t - \omega_b\tau)}{\tau^{2\alpha_t-2\alpha_s}(t-\tau)^{4\alpha_s}}d\tau \ . \end{aligned} \tag{47}$$

The integrals in (47) can be expressed in terms of hypergeometric functions. Here we drop this expression for simplicity. The result (47) shows that for nonzero $\alpha_s$, oscillating terms decay in time and for large $t$ the voltage $\langle V(t) \rangle$ tends to the constant value $\langle \bar{V} \rangle$. For $\alpha_s \to 0$, however, terms that oscillate with a frequency $\omega = 2\omega_b$ survive even at $t \to \infty$. For $\alpha_s = 0$ and $T = 0$ we get

$$\langle V(t) \rangle = \frac{\pi^2\Delta^2}{2e\Gamma(2\alpha_t)}\frac{\omega_b^{2\alpha_t-1}}{\omega_c^{2\alpha_t}} \times [1 + d_1\cos(2\omega_b t) + d_2\sin(2\omega_b t)] \ , \tag{48}$$

where the constants $d_1$ and $d_2$ depend on the initial state. The existence of coherent voltage oscillations (48) for $\alpha_s = 0$ is a result of a superposition of SET and CPT effects: SET leads to an effective $e$-periodic Brillouin zone (and therefore to the frequency of oscillations $2\omega_b$ instead of $\omega_b$), while the current across the junction is transferred by CPT (i.e., by the mechanism that does not destroy voltage coherence) and thus voltage oscillations (48) do not decay in time.

Our analysis also allows us to calculate the voltage-voltage correlation function $S_v(t)$, the definition of which is now slightly different from (38). As before, in this section one should bear in mind that now $\langle V \rangle$ is the expectation value of the voltage operator rather than the average over realizations of the stochastic variable [as in Eq. (38)]. Besides to exclude the dependence of $S_v(t)$ on the initial state, one has to take the limit $t_0 \to \infty$. Therefore, we define

$$S_v(t) = (\mathrm{Re}\langle \overline{V(t_0+t)V(t_0)} \rangle - \langle V \rangle^2)|_{t_0 \to \infty} \ ,$$

$$\langle V(t_0+t)V(t_0) \rangle = \mathrm{tr}(\hat{V}(t)\hat{V}\hat{\rho}(t_0)) \ .$$

We can evaluate the correlation function $\langle V(t_0+t)V(t_0)\rangle$ by means of Eq. (A11) for $\langle V(t)\rangle$ if we substitute the kernel of the operator $\hat{V}\hat{\rho}(t_0)$ instead of the initial density matrix. Expanding in powers of $\Delta$, we get

$$S_v(t)=\Delta^2 S_2(t)+\Delta^4 S_4(t) \ . \tag{49}$$

After a simple calculation, we obtain a general expression for the leading term in (49),

$$S_2(t)=\frac{1}{2}\left[\frac{\pi}{e}\right]^2 \exp(-f(t))\cos(k_s(t))\cos(\omega_b t) \ . \tag{50}$$

For Ohmic dissipation $Z_s(\omega)=R_s$ and $t \gg 1/\omega_0$ Eq. (50) yields

$$S_2(t)=\frac{\pi^2}{2e^2}\left[\frac{2\pi T}{\omega_0}\right]^{2\alpha_t+2\alpha_s}\cos(\pi\alpha_s) \times \exp[-2\pi(\alpha_t+\alpha_s)Tt]\cos(\omega_b t), \quad t \gg 1/T \ , \tag{50a}$$

$$S_2(t)=\frac{\pi^2}{2e^2}\exp[-2\gamma(\alpha_s+\alpha_t)] \times \cos(\pi\alpha_s)\frac{\cos(\omega_b t)}{(\omega_0 t)^{2\alpha_t+2\alpha_s}}, \quad t \ll 1/T \ , \tag{50b}$$

where $\gamma \simeq 0.577$ is the Euler constant. Note that for $\alpha_t \ll 1$ and $\alpha_s=0$, result (50a) coincides with that obtained in Ref. 28 perturbatively in $\alpha_t$. On the other hand, result (50b) demonstrates that perturbation theory of Ref. 28 fails in the low-temperature limit and a power-law (rather than exponential) decay of the correlation function takes place at $T=0$. For $\alpha_s+\alpha_t<1/2$, the Fourier transform of $S_2$ reads

