Quantum particle in a washboard potential. I. Linear mobility and the Einstein relation

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The Einstein-Kubo relation between the diffusion constant D and the linear mobility ν is investigated for a dissipative quantum particle in a potential $V(x)=-Fx+V_0\cos(k_0x)$. This system is directly related to a current-biased Josephson junction. It is shown that $D=kT\nu$ holds to all order in V_0 for a general dissipation spectrum $J(\omega)$, as long as $J(\omega)$ is linear near $\omega\to 0$ and is not pathological. A generalized version of the Einstein relation is shown to hold also for sub-Ohmic dissipation, $J(\omega)\sim\omega^s$, $0<s<1$ at $\omega\to 0$ where the motion is subdiffusive. The super-Ohmic case $1<s<2$ is also discussed.

I. INTRODUCTION

We continue here our study of the dynamics of a quantum particle moving in a periodic cosine potential with dissipation.[1-3] The mobility of the particle subject to an external driving force is of special interest—it corresponds directly to the I-V characteristic of a current-biased Josephson junction.[4-6] In this paper we investigate the Einstein relation, which links the linear mobility ν to the diffusion constant D, for more general environments than before. In a companion paper we consider the general nonlinear mobility for the case of an Ohmic dissipation with small viscosity.

Although Kubo[7] has presented a general formal derivation of linear response theory to which the Einstein relation belongs, there are problems with the derivation and even the precise formulation of the mobility for the type of systems considered here, see Ref. 2. We therefore follow the approach of Ref. 2, start with a localized initial state, and define the mean velocity v_F in the presence of a constant driving force F via

$$v_F=\lim_{t\to\infty}\frac{\langle\hat{x}\rangle(t,F)}{t}\ . \tag{1}$$

The $\langle\cdots\rangle$ is the average with respect to the reduced density matrix of the particle in contact with the dissipative environment, in the presence of the external field F. The linear mobility is given by $\nu=\lim_{F\to 0}[v_F/F]$. Likewise we define the diffusion constant via

$$D=\lim_{t\to\infty}\frac{\langle\hat{x}^2\rangle(t,F=0)}{2t}\ , \tag{2}$$

where the time evolution now takes place without any external field.

The usual formulation of the Einstein relation $D=kT\nu$ implicitly assumes that the limits defined above exist with D and $\nu>0$ for $T>0$. This clearly depends on the nature of the environment, which is here represented by a boson bath. We are interested in the case where the dissipation spectrum of the environment $J(\omega)\sim\omega^s$ for small ω. We shall see below that for the free case, i.e., no cosine potential, both $\langle\hat{x}\rangle(t,F)$ and $\langle\hat{x}^2\rangle(t,F=0)$ increase in time like t^s. Thus it is only when $s=1$, corresponding to Ohmic-like dissipation near $\omega=0$, that the above limits are well defined. Nevertheless it is easy to show, in the free case, the existence of a generalized Einstein relation between $\lim_{F\to 0}\langle\hat{x}\rangle(t,F)/F$ and $\langle\hat{x}^2\rangle(t,F=0)$ when $0<s<2$. Problems arise, however, in the presence of the cosine potential, $V_0\cos k_0x$. We show that a generalized Einstein relation still hold for $s\le 1$, although only for $s<1$ does the zeroth order in V_0 contribute to the long-time behavior of $\langle\hat{x}\rangle(t)$.

The Einstein-Kubo relation was studied recently by one of the authors[3] where it was proved that it holds to all orders in V_0 for strict Ohmic dissipation, i.e., $J(\omega)=\eta\omega$. A similar result was obtained by Weiss *et al.*[8] for a quantum particle moving in a one-dimensional periodic lattice, a tight-binding version of the continuous model considered here. The analysis in Ref. 8 also included certain cases of super-Ohmic ($s>1$) and sub-Ohmic ($s<1$) dissipation with, however, no modifications in the Einstein relation. This is very different behavior from that found for the continuous problem in this paper where an Ohmic behavior at small frequencies is required in order to have a well-defined linear mobility and diffusion constant. The difference is, we believe, due to the fact that the two models work in different regimes of parameters. Although there exists a dual transformation between the mobility of the two systems[4] in the Ohmic damping case, a general mapping between the two is impossible. In fact since the tight-binding approximation ignores the excited states at each lattice site, it might be insensitive to some aspects of the quantum dynamics seen in the continuous model. Note also that the proof of the Einstein relation presented in Ref. 8 is different from ours; it uses a rather abstract formalism in which the original double-time path integral is employed. It is not clear that the formalism can work in our model.

