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Perturbation theory of superconducting micronetworks: Second-order and self-induction effects

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The perturbation method to treat the nonlinear Ginzburg-Landau equations for micronetworks previously derived is extended to higher orders in the perturbation parameters, the temperature ΔT and magnetic field ΔH measured from the phase boundary. In this paper these two parameters are treated independently and the perturbation equations to all orders are analyzed. The theory allows for consideration of self-inductive effects. The importance of these for the interpretation of experimental results in micronetworks of type-I and type-II materials is pointed out. The Landau critical points have been discussed. Expressions for the relevant thermodynamic quantities are obtained. In particular, it is shown that the phase boundary satisfies the Ehrenfest-Keesom relations for second-order transitions. The theory has been explicitly worked out for simple geometries and compared with exact results, which have been obtained numerically. The lines separating different modes have been evaluated to second order for these cases.

I. INTRODUCTION

Superconductive micronetworks in a magnetic field[1,2] show second-order phase transitions between normal and superconductive states whose characteristics depend on the geometry of each specimen. Regular as well as disordered structures have been studied.[3] Details of the phase boundary can be predicted using the linearized version of the Ginzburg-Landau equations (GLE) derived by de Gennes and Alexander[1] for these systems.

Properties of the superconductive state can only be described either numerically or by approximate methods. To this end, Fink, Rodrigues, and López have devised a perturbation scheme[4] and Wang, Rammal, and Pannetier[5] have applied variational calculations. The lowest-order calculation of Ref. 4 made use, for simplicity, of a single perturbation parameter linking together the temperature and magnetic field differences ΔT and ΔH measured from the phase-transition boundary. It can be seen that such a scheme cannot be easily extended to higher orders.

In this paper we present results of a perturbation approach valid to all orders, using ΔT and ΔH as independent perturbation parameters. Relevant physical quantities are expanded in powers of two variables x and y directly related to ΔT and ΔH. Perturbative equations are obtained by substituting this expansion into both GLE.

The perturbation equations obtained can be applied to any network, finite or infinite. At the end of the paper we compare with exact results the case of the bare ring and a particular two-loop circuit.[1] For infinite circuits the situation is still simple in the case of periodic boundary conditions. At present, work on the infinite ladder is in progress.

The approach and general scope of this paper can be compared with Abrikosov's theory of the mixed state in type-II superconductors. The differences lie on the fact that here we vary both T and H and also in that the topology of the network imposes special conditions not present in the bulk case. The perturbation equations for networks contain three combinations of averages of the order parameter rather than the single Abrikosov β_A of the bulk case. There appear averages of the squared order parameter and of its fourth power as for the bulk case, but also combinations containing the currents along branches and the induction coefficients. These enter through a renormalization of the Landau-Ginzburg constant κ that could help one understand some recent experimental results[8] which show different behavior for Al or In samples.

The consistency of the perturbation expansions has been checked by showing that the equations obtained are special cases of general thermodynamic relations. In particular, we shall show how the line giving the phase boundary as obtained through perturbation theory is a special case of the Ehrenfest-Keesom relations for second-order transitions.

In Sec. II, we give the basic perturbation equations. Section III discusses the normalization conditions. Section IV is devoted to an analysis of the self-inductive effects and their contribution to the energy. In Section V, we present explicit expressions within first-order perturbation for relevant physical quantities. A discussion of Landau critical points and of the relative stability of different modes is given in Sec. VI. Section VII is devoted to the development of the second-order perturbation formalism and Sec. VIII contains the analysis of simple physical systems and comparisons with exact numerical calculations. Section IX summarizes the conclusions and gives some perspectives for future work. The Appendices contain explicit second-order perturbation equations. These have been given in enough detail so they can be used by any reader interested in applying them.

II. PERTURBATION EQUATIONS

Throughout this paper we shall use normalized quantities as in Ref. 4, calling

$$\Delta=\frac{\psi_{\rm conv}}{\psi_\infty}\ ,\quad \mathbf{H}=\frac{\mathbf{H}_{\rm conv}}{\sqrt{2}H_c}\ ,\quad \mathbf{j}=\frac{4\pi\lambda}{c\sqrt{2}H_c}\mathbf{j}_{\rm conv}\ ,$$

$$\mathbf{A}=\frac{\mathbf{A}_{\rm conv}}{\sqrt{2}H_c\lambda}\ ,\quad \phi=\frac{2\pi\phi_{\rm conv}}{\Phi_0}\ ,\quad \Delta G=\frac{4\pi}{H_c^2S\xi_0}\Delta G_{\rm conv}\ .$$

Here the label "conv" means quantities in conventional cgs units. ψ_∞ is the zero-field bulk order parameter, S the uniform cross section of the wires, and Φ_0 the flux quantum. All lengths will be normalized by ξ_0, the coherence length at the phase boundary.

We refer the perturbation parameters to a point at the phase boundary choosing x such that $x^2=\Delta T/(T_c-T_0)$, where $\Delta T=T_0-T$, T_0 being the phase-transition temperature of the network and T_c the critical temperature of the bulk material. Defining $\tau=1-T/T_c$, we have $x^2=\Delta\tau/\tau_0$.

The other perturbation parameter is related to the magnetic field. Calling $\mathbf{A}_0$ the vector potential of the applied field at the phase-transition boundary, the total vector potential $\mathbf{A}$ can be written as

$$\mathbf{A}=\mathbf{A}_e+\mathbf{A}_i=\mathbf{A}_0+\mathbf{A}_1+\mathbf{A}_i\ , \tag{1}$$

where $\mathbf{A}_1$ is the departure of the applied (external) field $\mathbf{A}_e$ from its value at the phase boundary and $\mathbf{A}_i$ the contribution from the induced (internal) currents. We choose as the other expansion parameter the quantity y such that $y^2=1/\xi\oint_c A_1(s)ds=\Delta\phi$, where ξ is the GL coherence length and c is a reference loop of area S_c in the network. It should be noted that y^2 is the incremental external magnetic flux $\Delta\phi$ through c and is temperature independent.

Following Ref. 4 we define a complex modified order parameter (mop) for branch ab by (see Fig. 1)

$$f(a,s)=e^{i\gamma(a,s)}\Delta(s)\ , \tag{2}$$

where $\gamma(a,s)=(1/\xi)\int_0^s A(s')ds'$. This mop simplifies the equations although it depends on the definition of a reference point (a) and on the choice of the route. For each branch it is defined from a starting node a, for which $s=0$.

As stated above, we assume an expansion of all physical quantities in powers of x and y. In particular,

$$f(a,s)=\sum_{k,l}f_{kl}(a,s)x^ky^l$$
$$(k+1=1,3,5,\ldots,2p-1,\ldots) \tag{3}$$

and similar expansions are defined for $\Delta(s)$, j, $A_i(s)$, and ΔG with coefficients $\Delta_{kl}(s)$, j_{kl}, $A_{kl}(s)$, and G_{kl}, respectively. Here $p\geq 1$ stands for the perturbation order to be considered. We will call "zero"-order perturbation the results of the linearized theory.[1,2] For the complex conjugate of the mop f, it can be seen that

$$f^*(a,s)=\sum_{k,l}\sigma_l f^*_{kl}(a,s)x^ky^l\ ,$$

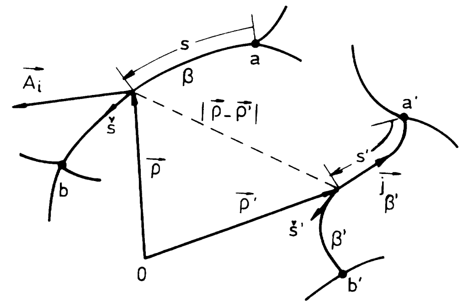

FIG. 1. Branches ab and $a'b'$ of the network and geometry used in the calculation of the self- and mutual induction effects.

where $\sigma_l=\sigma_0$ or σ_1 according to whether l is even or odd. In turn, $\sigma_0=\mathrm{sgn}(\Delta\tau)$ and $\sigma_1=\mathrm{sgn}(\Delta\phi)$.

For the coherence length $\xi(T)=\xi(x)$ and penetration depth $\lambda(T)=\lambda(x)$, we have

$$\xi=1-\frac{x^2}{2}+\frac{3}{8}x^4-\cdots=\sum_k(-)^kC_kx^{2k}\ , \tag{4}$$

$$\lambda^{-2}=\frac{1+x^2}{\kappa^2}\ , \tag{5}$$

where κ is the temperature-independent GL constant.