$$S_2(\omega)=\frac{\pi}{4}\left[\frac{\pi}{e}\right]^2 \frac{1}{\Gamma(2\alpha_s+2\alpha_t)\cos(\pi\alpha_t)} \times \left\{ \frac{|\omega-\omega_b|^{2(\alpha_s+\alpha_t)-1}}{\omega_c^{2(\alpha_s+\alpha_t)}} + \frac{|\omega+\omega_b|^{2(\alpha_s+\alpha_t)-1}}{\omega_c^{2(\alpha_s+\alpha_t)}} \right\}, \tag{51}$$

while, for $\alpha_s+\alpha_t>1/2$, peaks at $\omega=\pm\omega_b$ disappear and $S_2(\omega)$ monotonically decreases with increasing $\omega$. Note that, in the limit $\alpha_t=0$, the power-law dependence $S_v(\omega)\propto|\omega\pm\omega_b|^{2\alpha_s-1}$ has been also derived in Ref. 15 on the basis of duality arguments. For $\alpha_t \neq 0$, these arguments are insufficient to obtain result (51).

It is also interesting to analyze the next-order term, $S_4(t)$, in (49). The general expression for $S_4(t)$ is quite complicated and we will not specify it here. In the case of Ohmic external impedance $\alpha_s \neq 0$ the function $S_4(t)$ decays in time, showing damped oscillations with frequency $2\omega_b$. In the limit $\alpha_s=0$, the Fourier transform of $S_4(t)$ contains narrow peaks at frequencies $\pm 2\omega_b$ plus uninteresting regular terms that give a small correction to result (51). Dropping these terms, we get, for $\alpha_s=0$ and $T \to 0$,

$$S_4(\omega)=\frac{\pi}{2}\left\{\frac{\pi^2}{2e}\frac{1}{\Gamma(2\alpha_t)}\frac{\omega_b^{2\alpha_t-1}}{\omega_c^{2\alpha_t}}\right\}^2 \times[\delta(\omega-2\omega_b)+\delta(\omega+2\omega_b)] \ . \tag{52}$$

In complete agreement with Eq. (48), result (52) demonstrates that, in the absence of Ohmic dissipation and at $T=0$, phase correlation of Bloch oscillations with a frequency $\omega=2\omega_b$ survives even as $t\to\infty$ and when the corresponding linewidth is equal to zero.

### C. Voltage steps

In the presence of an external ac drive

$$I_x(t)=I_x+I_0\cos(\Omega t) \ ,$$
$$V_x(t)=\int \hat{Z}_s(t-\tau)I_x(\tau)d\tau =Z_s(0)I_x+|Z_s(\Omega)|I_0\cos[\Omega t+\arg Z_s(\Omega)] \ ,$$

resonances between Bloch oscillations and the external ac signal occur, leading to peculiarities (voltage steps) on the $I$-$V$ curves in the vicinity of the points $n\omega_b=k\Omega$.[2,3,20] This effect is a subject of a substantial experimental activity because it might serve as a convincing demonstration of the existence of Bloch oscillations (see, e.g., Refs. 8 and 9)].

Proceeding naively one can treat the effect of voltage steps within the framework of a perturbation theory. For example, for $E_J \gg E_c$ one can make use of the expression for $\langle V(t)\rangle$ derived perturbatively in $\Delta$ and get

$$\langle \bar{V}(I_x(t))\rangle=\sum_k J_k^2\left[\frac{\pi I_0}{e\Omega}\right]\left\langle \bar{V}\left[I_x-\frac{ke\Omega}{\pi}\right]\right\rangle , \tag{53}$$

where $J_k(x)$ are the Bessel functions and $\langle \bar{V}(I_x)\rangle$ was defined in (46). For $\alpha_t=0$, $Z_s(\omega)=R_s$, and large $T$, this result was previously discussed by Odintsov.[29] Below we shall see that the corresponding validity condition for (53) reads

$$T \gg \tilde{\Delta}_r \sim \tilde{\Delta}(\tilde{\Delta}/\omega_0)^{\alpha_s/(1-\alpha_s)}, \quad \tilde{\Delta}=\Delta J_{-k}(\pi I_0/e\Omega) \ .$$

For lower temperatures $T \lesssim \tilde{\Delta}_r$ expression (53) describes the $I$-$V$ curve not very close to the resonant points $|\omega_b-k\Omega| \gtrsim \alpha_s\tilde{\Delta}_r$. In the vicinity of these points, perturbation theory fails and one should use a different technique to recover the whole $I$-$V$ curve. For example, in the limit of a high external impedance, a quasiclassical-Langevin equation for the quasicharge (30) can be applied to this problem in the same spirit as it has been done in Ref. 30 for the phase variable of "classical" Josephson junctions. Here we develop another technique that is not based on a quasiclassical approximation for $q(t)$.

