Exact results for a finite-sized spherical model of ferromagnetism at the borderline dimensionality 4

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We report exact results on the zero-field susceptibility $\chi(T;L)$, the correlation length $\xi(T;L)$, and the "singular" part of the specific heat $c^{(s)}(T;L)$ of a finite-sized spherical model of ferromagnetism subjected to periodic boundary conditions. We take the total dimensionality of the system d to be 4 and deal with the geometry $L^{4-d'}\times\infty^{d'}$ $(d'\leq 2)$. In the region of first-order phase transition $(T<T_c)$, our results are formally the same as in other cases with $d>2$. The "core" region $(T\simeq T_c)$, however, is characterized by the appearance of factors involving ln$L$, which appear only when $d=4$. The relationship between these results and the corresponding ones following from the hyperscaling regime as $d\to 4-$ or from the mean-field regime as $d\to 4+$ is explored, and a formulation in terms of the finite-size scaling theory is presented.

I. INTRODUCTION

Spurred by the seminal work of Fisher and collaborators,[1] considerable progress has been made in the study of finite-size effects in systems undergoing phase transitions and in the formulation of a scaling theory that enables one to understand the origins and implications of these effects in a systematic manner; for a review of these developments, see Barber[2] and Privman.[3] The systems considered in most of these studies are the ones with O(n) symmetry, whose upper critical dimension $d_>$ is 4, while the lower critical dimension $d_<$ is 1 in the case of *discrete* symmetry ($n=1$, Ising model), 2 in the case of *continuous* symmetry ($n\geq 2$, of which the spherical model, with $n\to\infty$, is the extreme example); at the same time, the boundary conditions imposed on the system have been generally periodic, though occasionally nonperiodic boundary conditions have also been considered. While the evaluation of finite-size effects for systems with arbitrary n and confined to general geometries, such as $L^{d-d'}\times\infty^{d'}$ (with $d'\leq d_<$ and d unrestricted) presents some serious difficulties of analysis, special cases, notably the "block" geometry ($d'=0$) and the "cylinder" geometry ($d'=1$) turn out to be relatively more tractable; in the case of the spherical model, however, a host of analytical results have been obtained for general $d'\leq 2$ and d such that either $2<d<4$ or $d>4$. All in all, explicit results for the borderline dimensionality 4 are conspicuously few and far between.

Some of the major results pertaining to $d=4$ that we know of are the ones derived by Brézin[4] on the correlation length ξ of an O(n) model with $n\gg 1$, confined to geometry $L^{4-d'}\times\infty^{d'}$ with $d'=0$ or 1, by Luck[5] on the same quantity ξ but with $n\to\infty$ and $d'=1$, by Rudnick, Guo, and Jasnow[6] on the specific heat of an Ising model with $d'=0$, and by Shapiro and Rudnick[7] on the magnetic susceptibility of the spherical model, again with $d'=0$. A common feature of these results is the appearance, along with algebraic terms that characterize the behavior of the system for $2<d<4$ or for $d>4$, of *logarithmic* factors of the form $[\ln(L/a)]^x$ where a is a microscopic length, such as the lattice constant, of the system while x is an exponent that depends on the physical quantity under consideration but is otherwise universal. The purpose of the present paper is to report exact results on the correlation length, the magnetic susceptibility, and the "singular" part of the specific heat of a spherical-model system in geometry $L^{4-d'}\times\infty^{d'}$ $(d'\leq 2)$, which covers not only the block and cylinder geometries as special cases but also the "slab" geometry ($d'=2$) that throws in further factors of the form $\ln\ln(L/a)$.

In Sec. II we summarize the basic elements of our approach and quote explicit results for the aforementioned quantities for a finite-sized spherical-model system in four dimensions—in regions of both *first-order* $(T<T_c)$ and *second-order* $(T\simeq T_c)$ phase transitions; wherever possible, we compare our results with the ones obtained by the previous authors and find complete agreement with them. In Sec. III we explore the relationship between the present results, especially the ones at $T=T_c$, and the corresponding ones following from the hyperscaling regime $(2<d<4)$ as $d\to 4-$ or from the mean-field regime $(d>4)$ as $d\to 4+$. This enables us to understand the manner in which factors involving ln$L$ emerge in lieu of ones involving $1/(4-d)$ in the former regime or those involving $1/(d-4)$ in the latter. In view of the fact that the spherical model has generally served as a useful guide for systems with arbitrary n, especially the ones with $n\geq 2$, we hope that the relationships explored here will be of help in foreseeing similar situations in other models as well. Finally, in Sec. IV, we present a finite-size scaling hypothesis appropriate to dimensionality 4, to which the results reported in this paper are found to conform.