For the circulation of the external applied vector potential we have the relation which follows from the definition of the perturbation parameter y:

$$\gamma_e(a,s)=\frac{1}{\xi}\int_0^s A_e(s')ds'=\gamma_0(a,s)+\gamma_1(a,s)y^2\ ,$$

where

$$\gamma_0(a,s)=\frac{1}{\xi}\int_0^s A_0(s')ds'$$

and

$$\gamma_1(a,s)=\frac{1}{\xi\Delta\phi}\int_0^s A_1(s')ds'\ .$$

It is clear that $\gamma_1(a,s)$ does not depend on the strength of $\Delta\phi$ but on the configuration of the incremental field $\mathbf{A}_1$ and on the geometry of the network. Taking this into account, the expansion of $\gamma(a,s)$, the circulation of the total vector potential, is

$$\gamma(a,s)=\sum_{k,l}\gamma_{kl}(a,s)x^ky^l\ ,$$

where

$$\gamma_{00}(a,s)=\gamma_0(a,s)\ ,$$
$$\gamma_{02}(a,s)=\gamma_1(a,s)+\int_0^s A_{02}(s')ds'$$

Otherwise

$$\gamma_{kl}(a,s)=\sum_{2m\leq k}(-)^{m-1}\frac{C_m}{2m-1}\int_0^s A_{k-2m,l}(s')ds'\ .$$

It should be noticed that, in general, we have

$$f_{kl}(a,s)=e^{i\gamma_0(a,s)}\sum_{\substack{p\le k\\ q\le l}}\Gamma_{pq}(a,s)\Delta_{k-p,l-q}(s)\ , \tag{6}$$

where Γ_{pq} can be readily obtained from γ_{pq}, as indicated in Appendix A.

In order to arrive at the perturbation equations we must replace the above expansions in both GLE

$$\xi^2\ddot{f}+(1-|f|^2)f=0\ , \tag{7}$$

$$\nabla\times(\nabla\times\mathbf{A})=\frac{1}{\lambda^2}\mathbf{j}\ , \tag{8}$$

where $\mathbf{j}$ in terms of f is given by

$$\mathbf{j}=i\frac{\xi}{2}(f^*\dot{f}-f\dot{f}^*)\hat{\mathbf{s}}\ , \tag{9}$$

$\hat{\mathbf{s}}$ being the unit vector along the wire. The dot stands for the derivative with respect to s. From the first GL equation we obtain, for a given branch ab,

$$\ddot{f}_{kl}(a,s)+f_{kl}(a,s)=\omega_{kl}(a,b,s)\ , \tag{10}$$

where

$$\omega_{kl}(a,b,s)=\sum_{\substack{p\le k,q\le l\\ m\le k-p\\ n\le l-q}}\sigma_q f_{k-p-m,l-q-n}f^*_{pq}f_{mn} - \sum_{\substack{p\le k-2\\ (k-p\ \text{even})}}(-)^{(k-p)/2}(\omega_{pl}-f_{pl})\ . \tag{11}$$

The dependence of ω_{kl} on b comes through the lower order mop's at the nodes according to Eqs. (13a) and (13b). We shall leave the analysis of the perturbation equations that follows from (8) for the next sections.

If the network has n nodes, Eq. (10) has to be solved using the boundary conditions

$$\sum_{(\beta)}\dot{f}_{kl}(a,0)=0\quad (a=1,\ldots,n)\ , \tag{12}$$

where (β) stands for all branches connected to a given node a.

As in Ref. 4, with this perturbation scheme we can solve Eq. (10) by a variation of the constants method:

$$f_{kl}(a,s)=C_{kl}(a,b,s)\sin(L-s)+D_{kl}(a,b,s)\sin s\ , \tag{13a}$$

where L is the length of the branch and

$$C_{kl}(a,b,s)=\frac{f_{kl}(a,0)-Z_{kl}(a,b,s)}{\sin L}\ ,$$

$$D_{kl}(a,b,s)=\frac{f_{kl}(a,L)-Z_{kl}(b,a,L-s)}{\sin L}\ , \tag{13b}$$

$$Z_{kl}(a,b,s)=\int_0^s\omega_{kl}(a,b,s')\sin s'ds'\ .$$

Note that ω_{kl} has the important property $\omega_{kl}(b,a,L-s)=\omega_{kl}(a,b,s)$.

When the form (13a) for the mop is replaced in the boundary conditions (12), a set of equations determining the mop's at the nodes $f_{kl}(a,0)$ in a given order (k,l) in perturbation theory is obtained. In terms of the normalized order parameter $\Delta(a)$, at each node a we have

$$\Delta_{kl}(a)\sum_{(\beta)}\cot L_\beta-\sum_{(\beta)}\frac{e^{i\gamma_0(a,L_\beta)}\Delta_{kl}(\beta)}{\sin L_\beta}=\Pi_{kl}(a)\qquad (a=1,\ldots,n)\ , \tag{14}$$

where

$$\Pi_{kl}(a)=\sum_{(\beta)}\frac{1}{\sin L_\beta}\left[e^{i\gamma_0(a,L_\beta)}\sum_{\substack{p<k\\ q<1\\ (p+q>0)}}\Gamma_{pq}(a,L_\beta)\Delta_{k-p,l-q}(\beta)-Z_{kl}(\beta,a,L_\beta)\right]\ . \tag{15}$$

These equations are similar to Eqs. (13) of Ref. 4, except that now we have two independent expansion parameters x and y.

III. NORMALIZATION OF THE ORDER PARAMETER

The lowest-order equations (10) or the discrete form (14) are homogeneous equations and do not fix the amplitude of the order parameter, a situation similar to that encountered by Abrikosov in the analysis of the mixed state of type-II superconductors. As discussed in Ref. 4, the orthogonality condition in the form used by Abrikosov for the differential equation cannot be applied to networks.

For each order of perturbation theory, Eqs. (14) can be written as

$$\underline{A}\boldsymbol{\nabla}_{kl}=\boldsymbol{\Pi}_{kl}\ ,\quad \underline{A}=[\mathcal{A}(a,b)]\ ,$$

where $\underline{A}$ is an $n\times n$ Hermitian matrix with components $\mathcal{A}(a,b)$. To lowest order we have $\boldsymbol{\Pi}_{10}=\boldsymbol{\Pi}_{01}\equiv\mathbf{0}$ and the corresponding equations are homogeneous, identical to the de Gennes–Alexander linearized equations.[1] Calling $\boldsymbol{\Delta}_0$ the solutions of the homogeneous equations, the complex scalar product of the vectors $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Pi}_{kl}$ vanishes: $(\boldsymbol{\Delta}_0,\boldsymbol{\Pi}_{kl})=0$, a relation similar to Abrikosov's for the mixed state. The ensuing relations for the modified order parameter are

$$\sum_\beta\left\{\sum_{\substack{p\le k\\ q\le l\\ (p+q>0)}}\Gamma_{pq}(a,L_\beta)[f_{k-p,l-q}(\beta,0)\dot{f}^*_0(\beta,0)-f^*_0(\beta,0)\dot{f}_{k-p,l-q}(\beta,0)]-\int_0^{L_\beta}f^*_0(a,s)\omega_{kl}(a,\beta,s)ds\right\}=0\ , \tag{16}$$

where the summation is over all branches. Here $f(b,s)$ stands for f measured from the node b and $f_0(a,s)$ is the "zero"-order mop. These relations allow us to determine the amplitude of the order parameter at the different nodes and, from Eq. (13), the amplitude at any place in the network.

We give now the explicit form of Eq. (16) for the lowest order. The first amplitudes to be determined correspond to the normalization of the "zero"-order solution. Choosing a reference node $a=1$ at which $f_0(1,0)=1$, all other amplitudes associated with the linearized GLE are determined from the homogeneous Eqs. (14). The physically relevant order parameter to lowest order is

$$f_1(a,s)=f_{10}(a,s)x+f_{01}(a,s)y ,$$

where $f_{10}(a,s)=\alpha_{10}f_0(a,s)$ and $f_{01}(a,s)=\alpha_{01}f_0(a,s)$, α_{10} and α_{01} being constant amplitude factors. The normalization conditions allow us to determine α_{10} and α_{01}. For this order the values of k and l to be used in Eq. (16) are such that $k+l=3$. From Eq. (11) we obtain explicit expressions for ω_{30}, ω_{21}, ω_{12}, and ω_{03} in terms of f_{10} and f_{01}. Now, in terms of these variables we get from Eq. (16) four relations:

$$2i\alpha_{10}\sum_\beta \Gamma_{20}(a,L_\beta)j_{0\beta}+\sigma_0|\alpha_{10}|^2\alpha_{10}B_3^2-\alpha_{10}B_1=0 , \tag{17a}$$

$$2i\alpha_{01}\sum_\beta \Gamma_{20}(a,L_\beta)j_{0\beta}+2i\alpha_{10}\sum_\beta \Gamma_{11}(a,L_\beta)j_{0\beta} +(2\sigma_0|\alpha_{10}|^2\alpha_{01}+\sigma_1\alpha_{10}^2\alpha_{01}^*)B_3^2-\alpha_{01}B_1=0 , \tag{17b}$$

$$2i\alpha_{10}\sum_\beta \Gamma_{02}(a,L_\beta)j_{0\beta}+2i\alpha_{01}\sum_\beta \Gamma_{11}(a,L_\beta)j_{0\beta} +(2\sigma_1|\alpha_{01}|^2\alpha_{10}+\sigma_0\alpha_{01}^2\alpha_{10}^*)B_3^2=0 , \tag{17c}$$

$$2i\alpha_{01}\sum_\beta \Gamma_{02}(a,L_\beta)j_{0\beta}+\sigma_1|\alpha_{01}|^2\alpha_{01}B_3^2=0 , \tag{17d}$$

where the quantities B_1 and B_3^2 are defined in Eqs. (30) and

$$j_{0\beta}=(i/2)[f_0^*(a,0)f_0(a,0)-f_0(a,0)f_0^*(a,0)]$$

is the "zero"-order current density in each branch. Since the α's are complex, we need the four equations (17) to determine α_{10} and α_{01}.