Let us take $\alpha_t=0$ for simplicity. It is convenient to make a shift of the variable $q$,

$$q \to q+\frac{me}{\pi n}\Omega t+\frac{I_0}{\Omega}\sin(\Omega t)$$

in Eqs. (27)–(29). Then making use of the identity

$$\cos(x+z\sin\varphi)=\sum_{p=-\infty}^{\infty} J_{-p}(z)\cos(x-p\varphi) ,$$

we rewrite the ground-state energy $E_0$ in the form

$$E_0\left[q+\frac{ke}{\pi n}\Omega t+\frac{I_0}{\Omega}\sin(\Omega t)\right]=\sum_{m=1}^{\infty}\sum_{p=-\infty}^{\infty} a_m J_{-p}\left[\frac{\pi I_0}{e\Omega}m\right]\cos\left[\frac{\pi m}{e}q+\left[\frac{km}{n}-p\right]\Omega t\right] . \tag{54}$$

Let us now fix the numbers $n$ and $k$ and assume the new (shifted) variable $q$ to be slow enough to describe the problem within the adiabatic approximation. The corresponding condition reads $\dot{q}/e \ll \Omega/m_{\max}$ or, equivalently,

$$|\omega_b-(k/n)\Omega| \ll \Omega/m_{\max} , \tag{55}$$

$m_{\max}$ is the maximal number for which the Fourier coefficients $a_m$ still give a nonvanishing contribution. In the limit $E_J \gg E_c$ it is $m_{\max}=n=1$. In the opposite limit $E_J \ll E_c$ we have $a_m \propto 1/m^2$ and thus again essential values of $m$ are of order one. Provided the condition (55) is fulfilled the terms with $p/m \neq k/n$ in the expression for $E_0$ Eq. (54) oscillate with frequencies that exceed $\Omega/m_{\max}$. Therefore, in the main approximation, these terms can be neglected and, we get

$$\int_0^t E_0\left[q+\frac{ke}{\pi n}\Omega\tau+\frac{I_0}{\Omega}\sin(\Omega\tau)\right]d\tau \simeq \int_0^t E_{\text{eff}}(q)d\tau \tag{56}$$

and

$$E_{\text{eff}}^{kn}(q)=\sum_{m=1}^{\infty} a_{mn}J_{-mk}\left[\frac{\pi I_0}{e\Omega}mn\right]\cos\left[\frac{\pi mn}{e}q\right] . \tag{57}$$

Substituting (56) and (57) into (29) and splitting $dE_0/dq$ as

$$dE_0/dq=dE_{\text{eff}}^{kn}/dq+(dE_0/dq-dE_{\text{eff}}^{kn}/dq) ,$$

we present the expectation value $\langle V(t)\rangle$ in the vicinity of the points $n\omega_b=k\Omega$ as a sum of two terms

$$\langle V(I_x(t))\rangle=V_{kn}(t)+V'(t) , \tag{58}$$

where

$$V_{kn}(t)=\int dq\frac{dE_{\text{eff}}^{kn}}{dq}f_{\text{eff}}(q,t) , \tag{59}$$

$$f_{\text{eff}}(q,t)=\int dq_{i1}dq_{i2}\rho_0(q_{i1},q_{i2})\int_{q_{i1}}^{q}\int_{q_{i2}}^{q} Dq_1Dq_2\exp(iS_{\text{eff}}[q_1,q_2]) ,$$

$$S_{\text{eff}}[q_1,q_2]=\int_0^t\left[-E_{\text{eff}}^{kn}(q_1)+E_{\text{eff}}^{kn}(q_2)+Z_s(0)\left[I_x-\frac{ke}{\pi n}\Omega\right]q^-\right]d\tau+\int_0^t d\tau\int_0^t ds\left[-q^-(\tau)\hat{Z}_s(\tau-s)\dot{q}^+(s)+\frac{i}{2}q^-(\tau)G_q(\tau-s)q^-(s)\right] , \tag{60}$$

and $V'(t)$ is the contribution of the term $\langle(dE_0/dq-dE_{\text{eff}}^{kn}/dq)\rangle$. After a direct comparison between Eqs. (27)–(29) and Eqs. (58)–(60), we can conclude that $V_{kn}(t)$ coincides with the expectation value of the voltage operator for a fixed external bias $V_x=Z_s(0)(I_x-ke\Omega/\pi n)$ and the ground-state energy $E_{\text{eff}}^{kn}(q)$. Therefore, the time average $\overline{V}_{kn}$ represents the corresponding dc $I$-$V$ curve with the origin shifted to the point $I_x=ke\Omega/\pi n$. Making use of the results (31) and (37) in the limit (36) we find