II. A FINITE-SIZED SPHERICAL MODEL IN FOUR DIMENSIONS

We consider a spherical-model system confined to geometry $L^{4-d'} \times \infty^{d'}$ $(d' \leq 2)$ and subjected to periodic boundary conditions. Following the procedure laid down in previous publications,[8,9] the "singular" part of the free-energy density of the system in zero field turns out to be

$$f^{(s)}(T;L) = \frac{Ty^4}{\pi^2 L^4}\left[\frac{1}{2}\ln\left[\frac{L}{2ay}\right] + \frac{1}{4}|C_4| - \frac{1}{8} - \mathcal{K}(1|d^*;y) - \mathcal{K}(2|d^*;y)\right], \quad (1)$$

where y is the *scaled* length parameter,

$$y = \tfrac{1}{2}(L/a)\phi^{1/2} \quad [\phi = (\lambda/J) - 8], \quad (2)$$

a is the lattice constant, λ is the spherical field, while J is the nearest-neighbor interaction parameter for the spins constituting the system; the constant C_4 comes from the theory of the bulk system and is approximately equal to[10] $-4.7920\ldots$. The functions $\mathcal{K}(\nu|d^*;y)$ are defined as

$$\mathcal{K}(\nu|d^*;y) = \sum_{\mathbf{q}(d^*)}{}' \frac{K_\nu(2yq)}{(yq)^\nu} \quad [q = (q_1^2 + \cdots + q_{d^*}^2)^{1/2} > 0], \quad (3)$$

where $K_\nu(z)$ are modified Bessel functions, while $d^*(=4-d')$ is the number of dimensions in which the system is finite. The parameter $y(T;L)$, which is crucial to our analysis, is determined by the constraint equation

$$K_c \tilde{t} = \frac{a^2 y^2}{8\pi^2 L^2}\left[2\ln\left[\frac{L}{2ay}\right] + |C_4| - 2\mathcal{K}(1|d^*;y)\right], \quad (4)$$

where

$$\tilde{t} = (K_c - K)/K_c = (T - T_c)/T \quad [K = J/T, K_c = J/T_c], \quad (5)$$

T_c being the critical temperature of the corresponding bulk system. Once $y(T;L)$ is known, the various quantities of interest, such as the correlation length $\xi(T;L)$, the magnetic susceptibility per unit volume $\chi(T;L)$, and the "singular" part of the specific heat per unit volume $c^{(s)}(T;L)$ can be obtained from the formulas

$$\xi = \frac{L}{2y}, \quad (6)$$

$$\chi = \frac{L^2}{8Ja^6 y^2}, \quad (7)$$

and

$$c^{(s)} = -\frac{32\pi^2 K^2/a^4}{2\ln(L/2ay) + |C_4| - 1 + 2\mathcal{K}(0|d^*,y)}. \quad (8)$$

We shall now examine these results in different regimes of the variables T and L.

A. Case 1: $(T < T_c,\ L \to \infty)$

In this region—the region of the *first-order* phase transition—the correlation length ξ is known to be much greater than L; the parameter y would, therefore, be much less than unity. The functions $\mathcal{K}$ appearing in Eqs. (4) and (8) would then assume the asymptotic forms[11]

$$\mathcal{K}(1|4-d';y) \approx \begin{cases} \frac{1}{2}\pi^{(4-d')/2}\Gamma((2-d')/2)y^{-(4-d')} \quad (d' < 2), & (9a) \\ \frac{1}{2}\pi y^{-2}(\ln(1/y^2) - \ln\{[\Gamma(1/4)]^4/4\pi^3\}) \quad (d' = 2), & (9b) \end{cases}$$

and

$$\mathcal{K}(0|4-d';y) \approx \tfrac{1}{2}\pi^{(4-d')/2}\Gamma((4-d')/2)y^{-(4-d')} \quad (d' \leq 2). \quad (10)$$