It is possible to verify the Einstein relation directly by experiment as well (though it is often *assumed* in condensed matter physics). For example, in a current-biased Josephson junction the external force F corresponds to the bias current I, whereas the voltage across the junction corresponds to $d\hat{x}/dt$ $[V(t)=\Phi_0\dot{\phi}(t)/2\pi$ where $\Phi_0\equiv e/2h$ is the flux quantum and $\phi(t)$ is the Josephson phase]. Therefore for the Ohmic-like dissipation the Einstein relation reduces to

$$\lim_{t\to\infty}\frac{\left[\int_0^t dt' V(t')\right]^2}{2t}=\frac{kT}{R}\ , \tag{3}$$

where both $V(t)$ and the linear resistance R are measured at $I=0$ (we consider here the states $|\phi\rangle$ and $|\phi+2\pi\rangle$ to be different). Note that the quantities in both sides of (3) are *experimentally* measurable. It would thus be interesting to check this relation.

The outline of the paper is as follows. In Sec. II we review briefly the general approach of Ref. 2 where we developed a general real-time description of the system in terms of the Wigner distribution. Then in Sec. III we investigate the generalized Einstein relation for free Brownian motion. In Sec. IV we add the periodic cosine potential and show that the same relations also hold in the Ohmic and sub-Ohmic dissipation with $0<s\le 1$ for every order in V_0, but do not prove the convergence of the series.

II. GENERAL FORMALISM

The total Hamiltonian of our system consists of three parts,

$$\hat{H}=\hat{H}_p+\hat{H}_c+\hat{H}_e\ , \tag{4}$$

where $\hat{H}_p$ is the particle's Hamiltonian,

$$\hat{H}_p=\hat{p}^2/2m+V_0\cos(k_0\hat{x})-Fx\ . \tag{5}$$

$\hat{H}_e$ is that of the environment (a boson bath),

$$\hat{H}_e=\sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k \tag{6}$$

and

$$\hat{H}_c=\hat{x}\sum_k C_k(\hat{a}_k^\dagger+\hat{a}_k)+\hat{x}^2\sum_k\frac{C_k^2}{\hbar\omega_k}\ , \tag{7}$$

the coupling between the two subsystems. Note that a counter term is included in $\hat{H}_c$ to cancel the adiabatic potential shift induced by the coupling. The damping spectrum

$$J(\omega)=\pi\sum_k C_k^2[\delta(\omega-\omega_k)-\delta(\omega+\omega_k)] \tag{8}$$

will be taken, in the thermodynamic limit, to be a piecewise continuous function.

Given an initial density matrix $\hat{d}(0)$ at times 0, we define a reduced density matrix at time t for the particle alone by $\hat{\rho}(t)=\mathrm{Tr}_e[\hat{d}(t)]$ (here the subscript "e" indicates the "environment") and introduce the coordinate representation

$$\rho(Q,r,t)=\langle Q+(r/2)|\hat{\rho}(t)|Q-r/2\rangle\ . \tag{9}$$

Assuming an initial state of the product type,

$$\hat{d}(0)=\frac{\hat{\rho}(0)\exp(-\beta\hat{H}_e)}{\mathrm{Tr}_e[\exp(-\beta\hat{H}_e)]}\ , \tag{10}$$

and a switch-on of the coupling at $t=0^+$, the final Wigner distribution[9] at time $t>0$ is given by, see Ref. 2,

$$\begin{aligned}\omega(Q_f,P_f,t)&=\int_{-\infty}^{\infty}\frac{dr_f}{2\pi\hbar}\rho(Q_f,r_f,t)\exp\left[-\frac{i}{\hbar}P_f r_f\right]\\ &=\Bigg\langle\delta(Q_f-Q_0(t))\delta(P_f-P_0(t))\\ &\quad+\sum_{n=1}^{\infty}\left[\frac{V_0}{\hbar}\right]^n\int_0^t dt_n\int_0^{t_n}dt_{n-1}\cdots\int_0^{t_2}dt_1\\ &\quad\times\sum_{\{\sigma_j=\pm1\}}\delta(Q_f-Q_n(t))\delta(P_f-P_n(t))\prod_{j=1}^n\sigma_j\sin[k_0Q_n(t_j)]\Bigg\rangle_1\ .\end{aligned} \tag{11}$$