$$\overline{V}_{kn}=Z_s(0)(I_x-ke\Omega/\pi n), \quad |I_x-ke\Omega/\pi n|<V_c^{kn}/Z_s(0) , \tag{61a}$$

$$\overline{V}_{kn}=\sum_{m=1}^{\infty}\frac{\pi^3m^2n^2a_{mn}^2J_{mk}^2(\pi I_0mn/e\Omega)}{2e^3(n\omega_b-k\Omega)}\text{Re}\frac{1}{Z_s(mn\omega_b-kme\Omega)}, \quad |I_x-ke\Omega/\pi n| \gg V_c^{kn}/Z_s(0) , \tag{61b}$$

with further simplifications as in Eqs. (37a) and 37(b). Here we denote $V_c^{kn}=(dE_{\text{eff}}^{kn}/dq)_{\max}$. For $E_J \gg E_c$ one can also go beyond the limit $T \ll eV_c^{kn}$, $\text{Re}Z_s \gg r_q$ (36). In this case we have $n=1$ and $\overline{V}_{k1}$ is given by Eq. (61a) for

$$|I_x-ke\Omega/\pi n| \lesssim \tilde{\Delta}_r/Z_s(0), \quad T \ll \tilde{\Delta}_r$$

and by the equation

$$\overline{V}_{k1}=J_k^2\left[\frac{\pi I_0}{e\Omega}\right]\left\langle\overline{V}\left[I_x-\frac{ke\Omega}{\pi}\right]\right\rangle \tag{61c}$$

for

$$\max(Z_s(0)|I_x-ke\Omega/\pi n|,\ T) \gg \tilde{\Delta}_r .$$

The term $V'$ describe the tails of the steps $n'\omega_b=k'\Omega$, $k'/n' \neq kn$. Provided condition (55) is fulfilled, this term is small and can be neglected. Within this accuracy one can describe the whole $I$-$V$ curve as a linear combination of independent contributions from all steps and get

$$\langle \bar{V}[I_x(t)]\rangle = \langle \bar{V}(I_x)\rangle + \sum_{\substack{\text{all}\\\text{steps}}} \bar{V}_{kn} , \qquad (62)$$

where $\langle \bar{V}(I_x)\rangle$ is the $I$-$V$ curve for $I_0=0$. Equations (61) allow to estimate the effective "size" of each step $\delta V_{kn}$. At $T\to 0$ we estimate $\delta V_{kn}\sim 2V_c^{kn}$ and $\delta I_{kn}\sim \delta V_{kn}/Z_s(0)$ and get $\delta V_{k1}\sim\tilde{\Delta}_r$ for $E_J \gg E_c$ and

$$\delta V_{kn}\sim (e/Cn^2)J_{-k}(\pi I_0 n/e\Omega) \quad \text{for } E_J \ll E_c .$$

We see that in both cases $E_J\gg E_c$ and $E_J\ll E_c$ the step amplitudes are proportional to $J_{-k}(\pi I_0 n/e\Omega)$ reaching their maximum at $I_0\sim e\Omega/n$. At $T\lesssim eV_c^{kn}$ the steps can cross the axis $V=0$ and thus zero-voltage states can be realized in the vicinity of the points $n\omega_b=k\Omega$.

It is also possible to include the case $\alpha_t\neq 0$ into consideration. For finite $Z_s(\omega)$, no new effects arise. For $1/Z_s(\omega)\to 0$ and $\alpha_t\neq 0$, an additional set of infinitely sharp horizontal voltage spikes appear at frequencies $2\omega_b=k\Omega$ (see, e.g., Ref. 3). These spikes correspond to resonances between the external radiation and coherent voltage oscillations (48).