For higher orders we have, from Eqs. (14) and (6), relations of the form

$$\begin{aligned} &f_{kl}(1,0)=\alpha_{kl} ,\\ &f_{kl}(a,0)=[\alpha_{kl}-X_{kl}(a)]f_0(a,0) \quad (a\neq 1) , \end{aligned} \tag{18}$$

and

$$f_{kl}(a,L)=[\alpha_{kl}-X_{kl}(b)+Y_{kl}(a,b,L)]f_0(a,L) . \tag{19}$$

Here

$$X_{kl}(a)=\frac{\begin{vmatrix} \mathcal{A}(2,2) & \cdots & \mathcal{A}(2,a-1) & \Pi_{kl}(2) & \mathcal{A}(2,a+1) & \cdots & \mathcal{A}(2,n)\\ \mathcal{A}(3,2) & \cdots & \mathcal{A}(3,a-1) & \Pi_{kl}(3) & \mathcal{A}(3,a+1) & \cdots & \mathcal{A}(3,n)\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \mathcal{A}(n,2) & \cdots & \mathcal{A}(n,a-1) & \Pi_{kl}(n) & \mathcal{A}(n,a+1) & \cdots & \mathcal{A}(n,n) \end{vmatrix}}{\begin{vmatrix} \mathcal{A}(2,2) & \cdots & \mathcal{A}(2,a-1) & \mathcal{A}(2,1) & \mathcal{A}(2,a+1) & \cdots & \mathcal{A}(2,n)\\ \mathcal{A}(3,2) & \cdots & \mathcal{A}(3,a-1) & \mathcal{A}(3,1) & \mathcal{A}(3,a+1) & \cdots & \mathcal{A}(3,n)\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ \mathcal{A}(n,2) & \cdots & \mathcal{A}(n,a-1) & \mathcal{A}(n,1) & \mathcal{A}(n,a+1) & \cdots & \mathcal{A}(n,n) \end{vmatrix}} \tag{20}$$

and $Y_{kl}(a,b,L)$ stands for the sum

$$Y_{kl}(a,b,L) = \sum_{\substack{p\le k\\ q\le l\\ (p+q>0)}} \Gamma_{pq}(a,L)[\alpha_{k-p,l-q}-X_{k-p,l-q}(b)] .$$

The constant amplitude factors α_{kl} can be determined using condition (16) as previously done in lowest order. Knowledge of $f_{kl}(a,0)$ and $f_{kl}(a,L)$ allows us to determine $f_{kl}(a,s)$ along any branch ab using Eq. (13). Note that, because of our definition of the mop, the value of f at a given node is not uniquely determined. It can be shown that this gives an ambiguity in the definition of the f's which is limited to a phase difference.

IV. SELF-INDUCTION EFFECTS

At the phase-transition boundary, the superconductive currents are zero and only the external magnetic field acts on the system. Away from the phase boundary the currents become finite, creating induced fields. It is important to take proper care of these and to study their contribution to the free energy.

The fields created by the currents are determined by the second GL equation [Eqs. (8) and (9)]. The total vector potential is the sum of the applied field $\mathbf{A}_e$ plus the induced field $\mathbf{A}_i$. This last part is formally given, as solution of Eq. (8), by (see Fig. 1)

$$\mathbf{A}_i(\boldsymbol{\rho})=\frac{1}{4\pi\lambda^2}\int_V \frac{\mathbf{j}(\boldsymbol{\rho}')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}d^3\rho' .$$

In general, we are interested in the component of $\mathbf{A}(\rho)$ along a given wire in the network. For $\mathbf{A}_i$ this is given by a sum over all branches of the network:

$$A_i(s)=\mathbf{A}_i(\rho)\cdot\hat{\mathbf{s}}=\frac{S}{4\pi\lambda^2}\sum_{\beta'} j_{\beta'}\int_0^{L_{\beta'}}\frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'}{|\rho-\rho'|}ds' . \quad (21)$$

Using Eqs. (21), (3), and (5), we can find the following relation between the perturbation coefficients for field and current density:

$$A_{kl}(s)=\frac{S}{4\pi\kappa^2}\sum_{\beta'}[(j_{\beta'})_{kl}+(j_{\beta'})_{k-2,l}]\int_0^{L_{\beta'}}\frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'}{|\rho-\rho'|}ds' .$$

j_{kl} are independent of s because of current conservation along each branch. In a given branch β,

$$j_{kl}=\frac{i}{2}\sum_{\substack{p\le k\\ q\le l}}\sum_{2m\le k-p}(-)^m\sigma_q C_m[f^*_{pq}(a,0)\dot{f}_{k-p-2m,l-q}(a,0)-f_{k-p-2m,l-q}(a,0)\dot{f}^*_{pq}(a,0)] . \quad (22)$$

The free energy of the system, measured from the normal state, is given by

$$\Delta G=S^{-1}\int_{\text{all space}}\left[-|\Delta|^2+\frac{|\Delta|^4}{2}+\xi^2\left|\left[i\nabla-\frac{\mathbf{A}}{\xi}\right]\Delta\right|^2+|\mathbf{h}-\mathbf{H}|^2\right]dV , \quad (23)$$

where $\mathbf{H}$ is the applied field and $\mathbf{h}$ is the total field, $\mathbf{h}=\lambda\nabla\times\mathbf{A}$. In terms of the mop

$$\Delta G=\sum_\beta\int_0^{L_\beta}\left[-|f|^2+\frac{|f|^4}{2}+\xi^2|\dot{f}|^2\right]ds+S^{-1}\int_{\text{all space}}|\mathbf{h}-\mathbf{H}|^2dV . \quad (24)$$

The inductive effects are contained in $\xi^2|\dot{f}|^2$ and also in the last term in (24). This can be written as

$$\Delta G_i=S^{-1}\int_{\text{all space}}|\mathbf{h}-\mathbf{H}|^2dV=S^{-1}\int_V \mathbf{j}\cdot\mathbf{A}_i dV ,$$

where V is the volume of the network and can be transformed into

$$\Delta G_i=\sum_\beta j_\beta\int_0^{L_\beta}A_i(s)ds=\frac{S}{4\pi\lambda^2}\sum_{\beta\beta'}j_\beta j_{\beta'}K_{\beta\beta'} , \quad (25)$$

where

$$K_{\beta\beta'}=\int_0^{L_\beta}\int_0^{L_{\beta'}}\frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'}{|\rho-\rho'|}ds'ds .$$

The geometrical factors $K_{\beta\beta'}$ are related to the self- and mutual inductances of the network. In fact, the sum over branches in (25) can be replaced by a sum over loops μ and transformed into

$$\Delta G_i=\frac{S}{\lambda^2}\left[\frac{1}{2}\sum_\mu\Lambda_\mu j_\mu^2+\sum_{\substack{\mu\\ \mu'>\mu}}M_{\mu\mu'}j_\mu j_{\mu'}\right] . \quad (26)$$

This is the Neumann formula for the magnetic energy of a system of linear currents. Λ_μ and $M_{\mu\mu'}$ are the self- and mutual induction coefficients

$$M_{\mu\mu'}=\frac{1}{2\pi}\oint_\mu ds\oint_{\mu'}ds'\frac{\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'}{|\rho-\rho'|} , \quad \Lambda_\mu=M_{\mu\mu} .$$

The total free-energy expansion is $\Delta G=\sum_{kl}G_{kl}x^ky^l$, where G_{kl} can be written as

$$G_{kl}=\sum_\beta\int_0^{L_\beta}\left[-|f|^2_{kl}+\frac{|f|^4_{kl}}{2}+(\xi^2|\dot{f}|^2)_{kl}\right]ds+\frac{S}{2\kappa^2}\sum_{\mu\mu'}\sum_{\substack{p\le k\\ q\le l}}[(j_\mu)_{k-p,l-q}+(j_\mu)_{k-p-2,l-q}]\times(j_{\mu'})_{pq}M_{\mu\mu'} . \quad (27)$$

The self-induction effects can be neglected whenever $S/\kappa^2\to 0$, i.e., when one assumes that the wires of the network are really one-dimensional elements or the material is extreme type II. When L/ξ_0 is small, the $M_{\mu\mu'}$'s and the self-induction effects can also be neglected.