The quantities $Q_n(t)$ and $P_n(t)$ are solutions of the following modified "classical Langevin process"

$$m\ddot{Q}_n+\int_0^t dt'\alpha_1(t-t')Q_n(t')=F+\alpha_1(t)Q_i+\frac{\hbar k_0}{2}\sum_{j=1}^n\sigma_j\delta(t-t_j)+\xi(t)\ , \tag{12}$$

$$P_n(t)=m\dot{Q}_n(t),\quad Q_n(0)=Q_i,\quad P_n(0)=P_i\ , \tag{13}$$

where F is the external force and $\xi(t)$ is Gaussian noise with covariance

$$\langle\xi(t)\xi(t')\rangle=\alpha_2(t-t')\ . \tag{14}$$

$\langle\cdots\rangle$ in (11) refers to averages over both the initial Wigner distribution of the particle $\omega(Q_i,P_i,0)$ and the Gaussian random process $\xi(t)$. The functions $\alpha_1(t)$ and $\alpha_2(t)$ are related to $J(\omega)$ by

$$\alpha_1(t)=2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{J(\omega)}{\omega}\cos(\omega t)\ , \tag{15}$$

$$\alpha_2(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}J(\omega)\hbar\coth\left[\frac{\beta\hbar\omega}{2}\right]\cos(\omega t)\ . \tag{16}$$

The solution of (12) can be written as

$$Q_n(t)=Q_0(t)+\frac{\hbar k_0}{2}\sum_{j=1}^{n}\sigma_j g(t-t_j)\ , \tag{17}$$

where $Q_0(t)$ is the solution in the absence of the δ forces,

$$Q_0(t)=Q_0^i(t)+Q_\xi(t)+\int_0^t dt' F g(t-t') \tag{18}$$

with

$$Q_0^i(t)=Q_i+\frac{P_i}{m}g(t)+\int_0^t dt'\alpha_1(t')Q_i g(t-t')\ , \tag{19}$$

$$Q_\xi(t)=\int_0^t dt'\xi(t')g(t-t')\ , \tag{20}$$

and $g(t)$ is the Green's function of the homogeneous part,

$$m\ddot{g}(t)+\int_0^t dt'\alpha_1(t-t')\dot{g}(t')=\delta(t) \tag{21}$$

with $g(t)=0$ for $t<0$. The Green's function $g(t)$ can be solved by Fourier transform,

$$g(t)=\int_{-\infty+i0^+}^{\infty+i0^+}\frac{d\omega}{2\pi}g(\omega)\exp[-i\omega t]\ , \tag{22}$$

where

$$g(\omega)=\frac{-1}{m\omega^2+i\omega\gamma(\omega)} \tag{23}$$

with

$$\gamma(\omega)=+i\omega\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{2J(\omega')/\omega'}{\omega^2-\omega'^2}\ . \tag{24}$$

The boundary condition for $g(t)$ is satisfied provided $g(\omega)$ is analytic, i.e., has no poles, in the upper half plane of ω. Poles if any would arise when the denominator of $g(\omega)$ vanishes, i.e.,

$$\omega^2\left[1+\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi m}\frac{2J(\omega')\omega'}{|\omega^2-\omega'^2|^2}\right]-|\omega|^4\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi m}\frac{2J(\omega')/\omega'}{|\omega^2-\omega'^2|^2}=0\ . \tag{25}$$

Clearly the solutions have to be real since $J(\omega')/\omega'$ is positive. For $t>0$, the integral over ω in (22) has to be closed in the lower half plane. Note that there are branch cuts on the real axis as well as possible poles sitting between the cuts where $J(\omega)$ vanishes. This then gives (using the fact that there are no poles in the lower half plane either)

$$g(t)=\theta(t)\left\{2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}J(\omega)|g(\omega+i0^+)|^2\sin(\omega t)+\sum_\alpha A_\alpha\exp[-i\omega_\alpha t]\right\}\ . \tag{26}$$

The second term on the right-hand side of (26) results from the poles on the real axis which preserve the information of the initial state of the particle. We shall return to this below where one finds that these poles break a certain analyticity essential for the proof of the Einstein relation. Therefore we shall require throughout this paper the absence of poles in $g(\omega)$. This imposes constraints on the damping spectrum $J(\omega)$ at higher frequencies.

III. FREE MOTION

We consider now the case when $V_0=0$. The long-time behavior of the corresponding Brownian motion is controlled by the small frequency part of $J(\omega)$, which we shall assume to have the form

$$J(\omega)=\eta\omega^s[1+\sigma(\omega)]\ . \tag{27}$$

When $s\le 0$ the damping is so strong that the particle is basically localized near its initial position. On the other hand, for $s\ge 2$ the damping is almost negligible and as $t\to\infty$ the particle behaves essentially as a free particle.[10] Therefore we shall confine ourselves to $0<s<2$.