Equations (4) and (9) enable us to write y as a function of T and L:

$$y(T;L) \approx \begin{cases} h_1[K_c|\tilde{t}|(L/a)^2]^{-1/(2-d')} \quad (d' < 2), & (11a) \\ h_2 \exp[-4\pi K_c|\tilde{t}|(L/a)^2] \quad (d' = 2), & (11b) \end{cases}$$

where

$$h_1 = [\Gamma((2-d')/2)/8\pi^{d'/2}]^{1/(2-d')} \quad (12)$$

and

$$h_2 = 2\pi^{3/2}/[\Gamma(\tfrac{1}{4})]^2. \quad (13)$$

The desired physical quantities now follow from Eqs. (6)–(8):

$$\xi(T;L)\approx\frac{L}{2}\begin{cases}(1/h_1)[K_c|\tilde{t}|(L/a)^2]^{1/(2-d')} & (d'<2)\,, & (14a)\\ (1/h_2)\exp[4\pi K_c|\tilde{t}|(L/a)^2] & (d'=2)\,, & (14b)\end{cases}$$

$$\chi(T;L)\approx\frac{L^2}{8Ja^6}\begin{cases}(1/h_1)^2[K_c|\tilde{t}|(L/a)^2]^{2/(2-d')} & (d'<2)\,, & (15a)\\ (1/h_2)^2\exp[8\pi K_c|\tilde{t}|(L/a)^2] & (d'=2)\,, & (15b)\end{cases}$$

and

$$c^{(s)}(T;L)\approx-\frac{32\pi^2K^2}{a^4}\begin{cases}[h_1^2/4\pi^2(2-d')][K_c|\tilde{t}|(L/a)^2]^{-(4-d')/(2-d')} & (d'<2)\,, & (16a)\\ (h_2^2/\pi)\exp[-8\pi K_c|\tilde{t}|(L/a)^2] & (d'=2)\,. & (16b)\end{cases}$$

Comparison with the previous results shows that the various expressions pertaining to the region of the first-order phase transition are formally the same, irrespective of whether d is less than,[9,12] equal to, or greater than 4.[13] One notable improvement on our earlier results consists in the explicit evaluation of the constant h_2 appearing in the case of the slab geometry ($d'=2$) which, in contrast to other geometries, obeys an exponential, rather than algebraic, law of approach towards the bulk behavior.

B. Case 2: ($T=T_c$, $L\to\infty$)

At the bulk critical point ($T=T_c$), the value y_c of the parameter y is determined by the relationship, see Eq. (4),

$$\mathcal{K}(1|4-d';y_c)-\ln(L/2ay_c)=\tfrac{1}{2}|C_4|\,. \quad (17)$$

One readily sees that, unlike the case $2<d<4$ where $y_c=O(1)$, y_c in the present case is much less than 1. We may, therefore, continue to use expressions (9) for the function $\mathcal{K}(1|4-d';y)$ and obtain asymptotically

$$y_c\approx\pi^{1/2}\begin{cases}[\Gamma((2-d')/2)/2\ln(L/a)]^{1/(4-d')} & (d'<2) & (18a)\\ [\ln\ln(L/a)/2\ln(L/a)]^{1/2} & (d'=2)\,. & (18b)\end{cases}$$

The corresponding expressions for ξ, χ, and $c^{(s)}$ turn out to be

$$\xi(T_c;L)\approx\frac{L}{(4\pi)^{1/2}}\begin{cases}[2\ln(L/a)/\Gamma((2-d')/2)]^{1/(4-d')}\\ (d'<2)\,, & (19a)\\ [2\ln(L/a)/\ln\ln(L/a)]^{1/2}\\ (d'=2)\,, & (19b)\end{cases}$$

$$\chi(T_c;L)\approx\frac{L^2}{8\pi Ja^6}\begin{cases}[2\ln(L/a)/\Gamma((2-d')/2)]^{2/(4-d')}\\ (d'<2)\,, & (20a)\\ 2\ln(L/a)/\ln\ln(L/a) \quad (d'=2)\,, & (20b)\end{cases}$$

and

$$c^{(s)}(T_c;L)\approx-32\pi^2K_c^2/(4-d')a^4\ln(L/a) \quad (d'\le 2)\,. \quad (21)$$