V. FIRST-ORDER PERTURBATION RESULTS

A. Order parameter and current density

From the normalization conditions (17), we obtain

$$|\alpha_{10}|^2=\sigma_0\frac{B_1}{D} , \quad |\alpha_{01}|^2=\sigma_1\frac{B_2}{D} , \quad D=B_3^2-\frac{S}{\kappa^2}B_4^2 . \quad (28)$$

The following relation also obtains

$$\sigma_0\alpha_{01}\alpha^*_{10}+\sigma_1\alpha^*_{01}\alpha_{10}=0 , \quad (29)$$

which is essential to ensure that $|f|^2$ is real. In these equations

$$B_1=\sum_\beta\int_0^{L_\beta}|f_0|^2ds , \quad (30a)$$

$$B_2=2\sum_\beta\gamma_1(a,L_\beta)j_{0\beta} , \quad (30b)$$

$$B_3^2=\sum_\beta\int_0^{L_\beta}|f_0|^4ds , \quad (30c)$$

$$B_4^2=\sum_\mu\Lambda_\mu j_{0\mu}^2+2\sum_{\substack{\mu\\ \mu'>\mu}}M_{\mu\mu'}j_{0\mu}j_{0\mu'} . \quad (30d)$$

Here $j_{0\mu}$ is the "zero"-order current density in loop μ. The coefficients B are similar to the β's obtained in Ref. 4, but B_4^2 as written above shows explicitly the self- and mutual induction effects.

One can determine from Eqs. (28) and (29) for both α_{10}

and α_{01} their square amplitude and relative phase. For the mop to first order we have explicitly

$$|f(a,s)|^2 \simeq \frac{[B_1(\Delta\tau/\tau 0)+B_2\Delta\phi]}{D}|f_0(a,s)|^2$$
$$=\alpha^2|f_0(a,s)|^2 \ . \quad (31)$$

For the current density using Eq. (24) we obtain similarly $j \simeq \alpha^2 j_0$.

The perturbation coefficients B can be interpreted in terms of different contributions to the lowest-order energy. So, taking $\xi \simeq 1$ at the phase boundary, the kinetic energy of the supercurrents is related to B_1:

$$\frac{\Delta\tau}{\tau_0}\frac{1}{2S}\int_V \Delta_0^*(i\nabla - \mathbf{A}_0)^2\Delta_0 dV = \frac{1}{2}\frac{\Delta\tau}{\tau_0}\sum_\beta \int_0^{L_\beta}|f_0|^2 ds = \frac{1}{2}\frac{\Delta\tau}{\tau_0}B_1 \ ,$$

the interaction energy of the currents with the external field increment is related to B_2:

$$\frac{1}{S}\int_V \mathrm{j}_0\cdot\mathbf{A}_1 dV = \tfrac{1}{2}\Delta\phi B_2 \ ,$$

the superconductive condensation energy is

$$\frac{1}{2S}\int_V |\Delta_0|^4 dV = \tfrac{1}{2}\sum_\beta \int_0^{L_\beta}|f_0|^4 ds = \frac{B_3^2}{2}$$

and the magnetic energy of the induced field is

$$\frac{S}{\kappa^2}\left[\frac{1}{2}\sum_\mu \Lambda_\mu j_{0\mu}^2 + \sum_{\substack{\mu \\ \mu'>\mu}} M_{\mu\mu'}j_{0\mu}j_{0\mu'}\right] = \frac{S}{2\kappa^2}B_4^2 \ .$$

Finally, $D/2$ can be identified with the equilibrium free energy of the system: $D/2 = -\Delta G_0$.

B. The Ehrenfest-Keesom equations for the phase-transition boundary

Equation (31) gives the order parameter at a point separated from the phase boundary a certain distance $\Delta\phi$ and $\Delta\tau$. If these values were chosen such that the end point is again on the phase boundary, the order parameter must vanish. This gives a relation between $\Delta\phi$ and $\Delta\tau$, which gives rise to a differential equation for the phase boundary; in terms of $L = L_{\mathrm{conv}}/\xi_0(T_0)$, this reads

$$\frac{dL}{d\phi_0} = -\frac{L}{2}\frac{B_2}{B_1} \ . \quad (32)$$

This equation can be identified with one of the Ehrenfest-Keesom relations[6] for second-order phase transitions. Written with the variables we are using, these relations are

$$\frac{1}{S_c}\left[\frac{d\phi}{d\tau}\right]_0 = \frac{\Delta\varepsilon}{\Delta\eta_\tau} \ , \quad \frac{1}{S_c}\left[\frac{d\phi}{d\tau}\right]_0 = \frac{1}{(1-\tau_0)}\frac{\Delta C_\phi}{\Delta\varepsilon} \ , \quad (33)$$

where ε is the coefficient for the thermal variation of the magnetization, η_τ is the isothermal susceptance, and C_ϕ is the specific heat at constant field, defined by

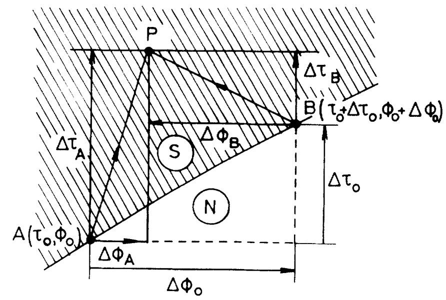

FIG. 2. Portion of the phase diagram indicating normal (N) and superconductive (S) regions. A point like P in region S can be reached from a point A at the phase-transition line through increments $\Delta\tau_A$ in the reduced temperature and $\Delta\phi_A$ in flux or from point B through increments $\Delta\tau_B$ and $\Delta\phi_B$.

$$\varepsilon = -\frac{1}{V}\left[\frac{\partial\mathcal{M}}{\partial\tau}\right]_\phi \ ,$$
$$\eta_\tau = \frac{S_c}{V}\left[\frac{\partial\mathcal{M}}{\partial\phi}\right]_\tau \ , \quad (34)$$
$$C_\phi = \frac{(\tau-1)}{V}\left[\frac{\partial\mathcal{S}}{\partial\tau}\right]_\phi \ .$$

In turn,

$$\mathcal{M} = -S_c\left[\frac{\partial G}{\partial\phi}\right]_\tau$$

and (35)

$$\mathcal{S} = \left[\frac{\partial G}{\partial\tau}\right]_\phi$$

are the total magnetic dipole moment and the entropy, respectively.

Since we are using two independent expansion parameters to describe the properties of the superconductive phase a given distance away from the phase boundary, the question arises whether this description is unique. As shown in Fig. 2, a point P in the superconductive phase can be reached from more than one point at the phase boundary. Neglecting second-order terms in $\Delta\tau$ and $\Delta\phi$, it can be seen that the Ehrenfest-Keesom equation (32) implies that Eq. (31) gives a unique result.

C. Free energy and thermodynamic relations

The free energy in first-order perturbation is

$$\Delta G \simeq -\frac{D}{2}\left[\left[\frac{B_1}{D}\right]\frac{\Delta\tau}{\tau_0} + \left[\frac{B_2}{D}\right]\Delta\phi\right]^2 = -\frac{\alpha^4 D}{2} \ . \quad (36)$$

This result, when the self-induction effects are neglected, reduces to Eq. (21) of Ref. 4.

Using Eqs. (34)–(36), we can calculate the values of the different thermodynamic variables in the superconductive state relative to the normal state, so explicit forms for $\Delta\mathcal{M}$, $\Delta\mathcal{S}$, $\Delta\varepsilon$, $\Delta\eta_\tau$, and ΔC_ϕ can be given in terms of B_1, B_2, and D. Since ΔG above refers to the whole system, $\mathcal{M}$ in (35) is the total magnetic moment of the network. For networks there is no meaning for a local magnetic moment so the relevant thermodynamic quantity is the susceptance and not the susceptibility.

For the sake of comparison with Abrikosov's theory for type-II bulk materials, we can give the expressions of the jump in the specific heat, thermal variation of magnetization coefficient, and magnetic susceptance across the phase boundary in cgs units:

$$(\Delta C_H)_{\rm bulk}=\frac{T_0}{16\pi^3\beta_A(2\kappa^2-1)}\frac{\Phi_0^2}{\xi^4(0)T_c^2}\,,\qquad (\Delta C_H)_{\rm network}=\frac{T_0}{16\pi^3\beta'_A(2\tilde{\kappa}^2-1)}\frac{\Phi_0^2}{\xi^4(0)T_c^2}\,, \tag{37a}$$

$$(\Delta\epsilon)_{\rm bulk}=\frac{1}{8\pi^2\beta_A(2\kappa^2-1)}\frac{\Phi_0}{\xi^2(0)T_c}\,,\qquad (\Delta\epsilon)_{\rm network}=\frac{1}{8\pi^2\beta''_A(2\tilde{\kappa}^2-1)}\frac{\Phi_0}{\xi^2(0)T_c}\,, \tag{37b}$$

$$(\Delta\eta_T)_{\rm bulk}=\frac{1}{4\pi\beta_A(2\kappa^2-1)}\,,\qquad (\Delta\eta_T)_{\rm network}=\frac{1}{4\pi\beta'''_A(2\tilde{\kappa}^2-1)}\,, \tag{37c}$$

where β_A is Abrikosov's factor, $\xi(0)$ is Gor'kov's coherence length, and

$$\beta'_A=2V\frac{B_4^2}{B_1^2}\,,\qquad \beta''_A=-2V\frac{B_4^2}{B_1S_cB_2}\,,\qquad \beta'''_A=2V\frac{B_4^2}{S_c^2B_2^2}\,,\qquad \tilde{\kappa}=\frac{1}{\sqrt{2S}}\frac{B_3}{B_4}\kappa\,. \tag{38}$$

We see that, instead of a single factor like Abrikosov's β_A, there are three different factors related to ratios of different contributions to the energy which are specific to the network structure. Besides, the topology and the geometry of the network require a renormalization of the GL constant κ. The factors β'_A, β''_A, and β'''_A are related by the Ehrenfest-Keesom equation (33).