Using (27) the small frequency limit of $g(\omega)$ given in (23) can be readily obtained,

$$g(\omega)=\frac{\sin(s\pi/2)}{\eta\omega^s\exp(-i\pi s/2)}[1+o(\omega)]\ , \tag{28}$$

which gives for large t

$$g(t)=\frac{\sin(s\pi/2)}{\eta\Gamma(s)}t^{s-1}[1+o(t)]\ . \tag{29}$$

Thus to leading order, $\langle\hat{x}\rangle(t,F)$ for the free motion in the presence of the external force F has the generalized form,[1,11]

$$\langle\hat{x}\rangle(t,F)=\langle Q_0(t)\rangle=v_0^{(s)}t^sF(1+o(t))\ ,\qquad v_0^{(s)}=\frac{\sin(s\pi/2)}{\eta s\Gamma(s)}\ , \tag{30}$$

where $\Gamma(s)$ is the usual Euler's gamma function. The calculation of $\langle x^2(t,F=0)\rangle$ can be similarly performed. For large t,

$$\langle\hat{x}^2\rangle(t,F=0)=\langle Q_0^2(t)\rangle\to\left\langle\left[\int_0^t dt' g(t-t')\xi(t')\right]^2\right\rangle=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hbar J(\omega)\coth\frac{\hbar\beta\omega}{2}|g(\omega+i0^+)\exp(-i\omega t)-\delta g(\omega)_t|^2\ , \tag{31}$$

where

$$\delta g(\omega)_t=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\frac{g(\omega'+i0^+)}{\omega+i0^+-\omega'}\exp(-i\omega' t)\ . \tag{32}$$

Note that in (32) the upper branch limit of $g(\omega'+i0^+)$ should be taken first. It is evident that $\delta g(\omega)_t$ is essential for the convergence of the integral (31). Using the small frequency limits (27) and (28), the long-time behavior of $\langle \hat{x}^2 \rangle (t,F=0)$ can be obtained (to leading order in t),

$$\langle \hat{x}^2 \rangle (t,F=0) = \langle Q_0^2(t) \rangle \longrightarrow 2D_0^{(s)} t^s [1+o(t)] \tag{33}$$

with the generalized diffusion constant (the subscript zero indicating the zeroth order in V_0)

$$D_0^{(s)} = \frac{kT}{\eta} \sin^2 \left[\frac{s\pi}{2} \right] \int_{-\infty}^{\infty} \frac{dx}{2\pi x^{1-s}} \left| \frac{\exp(-ix)}{x^s} - \int_{-\infty}^{\infty} \frac{dx'}{2\pi} \frac{\exp(-ix')}{(x+i0^+ - x')x'^s} \right|^2 . \tag{34}$$

For $s=1$ (34) reduces to the simple Ohmic limit with $D_0^{(1)} = kT\nu_0^{(1)} = kT/\eta$. For all $0<s<2$ a generalized Einstein relation holds in the sense that

$$\left. \frac{\langle \hat{x}^2 \rangle (t,F=0)}{2kT \langle \hat{x} \rangle (t,F)/F} \right|_{t\to\infty,\, F\to 0} \to \frac{D_0^{(s)}}{kT\nu_0^{(s)}} \quad \text{(independent of } T) . \tag{35}$$

We shall also use below the mean-square displacement after a long time for the classical Langevin process introduced above. It is defined by

$$C(t) = \lim_{t'\to\infty} \langle [Q_0(t+t') - Q_0(t')]^2 \rangle /2 . \tag{36}$$

The existence of the limit in the right-hand side of (36) depends on the behavior of $\delta g(\omega)_t$ given in (32). For long time and finite frequency,

$$\delta g(\omega)_t \sim (t^s/\omega t)[1+o(t)] . \tag{37}$$

It can be explicitly shown that for $s<2$ the long-time effect of $\delta g(\omega)_t$ vanishes [one needs also to take some care of the small frequency part of $\delta g(\omega)_t$]. The resulting $C(t)$ simply reads

$$C(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hbar J(\omega) |g(\omega + i0^+)|^2 [1-\cos\omega t] \coth \frac{\beta\hbar\omega}{2} . \tag{38}$$

The result coincides with $\tilde{C}(t) = \langle [\hat{x}(t) - \hat{x}(0)]^2 \rangle /2$ [$\hat{x}(t)$ is the coordinate operator in the Heisenberg picture] of the free motion obtained by genuine quantum-mechanical considerations.[10] It is interesting that if we define the generalized diffusion constant by $\tilde{D}_0^{(s)} = \lim_{t\to\infty} \tilde{C}(t)/t^s$ (as has been proposed in Ref. 10), then $\tilde{D}_0^{(s)} = skT\nu_0^{(s)}$. However, $\tilde{D}_0^{(s)}$ is in general not equal to $D_0^{(s)}$ for $s \neq 1$. Since $\tilde{C}(t)$ involves calculations of correlation functions to which our general formalism is not readily applicable (when the periodic potential is present), (33) will be employed throughout this paper.