## V. DISCUSSION AND CONCLUSIONS

Making use of real-time path-integral technique, we have developed a detailed analysis of quantum dynamics of both phase and charge variables of an ultrasmall tunnel junction, which interacts with an arbitrary external impedance $Z_s(\omega)$. For normal junctions, our analysis enables us to go beyond the perturbation theory in $\alpha_t$ developed in Refs. 11, 13, and 14 and to study the effect on the phase dynamics of both the junction shot noise and Gaussian fluctuations of the charge in the external circuit. Within the quasiclassical approximation for the phase $\varphi$, we calculate the junction current-voltage characteristics for various types of external impedance [see Eqs. (18)–(24)]. At low $T$, the offset on the $I$-$V$ curve is equal to the universal value $\Delta V=e/2C_{\text{eff}}$ for any $R_t$ and practically all reasonable $Z_s(\omega)$,

$$I=\frac{1}{R_t}\left[V-\frac{e}{2C_{\text{eff}}}\right],\quad C_{\text{eff}}=C+\lim_{\omega\to\infty}[i/\omega Z_s(\omega)] .$$

In the case of superconducting junctions, we have mostly concentrated on the Bloch-oscillation regime. We have evaluated the junction current-voltage characteristic for an arbitrary $Z_s(\omega)$. It reads

$$V=\frac{2e}{\pi C^2\omega_b}\sum_{m=1}^{\infty}\frac{1}{m^2}\text{Re}\frac{1}{Z_s(m\omega_b)},\quad E_J\ll E_c;\quad V=\frac{\pi^3\Delta^2}{2e^3\omega_b}\text{Re}\frac{1}{Z_s(\omega_b)},\quad E_J\gg E_c .$$

These equations show that, for both $E_J\ll E_c$ and $E_J\gg E_c$, the impedance at frequencies $\omega\sim\omega_b$ gives the main contribution to the $I$-$V$ curve, while substantially higher frequencies (e.g., of order of the band gap $\delta\sim E_J$ for $E_J\ll E_c$ and $\delta\sim\omega_0$ for $E_J\gg E_c$) are unimportant in the Bloch-oscillation regime $I<\max(I_{cr}^s,I_{cr}^t)$. We have also presented a direct nonperturbative calculation for the time-dependent excitation value of the voltage operator [Eqs. (45)–(48)] and voltage-voltage correlation functions. This calculation shows that a simple perturbative in $\alpha$ approach,[28] which leads to an exponential decay of correlation functions [Eq. (50a)] is applicable except for the case $T\to 0$. At $T=0$, perturbation theory is insufficient to describe the long-time properties of the system for nonzero $\alpha_s$ and $\alpha_t$. For $\alpha_s\neq 0$ and $\alpha_t\neq 0$ and at $T=0$, the main term $S_2(t)$ in the expression for the voltage-voltage correlation function shows damped oscillations with the frequency $\omega=\omega_b$ (a power-law decay)

$$S_2(t)\propto t^{-2\alpha}\cos(\omega_b t) ,$$

where $\alpha=\alpha_s+\alpha_t$ for $E_J\gg E_c$ and $\alpha=\alpha_s$ for $E_J\ll E_c$. This corresponds to an effective linewidth of order $\Gamma=\omega_0\exp(-1/2\alpha)$. For $\alpha_s=0$, the next-order term, $S_4(t)$, of the spectral function contains $\delta$-shaped peaks $\omega=\pm 2\omega_b$ (52), correspond to coherent voltage oscillations with $\Gamma=0$.

The expression for the expectation value of the voltage operator $\langle V(t)\rangle$, derived here illustrates similar features. It also provides information about the effect of voltage steps on the $I$-$V$ curve of an ac-driven junction. These steps take place in the vicinity of the points $n\omega_b=k\Omega$. Each of them repeats the $I$-$V$ curve of a dc-biased junction with an effective Brillouin zone $E_{\text{eff}}(q)$ (54). The amplitudes of these steps are proportional to

$$\delta V_{kn}\propto J_{-k}(\pi I_0 n/e\Omega) .$$

Note that here we describe the quantum dynamics of a Josephson junction, provided that both Zener tunneling and thermal activation to higher zones can be neglected. For $E_J\lesssim E_c$, however, the process of Zener tunneling can play an important role.[26,27] As has been shown in Ref. 26, it restricts the regime of Bloch oscillations to currents

$$I<I_{cr}^s\sim(\alpha_s E_J)^{2/3}\max[T^{1/3},(\alpha_s E_c)^{1/3}]$$

and contributes to the expression for the linewidth $\Gamma$,

$$\Gamma=\Gamma_{\text{lowest zone}}+(I/2e)\exp(-\pi e E_J^2/8IE_c) .$$

In the limit $T\to 0$, the last term in this expression dominates in the interesting parameter region $\alpha_s\ll E_J^2/E_c^2$.