The specific heat at constant magnetization can be obtained from the usual thermodynamic relation $C_{\mathcal{M}}-C_\phi=4\pi(1-\tau)\epsilon^2/\eta_\tau$; using Eqs. (37) we have, for the specific heat $\Delta C_{\mathcal{M}}$ in the superconductive state relative to the normal state, $\Delta C_{\mathcal{M}}=(1+4\pi)\Delta C_\phi$. As a check of the perturbation expansions we can see from Eqs. (37) and (32) that the thermodynamic quantities, independently calculated, satisfy the Ehrenfest-Keesom relations (33).

VI. PHYSICAL RESULTS

A. Landau critical points (LCP)

We can see from (31) and (36) that, whenever $D(\tau_0,\phi_0)=0$, there is a change in the behavior of the solutions. It can be seen from (31) that $|f|^2$ is always positive when the increments $\Delta\tau$ and $\Delta\phi$ bring us from a point at the phase boundary to another point inside the superconducting region. A special situation arises when $D(\tau_0,\phi_0)$ changes sign. In that case, in order to have a positive definite $|f|^2$, $\Delta\tau$ and $\Delta\phi$ must bring us out of the superconducting region and into the "normal" region. When this happens, the free energy as a function of $\Delta\tau$ (or $\Delta\phi$) undergoes a change in curvature, which indicates that the system goes through a Landau critical point (see Fig. 3). The transition changes to first order and the perturbation approach gives the supercooling limit.

The change in sign in D allows for the classification of the transitions as follows:

$$D>0\ \text{(second order)}\,,\qquad D=0\ \text{(LCP)}\,,\qquad D<0\ \text{(first order)}\,. \tag{39}$$

We see from (28) that S/κ^2 and B_4^2, which contain the self-induction effects, are important to determine the kind of transition to be expected. In a first-order transition the positive energy associated with the magnetic field of the induced currents overcomes the negative contribution from the condensation energy. Superconductivity is only possible with a finite and large order parameter so the system makes a jump to meet that condition.

We checked our results against exact calculations for a hollow cylinder.[7] Using the values given in Fig. 7 of Ref. 7, we obtain $T_{\rm LCP}\simeq0.961T_c$. This is slightly different from the result in Ref. 7 ($T_{\rm LCP}\simeq0.987T_c$) due to the fact that the exact calculation takes into account field expulsion effects (Meissner currents) which are neglected in the de Gennes–Alexander model which we use.

It can be seen from (28) that $1/\kappa^2$ amplifies the self-

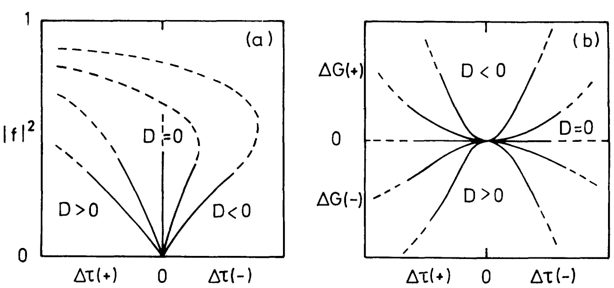

FIG. 3. Instabilities of (a) the order parameter and (b) free energy near a Landau critical point. The LCP is given by the condition $D=0$ (see text). $\Delta\tau(+)$ means a positive temperature change from the phase boundary.

inductive effects given by B_4^2. Recent experimental work on the critical currents and I-V characteristics in the resistive transition of micronetworks suggests a strong dependence on the type of material.[8] In particular, in Ref. 8, In and Al are used, which have κ values differing by a factor of 10. Although we have not addressed the question of critical currents in this paper, our results suggest that induction effects help understand the difference in the experimental results of Ref. 8 for In and Al.

B. Relative stability of different modes

In general, for any network, different condensation modes are characterized by the value of the integer n in the fluxoid quantization condition.[11] Within the superconductive region a given mode will be stable when its associated free energy is lower than that of any other mode. The boundaries separating different modes are determined by the condition $\Delta G^{(n)}(\phi,\tau)=\Delta G^{(n+1)}(\phi,\tau)$. Using the variable $l=L_{\rm conv}/\xi(T)$, it can be shown from (36) that, to first order in perturbation theory, the line along which this is satisfied obeys the equation

$$\phi=\phi_0+\left[\frac{B_1^{(n)}\sqrt{D^{(n+1)}}-B_1^{(n+1)}\sqrt{D^{(n)}}}{B_2^{(n)}\sqrt{D^{(n+1)}}-B_2^{(n+1)}\sqrt{D^{(n)}}}\right]\left[1-\frac{l^2}{L^2}\right]. \tag{40}$$

The stability lines drawn in Figs. 2 and 3 of Ref. 4 were obtained numerically from Eq. (40) neglecting self-inductive effects.

VII. SECOND-ORDER PERTURBATION FORMALISM

A. Second-order expansions

In the previous paragraphs we have discussed the lowest-order perturbation expansion in detail, having obtained explicit results for the coefficients in the expansion for the order parameter, current, vector potential, and free energy [see Eq. (3)].

From Ref. 4, we know a comparison of these results with exact calculations shows that very good agreement is obtained in first order for the slopes of current and order parameter as function of temperature or field. This is also true for the phase separation lines for different modes in the (H,T) plane.

In order to explore the superconducting region further apart from the phase boundary, it is necessary to go up to second-order terms. From these we expect to predict the curvature of the different variables quoted above as functions of field and temperature. Once the formalism is developed and checked against exact results in some simple cases, the theory is equipped with a trustful procedure to study more complicated systems.

The second-order contribution to the mop follows from Eqs. (11) and (13) with $k+l=3$. Detailed expressions for the coefficients $f_{kl}(a,s)$ are given in Appendix B. For the squared order parameter, the expansion up to second order is thus

$$|\Delta|^2\simeq\alpha^2|\Delta_0|^2+|\Delta|_{40}^2\left[\frac{\Delta\tau}{\tau_0}\right]^2+|\Delta|_{22}^2\frac{\Delta\tau}{\tau_0}\Delta\phi+|\Delta|_{04}^2(\Delta\phi)^2 \tag{41}$$

and for the currents the result is

$$j\simeq\alpha^2 j_0+j_{40}\left[\frac{\Delta\tau}{\tau_0}\right]^2+j_{22}\frac{\Delta\tau}{\tau_0}\Delta\phi+j_{04}(\Delta\phi)^2\ , \tag{42}$$

where α^2 is defined in Eq. (31). Note that $|\Delta|_{31}^2=|\Delta|_{13}^2=0$ and $j_{31}=j_{13}=0$, which ensures that the squared order parameter and the current density are always real.

The free energy reads

$$\Delta G\simeq-\frac{\alpha^4 D}{2}+G_{60}\left[\frac{\Delta\tau}{\tau_0}\right]^3+G_{42}\left[\frac{\Delta\tau}{\tau_0}\right]^2\Delta\phi+G_{24}\frac{\Delta\tau}{\tau_0}(\Delta\phi)^2+G_{06}(\Delta\phi)^3\ . \tag{43}$$

In this case, $G_{51}=G_{33}=G_{15}=0$, and ΔG is always real. The explicit expressions for the coefficients are given at the end of Appendix C. In the following sections we shall apply these perturbation calculations to simple systems and compare the ensuing results with numerical calculations.

B. Normalization of the order parameter

The normalization coefficients $\alpha_{kl}(k+l=3)$, which give the part of the mop which is proportional to the "zero"-order solution, are determined using the normalization conditions (16) with $k+l=5$. After some algebra, the relations (C1) given in Appendix C are obtained. In principle, the linear system of equations (C1) can be solved for α_{30}, α_{21}, α_{12}, and α_{03}, but this is not necessary because all physically observable quantities ($|\Delta|^2$, j, ΔG, etc.) depend on the linear combinations of the α_{kl} coefficients given by (C1) and it is possible to make use of these equations without any more algebra.

VIII. APPLICATION TO SIMPLE SYSTEMS

In this section we apply our approximation scheme to two simple nontrivial systems for which exact results are available. The first is the bare ring, the most elementary multiply connected system, for which flux quantization applies. The second one is a two-loop system which shares with the loop the quantization conditions and adds the complication of multiple modes at the phase boundary.

The exact solutions for this last system require numerical integration of the differential equations, which give rise to Jacobi elliptic functions.[9] Due to these complications, any approximate solution should be welcomed.

For more complex systems numerical integration is generally out of the question and so only perturbation or variational methods are of any applicability. It is in these systems that the formalism developed here can be used.