IV. EINSTEIN RELATION IN A PERIODIC POTENTIAL

A. General expressions

We now generalize the result of the last section to include the periodic cosine potential. It is a straightforward matter to obtain from the general formalism in Sec. II that

$$\left. \frac{\partial}{\partial F} \langle \hat{x} \rangle (t,F) \right|_{F\to 0} = \sum_{n=0}^{\infty} A_n(t) V_0^n = \left. \frac{\partial}{\partial F} \langle Q_0(t) \rangle \right|_{F\to 0} + \sum_{n=1}^{\infty} k_0 \hbar \left[\frac{iV_0}{\hbar} \right]^n \int_0^t dt_n g(t-t_n) \psi_{1,n}(t_n) \tag{39}$$

and

$$\langle \hat{x}^2 \rangle (t,F=0) = \sum_{n=0}^{\infty} B_n(t) V_0^n = \langle Q_0^2(t) \rangle + \sum_{n=1}^{\infty} k_0 \hbar \left[\frac{iV_0}{\hbar} \right]^n \int_0^t dt_n g(t-t_n) \psi_{2,n}(t,t_n) , \tag{40}$$

where

$$\begin{aligned} \psi_{1,n}(t_n) = \frac{k_0}{2} \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} dt_1 \sum_{\{\mu_j = \pm 1\}} & \left\{ \sum_{l=1}^{n} \mu_l \int_0^{t_l} dt' g(t_l - t') \right. \\ & \times \prod_{j=1}^{n-1} \sin \left[\frac{k_0^2 \hbar}{2} \sum_{k>j}^{n} \mu_k g(t_k - t_j) \right] \\ & \left. \times \exp \left[ik_0 \sum_{j=1}^{n} \mu_j \left[-\frac{\pi}{2} + Q_0^i(t_j) \right] - \frac{1}{2} \left\langle \left[k_0 \sum_{j=1}^{n} \mu_j Q_\xi(t_j) \right]^2 \right\rangle \right] \right\} \end{aligned} \tag{41}$$

and

$$\psi_{2,n}(t,t_n)=\int_0^{t_n}dt_{n-1}\int_0^{t_{n-1}}dt_{n-2}\cdots\int_0^{t_2}dt_1 \times \sum_{\{\mu_j=\pm 1\}}\left\{\left[-iQ_0^i(t)-i\frac{k_0\hbar}{2}\sum_{l=1}^{n-1}g(t-t_l)\cot\left[\frac{k_0^2\hbar}{2}\sum_{k>l}^{n}\mu_k g(t-t_l)\right] + \sum_{l=1}\mu_l k_0\langle Q_\xi(t)Q_\xi(t_l)\rangle\right]\prod_{j=1}^{n-1}\sin\left[\frac{k_0^2\hbar}{2}\sum_{k>j}^{n}\mu_k g(t_k-t_j)\right] \times\exp\left[ik_0\sum_{j=1}^{n}\mu_j\left[-\frac{\pi}{2}+Q_0^i(t_j)\right]-\frac{1}{2}\left\langle\left[k_0\sum_{j=1}^{n}\mu_j Q_\xi(t_j)\right]^2\right\rangle\right]\right\} \tag{42}$$

with $Q_0^i(t)$ containing the information of the initial state and $Q_\xi(t)$ denoting the random noise part of $Q_0(t)$ [see (19) and (20)]. To simplify these expressions, we first notice that for large t_j's,

$$-\frac{1}{2}\left\langle\left[k_0\sum_{j=1}^{n}\mu_j Q_\xi(t_j)\right]^2\right\rangle\to\frac{k_0^2}{2}\sum_{j,k=1}^{n}\mu_j\mu_k C(t_k-t_j)-\frac{k_0^2}{2}\sum_{j,k}\mu_k\mu_j\langle Q_\xi^2(t_k)\rangle . \tag{43}$$

Therefore one needs $\sum_j\mu_j=0$ (thus n =even) when $0<s<2$ for the integrands above to survive at long time. Once this is imposed, the other exponent $ik_0\sum_{j=1}^{n}\mu_j[-\pi/2+Q_0^i(t_j)]$ vanishes for large t_j's and $s<2$ since $Q_0^i(t_j)\sim t^{s-1}$. The preexponential term $-iQ_0^i(t)$ in (42) will then vanish too by symmetry ($\mu_j\to-\mu_j$). Let

$$F(\{t_j,\mu_j\})=\prod_{j=1}^{n-1}\sin\left[\frac{k_0^2\hbar}{2}\sum_{k>j}^{n}\mu_k g(t_k-t_j)\right]\exp\left[\frac{k_0^2}{2}\sum_{j=1}^{n}\mu_j\mu_k C(t_j-t_k)\right]\Bigg|_{\sum_j\mu_j=0} . \tag{44}$$