Results (61) and (62) of our paper are of particular importance in view of recent experimental observations of peculiarities on the $I$-$V$ curve in the vicinity of the points $\omega_b=k\Omega$.[8,9] The authors of Refs. 8 and 9 estimated the effective impedance of external leads as that of an $RC$ line: $Z_s(\omega)\simeq R_1$ for $\omega\ll 1/R_1C_1$ and $Z_s(\omega)\sim\sqrt{R_1/i\omega C_1}$ for $\omega\gg 1/R_1C_1$. Making use of typical experimental values of the parameters $R_1$, $C_1$, and $I_{cr}$,[8,9] we can conclude that, for $I<I_{cr}^s$, important frequencies $\omega\sim\omega_b$ are smaller than $1/R_1C_1$ and thus the effective constant $\alpha_s$ should be estimated as $\alpha_s=R_q/R_1\simeq 0.07$ (but not $\alpha_s\simeq 0.3$ as in Ref. 8). This in turn allows us to proceed with a comparison of theoretical predictions with experimental data.[8,9] For two samples with $V_c\approx 1$ $\mu$eV and $E_J\gtrsim E_c$, our estimation of the amplitude $\Delta$ yields $\Delta\approx 3$ $\mu$eV and thus we have $\Delta<T$. Then we can use a high-temperature result, $\delta I\simeq 2e\Gamma/\pi=4e\alpha_s T$, and find $\delta I\simeq 0.42$ nA (Pb) and $\delta I\simeq 0.24$ nA (Al). For both samples the measured value $\delta I$ (Refs. 8 and 9) is approximately twice as large as the theoretical one. For the Pb sample with $V_c\approx 9$ $\mu$eV and $E_J<E_c$, we have $T<eV_c$. Making use of the low-temperature result (61a), we estimate the effective "width" of the main step $(n=k=1)$ as $\delta I\lesssim 2V_c/Z_s(0)\approx 0.19$ nA. This value is in a good agreement with the experimental result of $\delta I_{\rm expt}\approx 0.21$ nA. The value of $I_{cr}^s$ (Ref. 26) for the samples with $E_J<E_c$ also agrees with the results of measurements.[8,9] Further experimental investigations are needed to provide more detailed verification of theoretical predictions. However, even available experimental data[8,9] clearly demonstrate the existence of Bloch steps (61) and (62) on the $I$-$V$ curve of an ac-driven Josephson junction.

We believe that the whole scope of results derived here as well as experimental findings[8,9] leave no room for doubts[19] in the validity of the general picture of Bloch oscillations given in Ref. 20 within the framework of a perturbation theory in $\alpha$. Indeed in the limit $T\to 0$ (but not for $T\gg\Delta$ as was suggested in Ref. 19) perturbation theory does not work even for $\alpha\ll 1$. However, this fact results only in a slower than exponential decay of correlation functions at $T=0$ and does not influence any of the theoretical predictions.[2,3]

After this work had been completed, we became aware of the paper by Nazarov and Odintsov[31] in which steps (53) were interpreted as a result of resonant interaction between external irradiation and incoherent tunneling of the phase variable $\varphi$. The authors[31] considered this resonant effect as an alternative mechanism for voltage steps. On the contrary, we believe that this is simply an alternative language to describe the same phenomenon of Bloch steps in the limit $E_J\gg E_c$. Indeed the interpretation of Ref. 31 works only for $E_J\gg E_c$, while Eqs. (61) and (62) are valid also for $E_J\lesssim E_c$, i.e., in the region where the picture of incoherent tunneling in the phase space becomes inadequate. Neither do we share the opinion of the authors[31] about the existence of another type of steps, which (according to Ref. 31) might be a result of phase coherence at very small $T$ and $\alpha_s$. We would like to emphasize in this context that, for $I>V_c/Z_s(0)$, phase coherence *does not exist* even at $T=0$ and for arbitrary small $\alpha_s$. Horizontal voltage steps at $\omega_b=k\Omega$ could be seen only if the condition

$$f(t_{\rm expt})=2(\alpha_s+\alpha_t)\ln\left[\frac{\omega_c}{\pi T}\sinh(\pi T t_{\rm expt})\right]\lesssim 1$$

is fulfilled, where $t_{\rm expt}$ is the experimental time. For typical values $T\sim 10$–$100$ mK and $t_{\rm expt}\sim 1$ sec one can hardly hope to achieve dissipation of order $\alpha\lesssim 1/2\pi T t_{\rm expt}\sim 10^{-10}$–$10^{-11}$ in any real experiment. Thus for $\alpha_s\neq 0$ there is only one type of voltage steps (16) and (62) with amplitudes proportional to $\delta V_{kn}\propto J_{-k}(\pi I_0 n/e\Omega)$ at low $T$ and $\delta V_k\propto J_k^2(\pi I_0/e\Omega)$ at $T\gg\Delta_r$ and $E_J\gg E_c$.