A. Superconducting bare ring

Fink and Grünfeld[9] have explicitly solved the GL equations for a simple wire ring of length L; $|f|$ is uniform around the ring and is simply related to temperature and magnetic flux. The self-induction coefficient $\Lambda=L[\ln(4L/\sqrt{\pi S})-\frac{7}{4}]/\pi$ is different from Fock's formula[10] because here the current is assumed to be uniformly distributed in the cross section S of the wire.

The first-order perturbation coefficients can be calculated from Eqs. (30): $B_1=B_3^2=L$, $B_2=2j_0=\mp 2$, $B_4^2=\Lambda j_0^2=\Lambda$. So, in first-order perturbation, we have

$$|f|^2\simeq\frac{L}{D}\frac{\Delta\tau}{\tau_0}\mp\frac{2}{D}\Delta\phi\ ,\quad j\simeq\mp\frac{L}{D}\frac{\Delta\tau}{\tau_0}+\frac{2}{D}\Delta\phi\ , \tag{44}$$

where $D=L-(S/\kappa^2)\Lambda$. If the self-induction effects are moderate, $D>0$, $|f|^2$ is always positive in the superconductive region and j changes sign according to the mode considered. The Ehrenfest-Keesom equation (32) now takes the simple form $dL/d\phi_0=\pm 1$ or $L=\pm\phi_0+\text{const}$ which is just the characteristic equation for the phase-transition lines.

For the free-energy difference in first-order perturbation, we have from (36)

$$\Delta G\simeq-\frac{L^2}{2D}\left[\frac{\Delta\tau}{\tau_0}\mp\frac{2}{L}\Delta\phi\right]^2 . \tag{45}$$

As discussed in Sec. VI, the condition for a LCP in this model is $D=0$, which in cgs units corresponds to

$$T_{\rm LCP}=T_c\left[1-\frac{\pi\lambda^2(0)}{S(\ln(4L/\sqrt{\pi S})-7/4}\right] .$$

The lines separating regions of stability for different modes (different values of n), as obtained from Eq. (51), are given by $\phi=\phi_0$ meaning that the lines are parallel to the T axis.

Now, we will consider second-order perturbation results. For a network with only one node, as is the case for the ring, the system of equations (14) is homogeneous to all orders. The second-order coefficients for the bare ring can be evaluated using the formulas in Appendix C.

Using (41), the formulas of Appendix B, and relations (C1), we have for $|f|^2$ up to second-order perturbation the result

$$\begin{aligned}|f|^2\simeq{}&\frac{L}{D}\frac{\Delta\tau}{\tau_0}\mp\frac{2}{D}\Delta\phi-\frac{(3L^3-6L^2D+7LD^2)}{4D^3}\left[\frac{\Delta\tau}{\tau_0}\right]^2\\&\pm\frac{(3L^2-4LD+3D^2)}{D^3}\frac{\Delta\tau}{\tau_0}\Delta\phi\\&-\frac{(3L-2D)}{D^3}(\Delta\phi)^2 .\end{aligned} \tag{46}$$

Using (42), we have for the current density in the wire up to second-order perturbation

$$\begin{aligned}j\simeq{}&\mp\frac{L}{D}\frac{\Delta\tau}{\tau_0}+\frac{2}{D}\Delta\phi\pm\frac{(3L^3-4L^2D+7LD^2)}{4D^3}\left[\frac{\Delta\tau}{\tau_0}\right]^2\\&-\frac{(3L^2-2LD+3D^2)}{D^3}\frac{\Delta\tau}{\tau_0}\Delta\phi\pm\frac{3L}{D^3}(\Delta\phi)^2 .\end{aligned} \tag{47}$$

Furthermore, the supercurrent velocity field is $q=\xi(d\theta/ds+d\gamma/ds)$ where θ is the phase of the normalized order parameter; in first-order perturbation we have

$$q\simeq\pm 1\mp\frac{L}{2D}\frac{\Delta\tau}{\tau_0}+\frac{\Delta\phi}{D} . \tag{48}$$

From Eqs. (46)–(48) we can verify that, neglecting powers higher than $(\Delta\tau,\Delta\phi)^2$, the relation $j=-q|f|^2$ holds.[11] Having calculated j, q, and $|f|^2$ independently, this relation proves the consistency of the perturbation expansion. At the phase boundary, j is zero but q remains finite, as shown by Eqs. (47) and (48).

Using formulas (C2) of Appendix C, we can calculate the free energy explicitly up to second order; from (46) and (47) it is possible to verify that, neglecting powers higher than $(\Delta\tau,\Delta\phi)^3$, the relation[11]

$$\begin{aligned}\Delta G&=-\frac{1}{2S}\int_V|\Delta|^4dV+\frac{S}{\lambda^2}\frac{\Lambda j^2}{2}\\&=-\frac{L}{2}\left[|f|^4-\frac{(L-D)}{L\xi^2}j^2\right]\end{aligned} \tag{49}$$

holds, which links the equilibrium free energy to the order parameter and current density. As above, this relation verifies the consistency of the perturbation expansion.

In Fig. 4, we have compared our results given by the perturbative equations with the results obtained from the theory of Fink and Grünfeld.[9] A detailed analysis of the

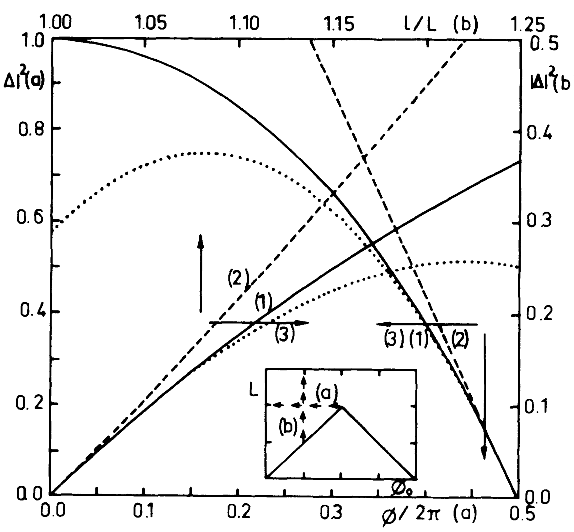

FIG. 4. Comparison of exact and perturbation theory results for a bare ring. $|\Delta|^2(a)$ shows the variation of the normalized order parameter at constant temperature as a function of flux. $|\Delta|^2(b)$ corresponds to the same quantity at constant flux and as a function of the variable $l/L=(T_c-T)/(T_c-T_0)$. The inset shows the paths in the phase diagram. In these plots $\kappa=1/\sqrt{2}$, and $D=0.9\pi$, $\phi_0=\pi$ for (a) and $D=0.4875\pi$, $\phi_0=\pi/2$ for (b). Lines (1) are the exact results of Ref. 9; lines (2) and (3) are first- and second-order calculations. The arrows point the scales to be considered in each case.

comparisons show that the perturbation approach gives quite good agreement with the exact results. In particular, if self-induction effects are neglected, the perturbation scheme to second order gives the exact results.

For the case of Fig. 4 we see that the perturbation follows the general trend of the exact result for a wide range of field or temperature variation. In Fig. 4(a), a change of 50% in the variable ϕ gives a departure of the order of 10% in $|\Delta|^2$.

B. Superconducting two-loop system

The superconducting two-loop system is a particular case of the Wheatstone bridge discussed in the literature.[1] The solutions in the linearized theory can be analyzed using group theory. The balanced Wheatstone bridge is a planar two-loop network with point symmetry group D_{2h} in the absence of an external magnetic field; when in a magnetic field, the system remains invariant only under those point operations that leave **H** invariant[12] and the new point symmetry group is C_{2h}. In fact, the only acceptable point groups for regular superconducting networks are of the form C_s, C_n, C_{nh}, and S_n in the Schönflies notation.[12] Recent theoretical work on the second-order phase transition of regular infinite networks has invoked mirror symmetries in $p4gm$ and $c2mm$ space groups to explain cusps in the phase boundary,[13] but the pseudovector character of **H** prevents such an explanation.

In the presence of a magnetic field, the two-loop system and the balanced Wheatstone bridge have the same symmetry C_{2h} (see Fig. 5). The two-loop system adds the advantage of equal length branches. The phase boundary is determined by two eigenvalues of the linearized equations which cross one another as shown in Fig. 5. The possible modes of $\Delta_0(s)$ transform under symmetry operations as the irreducible representations A_g (symmetric) or B_u (antisymmetric) of C_{2h}. The equations for the phase boundary lines are $\cos L = \pm(1+2\cos\phi_0)/3$, where L is the length of the branches (normalized by ξ_0) and ϕ_0 is the external magnetic flux linked by one elementary loop. The upper (lower) sign holds when the "winding number" n for the order parameter around the external ring is even (odd). For a complete group-theoretical analysis of networks see Ref. 13(b).

The first-order perturbation coefficients (30) can be calculated analytically. The Ehrenfest-Keesom equation (32) reads, in this case, $dL/d\phi_0 = \pm\frac{2}{3}\sin\phi_0/\sin L$ and, when integrated, gives the characteristic equation. The LCP can be found using condition (39) with the explicit form of the coefficients.