we can then rewrite (41) and (42) in the following form (for long times and even n)

$$\psi_{1,n}(t_n)=\frac{k_0}{2}\int_0^{t_n}dt_{n-1}\int_0^{t_{n-1}}dt_{n-2}\cdots\int_0^{t_2}dt_1\sum_{\{\mu_j=\pm1\}}\left\{\sum_{l=1}^{n}\mu_l\int_0^{t_l}dt' g(t_l-t')F(\{t_j,\mu_j\})\right\} , \tag{45}$$

while

$$\psi_{2,n}(t,t_n)=\psi_{2,n}^{(1)}(t_n)+\psi_{2,n}^{(2)}(t,t_n) , \tag{46}$$

with

$$\psi_{2,n}^{(1)}(t_n)=\frac{k_0}{2}\int_0^{t_n}dt_{n-1}\int_0^{t_{n-1}}dt_{n-2}\cdots\int_0^{t_2}dt_1\sum_{\{\mu_j=\pm1\}}\left\{\sum_{l=1}^{n}\mu_l\langle Q_\xi^2(t_l)\rangle F(\{t_j,\mu_j\})\right\} \tag{47}$$

and

$$\psi_{2,n}^{(2)}(t,t_n)=-k_0\int_0^{t_n}dt_{n-1}\int_0^{t_{n-1}}dt_{n-2}\cdots\int_0^{t_2}dt_1 \times\sum_{\{\mu_j=\pm1\}}\left\{\left[i\frac{\hbar}{2}\sum_{l=1}^{n-1}g(t-t_l)\cot\left[\frac{k_0^2\hbar}{2}\sum_{k>l}^{n}\mu_k g(t-t_l)\right]+\sum_{l=1}^{n}\mu_l C(t-t_l)\right]F(\{t_j,\mu_j\})\right\} . \tag{48}$$

One immediately notes the similarities between $\psi_{1,n}(t_n)$ and $\psi_{2,n}^{(1)}(t_n)$. In fact,

$$\lim_{t_n\to\infty}\psi_{1,n}(t_n)/\psi_{2,n}^{(1)}(t_n)=v_0^{(s)}/2D_0^{(s)} . \tag{49}$$

To proceed further, it is most convenient to write

$$\psi_{2,n}^{(2)}(t,t_n)=-\frac{1}{k_0}\int_0^{t_n}dt_{n-1}\int_0^{t_{n-1}}dt_{n-2}\cdots\int_0^{t_2}dt_1 \times\sum_{\{\mu_j=\pm1\}}\left\{\mu_n k_0^2 C(t-t_n)\prod_{k=1}^{n-1}\mu_k\mathrm{Im}[S_k]+\sum_{l=1}^{n-1}\mathrm{Im}[R(t-t_l)S_l]\prod_{k=1,k\neq l}^{n-1}\mu_k\mathrm{Im}[S_k]\right\}\Bigg|_{\sum_j\mu_j=0} , \tag{50}$$

where

$$S_k(\{t_j,\mu_j\})=\exp\left[\mu_k\sum_{j>k}^{n}\mu_j R(t_j-t_k)\right] \tag{51}$$

and

$$R(t)=\left[C(t)+\frac{i}{2}\hbar g(t)\right]k_0^2$$
$$=k_0^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hbar J(\omega)|g(\omega+i0^+)|^2\frac{\cosh(\beta\omega\hbar/2)-\cosh[(\beta\hbar/2-it)\omega]}{\sinh(\beta\omega\hbar/2)} . \tag{52}$$

In the last step of (52) we have used (26) and (38).

These expressions are analogous to those for the Ohmic damping spectrum[3] except for the dependence of $R(t)$ on the general damping spectrum $J(\omega)$. It is most important that $R(t)$ can be analytically continued into the striplike regime $-\beta\hbar\leq \mathrm{Im}t\leq 0$ in the lower half plane and satisfy

$$R(t-i\beta\hbar)=R^*(t^*) \quad \text{for } \beta\hbar\geq \mathrm{Im}t\geq 0 , \tag{53}$$

$$R(-i\tau)=R^*(-i\tau) \quad \text{for } \mathrm{Im}\tau=0 . \tag{54}$$

In what follows we shall show that for $t_n\to\infty$ the contribution due to $\psi_{2,n}^{(2)}(t,t_n)$ is negligible to the leading order in t when compared to that of $\psi_{1,n}^{(1)}(t,t_n)$ for sub-Ohmic and Ohmic-like dissipation. We then briefly discuss the case of super-Ohmic damping in which the integrals over t_j's diverge at long times.