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## APPENDIX

Let us substitute the kernel $J(t;\varphi_f,\varphi_i)$ Eqs. (6)–(8) into (1). Then, we get

$$\rho(t,\varphi_{1f},\varphi_{2f})=\int dq_f dq_i dv_i dv_f dw_i dw_f \int D\mathbf{q}\, D\mathbf{v}\, D\mathbf{w}\tilde{\rho}(t,\varphi_{1f},\varphi_{2f};\mathbf{q},\mathbf{v},\mathbf{w})\exp(i\tilde{S})\ , \qquad \text{(A1)}$$

where

$$\tilde{S}=\int_0^t V_x q^-(\tau)d\tau+W_q[q_1,q_2]+R_t W[v_1,v_2]+R_t W[w_1,w_2] \qquad \text{(A2)}$$

and $\tilde{\rho}(t)$ is the kernel of the operator

$$\tilde{\rho}(t)=\left\{\hat{T}\exp\left[-i\int_0^t \hat{H}[q_1,v_1,w_1]d\tau\right]\right\}\hat{\rho}(0)\left\{\hat{T}\exp\left[-i\int_0^t \hat{H}[q_2,v_2,w_2]d\tau\right]\right\}^\dagger . \qquad \text{(A3)}$$

The Hamiltonian $\hat{H}[q,v,w]$ was defined in Eqs. (10) and (11). Similarly to Ref. 16, we shall describe the junction dynamics in the adiabatic limit, i.e, we assume that $q$, $v$, and $w$ are "slow" variables and neglect both Zener tunneling and thermal activation to higher zones. This allows to confine our consideration to the lowest Brillouin zone. The corresponding eigenfunctions of the Hamiltonian $\hat{H}_0$ (Eq. 11) are $\psi_k(\varphi)=\exp(ik\varphi/2e)u_k(\varphi)$, $u_k(\varphi)=u_k(\varphi+2\pi)$, and $-e<k<e$. The Hamiltonian $\hat{H}[q,v,w]$ contains the $4\pi$-periodic terms $(\dot{v}/e)\cos(\varphi/2)$ and $(\dot{w}/e)\sin(\varphi/2)$. Therefore, the functions $\psi_k^{(1)}(\varphi)=\psi_k(\varphi)$, $\psi_k^{(2)}(\varphi)=\psi_{k+e}(\varphi)$ with $-e/2<k<e/2$ form the complete basis for our problem and the initial density matrix can be written in the form

$$\rho(0,\varphi_1,\varphi_2)=\int \rho_{ij}(0,k_1,k_2)\psi_{k_1}^i(\varphi_{1i})\psi_{k_2}^{j*}(\varphi_{2i})dk_1dk_2\ . \qquad \text{(A4)}$$

Here and below it is convenient for us to extend the interval of integration from $-e/2<k<e/2$ to $-\infty<k<\infty$. Accordingly, we define the kernel $\rho_{ij}$ (as well as the kernel of any other operator) outside the interval $-e/2<k<e/2$ as

$$\rho_{ij}(0,k_1,k_2)=\sigma^{\eta}\rho_{ij}(0,k_1=en,k_2+em)\sigma^{m}\ , \tag{A5}$$

where $\sigma=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The matrix $\rho_{ij}$ also obeys the condition

$$\int \mathrm{tr}(\rho_{ii}(k,k))dk=1\ . \tag{A6}$$

The density matrix of the final state is

$$\tilde{\rho}(t,\varphi_{1f},\varphi_{2f})=\int \rho(0,k_1,k_2)\psi(t,k_1,\varphi_{1f})\psi^*(t,k_2,\varphi_{2f})dk_1dk_2\ , \tag{A7}$$

where $\psi_k(t,\varphi)$ is the solution of the Schrödinger equation

$$i\frac{d}{dt}\psi_k(t;\varphi)=\hat{H}[q,v,w]\psi_k(t;\varphi),\quad \psi_k(0,\varphi)=\psi_k(\varphi)\ . \tag{A8}$$