To analyze the second-order effects, we note that the symmetry properties of the two-loop system imply that Eqs. (14) are homogeneous to all perturbation orders, i.e., $\Pi_{kl}(1)=\Pi_{kl}(2)=0$. We shall consider a two-loop system made of an extreme type-II material, a case whose exact solution can be obtained from numerical calculations.[14]

The coefficients $|\Delta|^2_{kl}$ in Eq. (41) for the nodes, the coefficients j_{kl} in Eq. (42) for the external branches, and the coefficients G_{kl} in Eq. (43) for the whole two-loop system have been calculated and the ensuing second-order

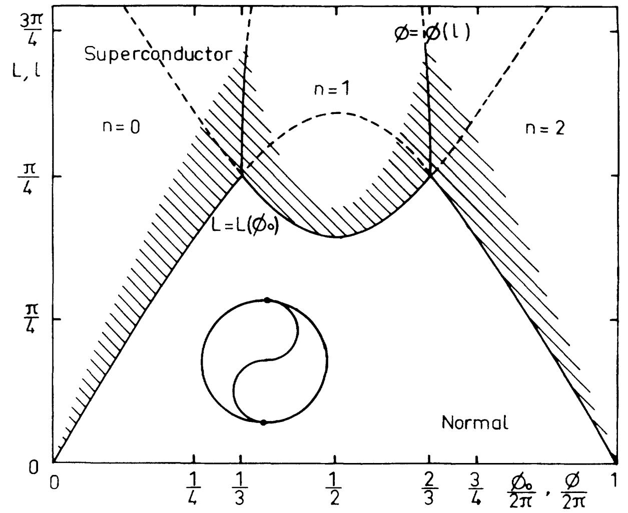

FIG. 5. Phase diagram for the two-loop system. The inset shows the micronetwork geometry. The shadowed region indicates where second-order perturbation results depart less than 15% from exact ones. Shown also are the equilibrium lines between modes (see text).

perturbation results have been checked by comparison with exact calculations made by Fink,[14] as can be seen in Fig. 6.

We have studied the stability line separating both modes in the superconductive region. This line begins at the point $\phi_0=2\pi/3$ on the phase boundary and goes ini-

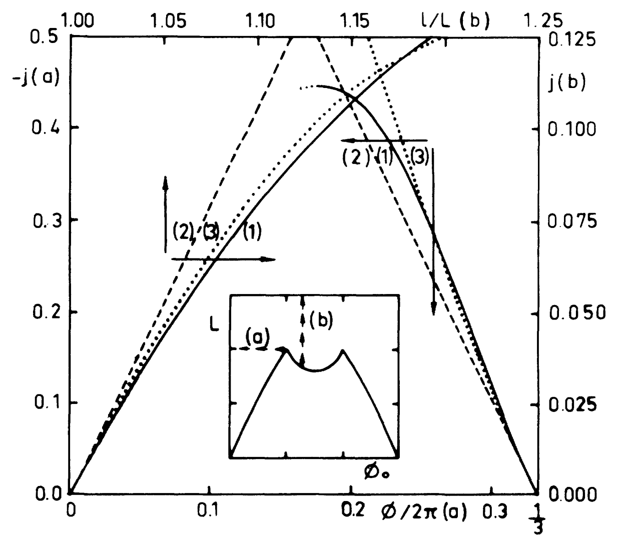

FIG. 6. Comparison of exact and perturbation theory results for the two-loop system. Shown is $j(a)$, the current density vs applied flux at fixed temperature for the mode $n=0$; in this plot the starting point is at $\phi_0=2\pi/3$. Also shown is $j(b)$, the current density vs $l/L=(T_c-T)/(T_c-T_0)$ at a fixed magnetic field for the mode $n=1$; in this plot $\phi_0=2.47$. Lines (1) are the exact results of Fink; lines (2) and (3) are first- and second-order perturbation results, respectively. The inset shows the paths in the phase diagram. Note that the second-order results reproduce the correct curvature at the starting points (see text). The arrows point to the scales to be considered in each case.

tially parallel to the temperature axis, then it bends into values of ϕ_0 greater than $2\pi/3$, as we have indicated in Fig. 5.

In the case of Fig. 6(a), we should point out that the perturbation scheme reproduces a peculiar feature of the exact solution. The overall curvature of the j vs ϕ line seems to be negative. Near $\phi/2\pi=\frac{1}{3}$, however, the curvature is slightly positive, a fact reproduced by the perturbation results. Because of this good behavior near $\frac{1}{3}$, the overall quality of the approximation seems to be poor. As a rule, second-order perturbation gives good agreement for the initial curvature. On the other hand, we see that the second-order result is a very good approximation for the case considered in (b).

IX. CONCLUSIONS AND PERSPECTIVES

We have shown how a consistent perturbation scheme can be devised to treat the superconductive phase of micronetworks. The procedure can be extended to all orders in ΔT and ΔH and applied to any type of network. The complications of the algebra suggest, however, that beyond second order some other procedure should be applied if additional physical properties of the networks are sought. In this sense we feel that a variational approach of the kind developed by Wang, Rammal, and Pannetier,[5] if self-induction effects are properly included, could be more adequate to describe the vortex structure. In this particular aspect it would also be desirable to have experimental results obtained through decoration techniques to compare with.

The subject of this paper is relevant to the theory of high-T_c superconductors considered as granular systems, in which there is an intrinsic network structure, with superconductive islands interconnected via weak links. This would bring us back to the original ideas of de Gennes and Alexander, who proposed their theory as a model for heterogeneous superconductors.

One of the theories[15] for granular high-T_c superconductors interprets the phase diagram of these materials using an averaged GL framework with renormalized coherence length and penetration depth. This approach is successful in describing the static properties of the magnetic flux within the material. Recent experimental results,[16] however, show that such an approach cannot account, neither qualitatively nor quantitatively, for the dissipative effects caused by flux motion induced by a transport current. It has been suggested[17] that both the depinning and the motion of the vortices are related to the topological characteristics of the intrinsic network, which are not included in an averaged theory. For this reason a possible extension of the present work is towards the inclusion of transport currents and a study of the induced vortex movement. Work in that direction is in progress.

ACKNOWLEDGMENTS

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APPENDIX A: THE Γ MATRIX

To second order the matrix $\Gamma_{pg}(a,s)$ in Eq. (6) is given by

Γ_{pq} ($p\downarrow$, $q\rightarrow$)	0	1	2	3	4
0	1	0	$i\gamma_{02}$	0	$i\gamma_{04}-\dfrac{\gamma_{02}^2}{2}$ . . .
1	0	$i\gamma_{11}$	0	$i\gamma_{13}-\gamma_{11}\gamma_{02}$	. . .
2	$i\gamma_{20}$	0	$i\gamma_{22}-\gamma_{20}-\gamma_{02}-\dfrac{\gamma_{11}^2}{2}$	. . .	
3	0	$i\gamma_{31}-\gamma_{20}\gamma_{11}$	. . .		
4	$i\gamma_{40}-\dfrac{\gamma_{20}^2}{2}$	. . .			
	. . .				

This can be reduced further by taking into account that the normalization conditions (16) imply $\gamma_{kl}\equiv 0$ for k (or l) odd.

APPENDIX B: SECOND-ORDER PERTURBATION FORMALISM

We will use the notation $f_0(a,0)\equiv a$, $f_0(a,L)\equiv b$, and $\dot{f}_0(a,0)=\dot{a}$. The general explicit form of $f_{kl}(a,s)$ for a branch is

$$\begin{aligned} f_{kl}(a,s)=&\alpha_{kl}f_0(a,s)\\ &-[Z_{kl}(a,b,s)+aX_{kl}(a)]\frac{\sin(L-s)}{\sin L}\\ &-\{Z_{kl}(b,a,L-s)\\ &\quad+b[X_{kl}(b)-Y_{kl}(a,b,L)]\}\frac{\sin s}{\sin L}\ . \end{aligned} \tag{B1}$$