B. Sub-Ohmic and Ohmic-like dissipation

It is necessary to check first the convergence of the integrals in these expressions. The integrands can be divided into many "neutral charge" clusters each of which has $\sum\mu_j=0$ (viewing each t_j as the position of an object with charge μ_j). Large distances between intracluster "charges" are exponentially suppressed by $C(t_j-t_k)$ (it behaves as $\sim kT|t_j-t_k|^s/\eta$ for large $|t_j-t_k|$). But this does not work for intercluster separations. In the latter case, the suppressions come from the sine factors in $F(\{t_j,\mu_j\})$ that connect neighboring clusters. Since the clusters are "charge" neutral and "compact," $\sum_j\mu_j$ inside the sine factors reduces to taking derivatives when the intercluster distances are large. The suppressions then have the form

$$\propto\frac{\partial}{\partial t_j}g(t_k-t_j) \quad \text{for } t_k>t_j , \tag{55}$$

where t_k and t_j belong to different clusters. Therefore the convergence is sufficient for the sub-Ohmic dissipation as well as for the Ohmic dissipation [$g(t)\to$const as $t\to\infty$ in the Ohmic case]. For $0<s\leq 1$ the preexponential factors in $\psi_{1,n}$ and $\psi_{2,n}$ do not affect the intercluster suppressions either except for the terms associated with the cotangent factors in $\psi_{2,n}^{(2)}$ which substitute

$$\sin\left[\frac{\hbar k_0^2}{2}\sum_{j=l+1}^{n}\mu_j g(t_j-t_l)\right] \to \cos\left[\frac{\hbar k_0^2}{2}\sum_{j=l+1}^{n}\mu_j g(t_j-t_l)\right] .$$

The right-hand side does not decay for large intercluster separation. In this case the convergence question is somewhat subtle: It results from summing over μ_j's with $j>l$ (l=even). If there is no correlation at all between the upper and lower parts the summation simply yields zero by symmetry ($\mu_j\to-\mu_j$, $j>l$). Taking this into account, the suppression will then have the form

$$\propto\left[\frac{\partial}{\partial t_l}g(t_j-t_l)\right]^2 \quad \text{or} \quad \propto\frac{\partial^2}{\partial t_l\partial t_j}[C(t_j-t_l)]$$

[the second quantity is related to the short-range part of $C(t)$'s]. The integral over t_l is thus also convergent for $s\leq 1$.

It remains to show that the contribution of $\psi_{2,n}^{(2)}$ does vanish to leading order in t for all n. This is done by converting the real-time integrals into imaginary-time integrals over a finite interval. It involves a number of contour deformations in the striplike regime $-\beta\hbar\leq\mathrm{Im}t\leq 0$ and repeated use of the analytic property of the function $R(t)$. In the case of sub-Ohmic and Ohmic dissipation, we can safely put the lower limit of the integrals to $-\infty$; we shall return to this point in the super-Ohmic dissipation where the integrals are not convergent. For the lowest order

$$\psi_{2,2}^{(2)}(t_2)\longrightarrow\frac{2}{k_0}\mathrm{Im}\left[\int_{-\infty}^{t_2}dt_1\{-k_0^2C(t-t_2)\exp[-R(t_2-t_1)]+R(t-t_1)\exp[-R(t_2-t_1)]\}\right] . \tag{56}$$

We note that the integrand is odd under $t_1\to t_1+i\beta\hbar$. Therefore a contour deformation

$$(-\infty\to t_2)\longrightarrow(-\infty+i\beta\hbar\to i\beta\hbar+t_2\to t_2)$$

for t_1 converts it to an integral over the imaginary axis,

$$\psi_{2,2}^{(2)}(t_2)=\frac{1}{k_0}\mathrm{Re}\left\{\int_{\beta\hbar}^{0}d\tau_1[-k_0^2C(t-t_2)+R(t-t_2-i\tau_1)]\exp[-R(-i\tau_1)]\right\} . \tag{57}$$

For the higher-order terms the procedure becomes much more involved. It contains basically step by step integrations over t_j's. The central idea here is to rearrange the integrands so that they are odd under some contour deformations similar to the one used above. This was first applied in Ref. 3 to the case of strict Ohmic dissipation. It is straightforward to generalize the procedure to our case. The resulting expression reads

$$\psi_{2,n}^{(2)}(t,t_n)=-\frac{1}{k_0}\int_{\beta\hbar}^{0}\frac{d\tau_{n-1}}{2}\cdots\frac{d\tau_1}{2}\theta\left[\beta\hbar-\sum_{j=1}^{n-1}\tau_j\right]$$
$$\times\sum_{\{\mu_j=\pm1\}}\mathrm{Re}\left[\left[\prod_{j=1}^{n}\mu_j\right]\left\{\sum_{j=1}^{n-1}\mu_j\left[R\left[t-t_n-i\sum_{k=j}^{n-1}\tau_k\right]-k_0^2C(t-t_{n-1})\right]\right\}\exp(E_n)\right]\Bigg|_{\sum_j\mu_j=0}, \tag{58}$$