We note that our result for $\xi(T_c)$, with $d'<2$, agrees with the corresponding results obtained earlier by Brézin[4] (for $n\gg1$ and $d'=0$ or 1) and by Luck[5] (for $n\to\infty$ and $d'=1$), viz.,

$$\xi(T_c;L)\approx\begin{cases}\text{const}\times L[\ln(L/a)]^{1/4} & (d'=0)\,,\\ L[\ln(L/a)/4\pi^2]^{1/3} & (d'=1)\,,\end{cases} \quad (22)$$

while the one for $\chi(T_c;L)$ agrees with the corresponding one obtained by Shapiro and Rudnick[7] (for $n=\infty$ and $d'=0$), viz.,

$$\chi(T_c;L)\sim L^2[\ln(L/a)^2]^{1/2} \quad (d'=0)\,. \quad (23)$$

The only comparison we could make of our result for the specific heat was the one with Rudnick, Guo, and Jasnow,[6] who had shown that for an Ising model ($n=1$) with $d'=0$

$$c^{(s)}(T_c;L)\sim[\ln(L/a)]^{1/3} \quad (d'=0)\,. \quad (24)$$

Comparing (21) with (24), and keeping in mind certain bulk results for this quantity,[14] we feel emboldened to conjecture that, for all $O(n)$ models in geometry $L^4\times\infty^0$,

$$c^{(s)}(T_c;L)\sim[\ln(L/a)]^{(4-n)/(n+8)}\,, \quad (25)$$

of which (21) and (24) may be regarded as special cases—with $n=\infty$ and $n=1$, respectively.

C. Case 3: ($T\gtrsim T_c$, $L\to\infty$)

In this region, first of all, there exists a buffer zone where y is of order unity; in terms of T, this corresponds to the condition, see (4),

$$K_ct=O((a/L)^2\ln(L/a)) \quad [t=(T-T_c)/T_c]\,. \quad (26)$$

Exact results pertaining to this zone will have to be obtained numerically; one may, nevertheless, note that here

$$\xi\sim L,\quad \chi\sim L^2,\quad c^{(s)}\sim[\ln(L/a)+\text{const}]^{-1}\,. \quad (27)$$

As T increases, y eventually becomes much greater than unity; in fact, y becomes $O(L/a)$ when $K_c t = O(1)$. The function $\mathcal{H}(1|4-d';y)$ then becomes negligible, and Eq. (4) gives

$$y \approx \frac{L}{a}\left[\frac{8\pi^2 K_c t}{\ln(1/t)+\text{const}}\right]^{1/2} \quad (0<t\ll 1) . \tag{28}$$

This reproduces the standard bulk results, namely,

$$\xi \approx \frac{a}{4\pi(2K_c)^{1/2}}\left[\frac{\ln(1/t)+\text{const}}{t}\right]^{1/2} , \tag{29}$$

$$\chi \approx \frac{\ln(1/t)+\text{const}}{64\pi^2 J a^4 K_c t} , \tag{30}$$

and

$$c^{(s)} \approx -\frac{32\pi^2 K_c^2}{a^4[\ln(1/t)+\text{const}]} . \tag{31}$$

If the influence of the functions $\mathcal{H}(\nu|4-d';y)$ is included, one obtains exponentially small finite-size corrections which are of the same general nature as the ones reported earlier for the cases $2<d<4$ and $d>4$.

III. APPROACHING $d=4$ FROM THE HYPERSCALING REGIME OR FROM THE MEAN-FIELD REGIME

It is instructive to see how, in the "core" region ($T \simeq T_c$), factors involving $\ln(L/a)$ arise as one approaches dimensionality 4 *either* from the hyperscaling regime (where $d<4$) or from the mean-field regime (where $d>4$). In either case, the value y_c of the parameter y is determined by the equation

$$(4\pi)^{d/2}\phi^{-(d-2)/2}w_d(\phi)+2\mathcal{H}((d-2)/2|d^*;y)=0 , \tag{32}$$

where $w_d(\phi)$ represents the leading term(s) in the expansion of the function

$$W_d(\phi)-W_d(0)=\tfrac{1}{2}\int_0^\infty (e^{-(1/2)\varphi x}-1)[e^{-x}I_0(x)]^d\,dx \tag{33}$$