In second-order perturbation, taking $k=m+2p$, $l=n+2q$, and $mn,pq=10,01$ we have

$$Y_{kl}(a,b,L)=i\alpha_{mn}\gamma_{2p,2q}(a,L)$$

and

$$Z_{kl}(a,b,s)=\frac{\sin L}{(ab^*-ba^*)}\times\{U_{kl}[aI_1(s)-a^*I_2(s)]-V_{kl}[aI_3(s)-a^*I_4(s)]\}\ , \qquad \text{(B2)}$$

where $U_{kl}=\sigma_q\alpha_{mn}|\alpha_{pq}|^2$, $V_{kl}=p\alpha_{mn}$, and

$$I_1(s)=\int_0^s|f_0|^4ds'\ ,\quad I_2(s)=\int_0^s|f_0|^2f_0^2ds'\ ,$$

$$I_3(s)=\int_0^s|f_0|^2ds'\ ,\quad I_4(s)=\int_0^s f_0^2ds'\ .$$

APPENDIX C: SECOND-ORDER PERTURBATION COEFFICIENTS

The second-order normalization relations can be explicitly written in the form

$$\sigma_0(\alpha_{10}\alpha_{30}^*+\alpha_{10}^*\alpha_{30})=-\frac{C_1}{D}-\frac{B_1C_4}{D^2}-\frac{B_1^2C_6}{D^3}\ ,$$

$$\sigma_0(\alpha_{01}\alpha_{30}^*+\alpha_{10}^*\alpha_{21})+\sigma_1(\alpha_{10}\alpha_{21}^*+\alpha_{01}^*\alpha_{30})=0\ ,$$

$$\sigma_0(\alpha_{10}\alpha_{12}^*+\alpha_{10}^*\alpha_{12})+\sigma_1(\alpha_{01}\alpha_{21}^*+\alpha_{01}^*\alpha_{21}) \qquad \text{(C1)}$$

$$=-\frac{C_2}{D}-\frac{(B_1C_5+B_2C_4)}{D^2}-\frac{2B_1B_2C_6}{D^3}\ ,$$

$$\sigma_1(\alpha_{10}\alpha_{03}^*+\alpha_{01}^*\alpha_{12})+\sigma_0(\alpha_{01}\alpha_{12}^*+\alpha_{10}^*\alpha_{03})=0\ ,$$

$$\sigma_1(\alpha_{01}\alpha_{03}^*+\alpha_{01}^*\alpha_{03})=-\frac{C_3}{D}-\frac{B_2C_5}{D^2}-\frac{B_2^2C_6}{D^3}\ ,$$

where

$$C_1=B_1+\frac{R_1}{2}\ ,$$

$$C_2=B_2+R_2\ ,$$

$$C_3=\frac{R_3}{2}\ ,$$

$$C_4=2R_4+\frac{S}{\kappa^2}R_5\ ,$$

$$C_5=2R_6+\frac{S}{\kappa^2}R_7\ ,$$

$$C_6=\tfrac{3}{2}R_8+\tfrac{3}{8}\frac{S}{\kappa^2}R_9+\tfrac{3}{8}\left[\frac{S}{\kappa^2}\right]^2R_{10}\ .$$

In turn,

$$R_1=(N_1+N_1)^*+(N_2+N_2^*)-N_3-N_4\ ,$$

$$R_2=i[N_5-(N_6-N_6^*)]\ ,$$

$$R_3=N_7\ ,$$

$$R_4=N_8-(N_9+N_9^*)-(N_{10}+N_{10}^*)-(N_{11}+N_{11}^*)+(N_{12}+N_{12}^*)\ ,$$

$$R_5=i[N_{13}-(N_{14}-N_{14}^*)]\ ,$$

$$R_6=-i[N_{15}-(N_{16}-N_{16}^*)]\ ,$$

$$R_7=N_{17}\ ,$$

$$R_8=(N_{18}+N_{18}^*)+(N_{19}+N_{19}^*)-N_{20}-N_{21}\ ,$$

$$R_9=-i[N_{22}-(N_{23}-N_{23}^*)]\ ,$$

$$R_{10}=N_{24}\ ,$$

and

$$N_1=\sum_\beta\left[\frac{\sin L_\beta}{ab^*-ba^*}\right]J_1(L_\beta)\ ,$$

$$N_2=2\sum_\beta\frac{a^*b^*\sin L_\beta}{(ab^*-ba^*)^2}I_3(L_\beta)I_4(L_\beta)\ ,$$

$$N_3=\sum_\beta\frac{(ab^*+ba^*)\sin L_\beta}{(ab^*-ba^*)^2}I_3^2(L_\beta)\ ,$$

$$N_4=\sum_\beta\frac{(ab^*+ba^*)\sin L_\beta}{(ab^*-ba^*)^2}|I_4(L_\beta)|^2\ ,$$

$$N_5=\sum_\beta\left[\frac{ab^*+ba^*}{ab^*-ba^*}\right]\gamma_1(a,L_\beta)I_3(L_\beta)\ ,$$

$$N_6=\sum_\beta\left[\frac{a^*b^*}{ab^*-ba^*}\right]\gamma_1(a,L_\beta)I_4(L_\beta)\ ,$$

$$N_7=\sum_\beta\left[\frac{ab^*+ba^*}{\sin L_\beta}\right]\gamma_1^2(a,L_\beta)\ ,$$

$$N_8=\sum_\beta\frac{(ab^*+ba^*)\sin L_\beta}{(ab^*-ba^*)^2}I_1(L_\beta)I_3(L_\beta)\ ,$$

$$N_9=\sum_\beta\frac{a^*b^*\sin L_\beta}{(ab^*-ba^*)^2}I_2(L_\beta)I_3(L_\beta)\ ,$$

$$N_{10}=\sum_\beta\frac{a^*b^*\sin L_\beta}{(ab^*-ba^*)^2}I_1(L_\beta)I_4(L_\beta)\ ,$$

$$N_{11}=\sum_\beta\left[\frac{\sin L_\beta}{ab^*-ba^*}\right]J_2(L_\beta)\ ,$$

$$N_{12}=\sum_\beta\frac{ab^*\sin L_\beta}{(ab^*-ba^*)^2}I_2^*(L_\beta)I_4(L_\beta)\ ,$$

$$N_{13}=\frac{1}{2\pi}\sum_{\beta,\beta'}\left[\frac{ab^*+ba^*}{ab^*-ba^*}\right]I_3(L_\beta)j_{0\beta'}K_{\beta\beta'}\ ,$$

$$N_{14}=\frac{1}{2\pi}\sum_{\beta,\beta'}\left[\frac{a^*b^*}{ab^*-ba^*}\right]I_4(L_\beta)j_{0\beta'}K_{\beta\beta'}\ ,$$

$$N_{15}=\sum_\beta\left[\frac{ab^*+ba^*}{ab^*-ba^*}\right]\gamma_1(a,L_\beta)I_1(L_\beta)\ ,$$

$$N_{16}=\sum_{\beta}\left[\frac{a^*b^*}{ab^*-ba^*}\right]\gamma_1(a,L_\beta)I_2(L_\beta)\ ,$$

$$N_{17}=\frac{1}{2\pi}\sum_{\beta,\beta'}\left[\frac{ab^*+ba^*}{\sin L_\beta}\right]\gamma_1(a,L_\beta)j_{0\beta'}K_{\beta\beta'}\ ,$$

$$N_{18}=\sum_{\beta}\left[\frac{\sin L_\beta}{ab^*-ba^*}\right]J_3(L_\beta)\ ,$$

$$N_{19}=2\sum_{\beta}\frac{a^*b^*\sin L_\beta}{(ab^*-ba^*)^2}I_2(L_\beta)I_1(L_\beta)\ ,$$

$$N_{20}=\sum_{\beta}\frac{(ab^*+ba^*)\sin L_\beta}{(ab^*-ba^*)^2}I_1^2(L_\beta)\ ,$$

$$N_{21}=\sum_{\beta}\frac{(ab^*+ba^*)\sin L_\beta}{(ab^*-ba^*)^2}|I_2(L_\beta)|^2\ ,$$

$$N_{22}=\frac{1}{2\pi}\sum_{\beta,\beta'}\left[\frac{ab^*+ba^*}{ab^*-ba^*}\right]I_1(L_\beta)j_{0\beta'}K_{\beta\beta'}\ ,$$

$$N_{23}=\frac{1}{2\pi}\sum_{\beta,\beta'}\left[\frac{a^*b^*}{ab^*-ba^*}\right]I_2(L_\beta)j_{0\beta'}K_{\beta\beta'}\ ,$$

$$N_{24}=\frac{1}{4\pi^2}\sum_{\beta,\beta',\beta''}\left[\frac{ab^*+ba^*}{\sin L_\beta}\right]j_{0\beta'}j_{0\beta''}K_{\beta\beta'}K_{\beta\beta''}\ .$$

In these formulas

$$J_1(L_\beta)=\int_0^{L_\beta}f_0^{*2}I_4ds\ ,$$

$$J_2(L_\beta)=\int_0^{L_\beta}|f_0|^2f_0^{*2}I_4ds\ ,$$

$$J_3(L_\beta)=\int_0^{L_\beta}|f_0|^2f_0^{*2}I_2ds\ .$$

The coefficients G_{kl} in expression (43) can now be written as

$$G_{60}=\frac{B_1C_1}{D}+\frac{B_1^2C_4}{2D^2}+\frac{B_1^3C_6}{3D^3}\ ,$$

$$G_{42}=\frac{(B_1C_2+B_2C_1)}{D}+\frac{(B_1^2C_5+2B_1B_2C_4)}{2D^2}+\frac{B_1^2B_2C_6}{D^3}\ ,$$

$$G_{24}=\frac{(B_2C_2+B_1C_3)}{D}+\frac{(B_2^2C_4+2B_1B_2C_5)}{2D^2}+\frac{B_2^2B_1C_6}{D^3}\ , \tag{C2}$$

$$G_{06}=\frac{B_2C_3}{D}+\frac{B_2^2C_5}{2D^2}+\frac{B_2^3C_6}{3D^3}\ ,$$

$$G_{51}=G_{33}=G_{15}=0\ .$$

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