where the exponent E_n is obtained via the recurrence relation,

$$E_k=E_{k-1}|_{t_{k-1}\to t_k+i\tau_{k-1}}+\mu_k\sum_{j>k}^{n}\mu_jR(t_j-t_k) \tag{59}$$

with

$$E_1=\mu_1\sum_{j>1}^{n}\mu_jR(t_j-t_1)\ . \tag{60}$$

Note that $\exp(E_n)$ is purely real. Recalling the expression for $R(t)$ one finds that for finite τ

$$k_0^2C(t)-\mathrm{Re}[R(t-i\tau)]=k_0^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hbar J(\omega)|g(\omega+i0^+)|^2\frac{\cosh[(\beta-2\tau)\omega\hbar/2]-\cosh(\beta\omega\hbar/2)}{\sinh(\beta\omega\hbar/2)}\cos\omega t$$
$$\sim t^{s-2}\ \text{for large}\ t\ . \tag{61}$$

Therefore

$$\int_0^t dt_n g(t-t_n)\psi_{2,n}^{(2)}(t,t_n)\sim t^{2s-2}\ . \tag{62}$$

We now show that this is indeed of higher order than the integrals over ψ_1. Since $\langle Q_\xi^2(t_j)\rangle\sim t_j^s$ and $\sum_j\mu_j=0$ has to be imposed all the time, we have from (47)

$$\psi_{2,n}^{(1)}(t,t_n)\sim t^{s-1}\ . \tag{63}$$

Therefore

$$\int_0^t g(t-t_n)\psi_{2,n}^{(1)}(t,t_n)\sim t^{2s-1}\ . \tag{64}$$

Finally, using (49) it is readily established that to leading order in t the nth-order coefficients of (39) and (40) in V_0 are related by

$$\frac{A_n(t)}{B_n(t)}=\frac{v_0^{(s)}}{2D_0^{(s)}}[1+o(t)]\ \ \text{for all even}\ n\ , \tag{65}$$

where $v_0^{(s)}$ and $D_0^{(s)}$ are given in (30) and (34). We therefore arrive at the conclusion that

$$\lim_{t\to\infty}\frac{\frac{\partial}{\partial F}\langle\hat{x}\rangle(t,F)|_{F\to0}}{\langle\hat{x}^2\rangle(t,F=0)}=\lim_{t\to\infty}\frac{\frac{\partial}{\partial F}\langle\hat{x}\rangle(t,F)_{F\to0}}{\langle\hat{x}^2\rangle(t,F=0)}\Bigg|_{V_0=0}=\frac{v_0^{(s)}}{2D_0^{(s)}}\ . \tag{66}$$

In the Ohmic limit, all the coefficients increase linearly in t, while for sub-Ohmic damping the coefficients of $n\neq0$ grow as t^{2s-1}, which is slower than that of the corresponding free Brownian motion.

It ought to be repeated that the result holds only when the finite-frequency part of $J(\omega)$ is not so pathological that poles appear in $g(\omega)$. Otherwise the analyticity of $R(t)$ as well as the convergence would be destroyed. Such a case might occur if the particle interacts strongly with a particular harmonic oscillator in the bath so that it is driven to move along with the oscillator. Such a case should be generally excluded at the point of modeling the heat bath since one would then rather consider the bare system as two dimensional.

C. Super Ohmic dissipation

The case of the super-Ohmic dissipation cannot be discussed within the same framework when the periodic potential is present. Though it seems that one can formally apply the calculation of the last subsection here as well, there are several inevitable problems due to the divergence inherent in the integrals $\psi_{1,n}$ and $\psi_{2,n}$. Note that the intercluster suppression of (55) is insufficient for $s>1$. Consequently, the lower integral limit of $\psi_{2,n}^{(2)}$ cannot be taken to infinity. One therefore is forced to consider the rather messy behavior of the integrands around $t=0$. Furthermore, $A_n(t)$ and $B_n(t)$ grow faster in time as n increases. This point can be checked by a simple argument. Consider, for example, the "charge" configuration $\mu_j=(-1)^j$. There are then $n/2$ neutral charge clusters. The connections between them decay as $|t_k-t_j|^{s-2}$. This leads to a dependence of $A_n(t)$ and $B_n(t)$ on time as $\sim t^st^{(s-1)n/2}/(n/2)!$.

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