for $\phi \ll 1$; here, $W_d(\phi)$ is the familiar Watson integral[10] that appears in the theory of the bulk system, while $I_0(x)$ is another modified Bessel function. Differentiating (33) with respect to ϕ, we obtain

$$W_d'(\phi)=-\tfrac{1}{4}\int_0^\infty e^{-(1/2)\varphi x}[e^{-x}I_0(x)]^d x\,dx \tag{34}$$

$$=-\tfrac{1}{4}\left[\int_0^\Lambda+\int_\Lambda^\infty\right]e^{-(1/2)\varphi x}[e^{-x}I_0(x)]^d x\,dx , \tag{35}$$

where Λ, for convenience, will be chosen to be much greater than unity. In the second integral, we may approximate $I_0(x)$ by $e^x/(2\pi x)^{1/2}$ and obtain for that part of $W_d'(\phi)$ the expression

$$-\frac{1}{(4\pi)^{d/2}}\phi^{-(4-d)/2}\Gamma((4-d)/2,\tfrac{1}{2}\phi\Lambda) , \tag{36}$$

where $\Gamma(a,x)$ denotes the incomplete Γ function. In the first part, we may set $\phi=0$ and write it as

$$-\tfrac{1}{4}\int_0^\Lambda [e^{-x}I_0(x)]^d x\,dx . \tag{37}$$

The foregoing expressions suffice to determine the leading term(s) of the function $W_d(\phi)-W_d(0)$ for *all d*.

(i) For $d<4$, the dominant contribution, as ϕ becomes very small, comes from (36)—with the incomplete Γ function replaced by the standard Γ function. An integration over ϕ then gives

$$w_d(\phi)\approx -\frac{1}{(4\pi)^{d/2}}|\Gamma((2-d)/2)|\phi^{(d-2)/2} \quad (d<4) . \tag{38}$$

Substituting (38) into (32), we obtain

$$\mathcal{H}((d-2)/2|d^*;y_c)\approx \tfrac{1}{2}|\Gamma((2-d)/2)| \quad (d<4) \tag{39}$$

$$\simeq \frac{1}{4-d} \quad (d \lesssim 4) . \tag{40}$$

Notice that y_c in this regime is ordinarily O(1) but becomes much less than 1 as $d\to 4-$.

(ii) For $d=4$, we require both (36) and (37). The incomplete Γ function now becomes the exponential integral,[15]

$$\Gamma(0,\tfrac{1}{2}\phi\Lambda)=E_1(\tfrac{1}{2}\phi\Lambda)\approx -\ln(\tfrac{1}{2}\phi\Lambda)-\gamma , \tag{41}$$

where γ is the Euler constant, while (37) takes the form[10]

$$-\tfrac{1}{4}\int_0^\Lambda [e^{-x}I_0(x)]^4 x\,dx \approx -\frac{1}{16\pi^2}[\ln(\tfrac{1}{2}\Lambda)+|C_4|-1+\gamma] , \tag{42}$$

C_4 being the same constant as in Eqs. (1) and (2). Collecting the various terms and integrating over ϕ, we now obtain

$$w_4(\phi)\approx -(\phi/16\pi^2)[\ln(1/\phi)+|C_4|] . \tag{43}$$

Substituting into (32), we get

$$\mathcal{H}(1|d^*;y_c)\approx \tfrac{1}{2}[\ln(1/\phi_c)+|C_4|] , \tag{44}$$

exactly as in (17).

(iii) For $d>4$, we may set $\phi=0$ in (34) itself and obtain on integration

$$w_d(\phi)\approx -w\phi , \tag{45}$$

where

$$w=\tfrac{1}{4}\int_0^\infty [e^{-x}I_0(x)]^d x\,dx \quad (d>4) . \tag{46}$$

Equation (32) now becomes

$$\mathcal{H}((d-2)/2|d^*;y_c)\approx \tfrac{1}{2}(4\pi)^{d/2}w\phi_c^{-(d-4)/2} \quad (d>4) \tag{47}$$