This solution can be written as

$$\psi_k(t)=c_1(t)\psi^{(1)}_{p(t)}+c_2(t)\psi^{(2)}_{p(t)}\ , \tag{A9}$$

where $p(t)k+q(t)-q(0)$ and the amplitudes $c_1$ and $c_2$ satisfy the equations

$$\begin{aligned} i\dot{c}_1&=E_0(p(t))c_1-\left\langle\psi_q\left|\left[\frac{\dot{v}(t)}{e}\cos\left[\frac{\varphi}{2}\right]+\frac{\dot{w}(t)}{e}\sin\left[\frac{\varphi}{2}\right]\right]\right|\psi_{q+e}\right\rangle c_2\ ,\\ i\dot{c}_2&=-\left\langle\psi_{q+e}\left|\left[\frac{\dot{v}(t)}{e}\cos\left[\frac{\varphi}{2}\right]+\frac{\dot{w}(t)}{e}\sin\left[\frac{\varphi}{2}\right]\right]\right|\psi_q\right\rangle c_1+E_0(p(t)+e)c_2\ , \end{aligned} \tag{A10}$$

with the initial conditions $c_1(0)=1$, $c_2(0)=0$. Here $E_0(p)$ is the ground-state energy for the Hamiltonian $\hat{H}_0$ (11).

Making use of the expression for the voltage operator in the $q$-representation $\hat{V}=dE_0/dq$ and combining (A1), (A7), and (A9) we get

$$\begin{aligned} \langle V(t)\rangle&=\int dq\left[\sigma_1(t,q)\frac{dE_0(q)}{dq}+\sigma_2(t,q)\frac{dE_0(q+e)}{dq}\right]\ ,\\ \sigma_{1,2}(t,q)&=\int dk_1dk_2dv_idv_fdw_idw_f\int_{k_1}^{q}Dq_1\int_{k_2}^{q}Dq_2\int_{v_i}^{0}Dv^-\int_0^{v_f}Dv^+\int_{w_i}^{0}Dw^-\int_0^{w_f}Dw^+\exp(i\tilde{S}[\mathbf{q},\mathbf{v},\mathbf{w}])Y_{1,2}\ ,\\ Y_1&=[c_1(t;1)c_1^*(t;2)\rho_{11}(0;k_1,k_2)+c_2^*(t;1)c_2(t;2)\rho_{22}(0;k_1,k_2)]\ ,\\ Y_2&=[c_2(t;1)c_2^*(t;2)\rho_{11}(0;k_1,k_2)+c_1^*(t;1)c_1(t;2)\rho_{22}(0;k_1,k_2)]\ . \end{aligned} \tag{A11}$$

In the limit $E_J\gg E_c$ one can simplify Eqs. (A10) reducing them to (44). Expanding the solution of Eqs. (44) in powers of $\Delta$ and then substituting it into (A11) we arrive at the result (45) where the parameters $A(t)$, $B$, and $C(t)$ depend on the initial density matrix:

$$\begin{aligned} A(t)=\frac{1}{2i}&\left\{\int\exp\left[i\left[w_bt+\frac{\pi q}{e}\right]\right]\rho^-\left[0,q+\frac{\pi}{2e}K(\infty),q-\frac{\pi}{2e}K(\infty)\right]dq\right.\\ &\left.-\int\exp\left[-i\left[w_bt+\frac{\pi q}{e}\right]\right]\rho^-\left[0,q-\frac{\pi}{2e}K(\infty),q+\frac{\pi}{we}K(\infty)\right]dq\right\}\ , \end{aligned} \tag{A12}$$

$$B=\int[\rho_{11}(0,q,q)+\rho_{22}(0,q,q)]dq|_{t=0}\ , \tag{A13}$$

$$\begin{aligned} C(t)=\frac{1}{2i}&\left\{\int\exp\left[i\left[w_bt+\frac{2\pi q}{e}\right]\right]\rho^+\left[0,q+\frac{\pi}{e}K(\infty),q-\frac{\pi}{e}K(\infty)\right]dq\right.\\ &\left.-\int\exp\left[-i\left[w_bt+\frac{2\pi q}{e}\right]\right]\rho^+\left[0,q-\frac{\pi}{e}K(\infty),q+\frac{\pi}{e}K(\infty)\right]dq\right\}\ , \end{aligned} \tag{A14}$$

$\rho^+=\rho_{11}+\rho_{22}$, $\rho^-=\rho_{11}-\rho_{22}$. As it was already pointed out, one can also use the expressions (45) and (A12)–(A14) for calculation of the voltage-voltage correlation function if one substitutes

$$\rho_{ii}\to(\hat{V}\hat{\rho}(t_0))_{ii}=-\frac{\pi\Delta}{2e}(-1)^i\sin\left[\frac{\pi q_i}{e}\right]\rho_{ii}(t_0,q_1,q_2)\ . \tag{A15}$$

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