$$\simeq 1/(d-4) \quad (d \gtrsim 4) . \tag{48}$$

Comparing (40), (44), and (48), and remembering that ϕ_c in a finite-sized system is essentially $O(a^2/L^2)$, we infer that the various physical quantities pertaining to the system at $T=T_c$ and in the hyperscaling regime ($d<4$) acquire a rather "critical" dependence on the parameter $(4-d)$ as $d\to 4-$, while for the system in the mean-field regime ($d>4$) they acquire a similar dependence on the parameter $(d-4)$ as $d\to 4+$; as we see it, this dependence is literally transformed into one on the parameter $[\ln(L/a)+\text{const}]$ when the system happens to be in the borderline dimensionality 4. The results reported in Sec. II are a clear testimony to this observation. We suspect that this pattern of behavior is not peculiar to the spherical model alone; it may well hold for all $n\ge 2$.

IV. THE FINITE-SIZE SCALING HYPOTHESIS AT $d=4$

In our study of O(n) models in higher dimensions, we showed that the singular part of the free-energy density of the system in geometry $L^{d-d'}\times\infty^{d'}$ and over a considerable range of temperatures (which includes the region $T\simeq T_c$) conformed to the scaling hypothesis[13]

$$f^{(s)}(T,H;L)\approx\frac{T}{L^d}v_3^{\lambda}Y\left[\frac{v_1}{v_3^{\mu_1}},\frac{v_2}{v_3^{\mu_2}}\right], \tag{49}$$

where v_1, v_2, and v_3 are the *scaled* variables appropriate to $d>4$,

$$v_1=\tilde{A}_1L^2\tilde{t},\quad v_2=\tilde{A}_2L^3H/T,\quad v_3=\tilde{A}_3L^{-\omega^*}, \tag{50}$$

$\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ are the nonuniversal scale factors, $\tilde{t}$ is an appropriate temperature parameter (such that, for $T\simeq T_c$, $\tilde{t}\simeq t$), $\omega^*(=d-4)$ is the so-called "anomalous" dimension of the system, while

$$\lambda=\frac{d'}{4-d'},\quad \mu_1=\tfrac{2}{3}\mu_2=\frac{2}{4-d'}. \tag{51}$$

Accordingly, in zero field, one may write

$$\chi(T;L)\approx\frac{\tilde{A}_2^2}{\tilde{A}_1\tilde{A}_3T|\tilde{t}|}\tilde{Y}_{\chi}(v) \tag{52}$$

and

$$c^{(s)}(T;L)\approx-\frac{1}{\tilde{A}_3}\left[T\frac{\partial}{\partial T}(\tilde{A}_1|\tilde{t}|)\right]^2\tilde{Y}_{c^{(s)}}(v), \tag{53}$$

where

$$v=v_1/v_3^{2/(4-d')}\sim\tilde{t}L^{2(d-d')/(4-d')}. \tag{54}$$

In keeping with these results, one may also write

$$\xi(T;L)\approx\frac{1}{(\tilde{A}_1|\tilde{t}|)^{1/2}}\tilde{Y}_{\xi}(v). \tag{55}$$

Case 1. For $T<T_c$ and $L\to\infty$, the scaling functions $\tilde{Y}_{\chi}$, $\tilde{Y}_{c^{(s)}}$, and $\tilde{Y}_{\xi}$, for $d'<2$, possess the asymptotic behavior

$$\tilde{Y}_{\chi}(v)\sim|v|^{(4-d')/(2-d')},\quad \text{as } v\to-\infty; \tag{56a}$$

$$\tilde{Y}_{c^{(s)}}(v)\sim|v|^{-(4-d')/(2-d')},\quad \text{as } v\to-\infty; \tag{56b}$$

$$\tilde{Y}_{\xi}(v)\sim|v|^{(4-d')/2(2-d')|},\quad \text{as } v\to-\infty, \tag{56c}$$

with the result that

$$\chi(T;L)\sim\frac{\tilde{A}_2^2}{\tilde{A}_3T}\left[\frac{\tilde{A}_1|\tilde{t}|}{\tilde{A}_3}\right]^{2/(2-d')}L^{2(d-d')/(2-d')}, \tag{57}$$

$$c^{(s)}(T;L)\sim-\left[\frac{\tilde{A}_1|\tilde{t}|}{\tilde{A}_3}\right]^{-(4-d')/(2-d')}\left[T\frac{\partial}{\partial T}(\tilde{A}_1|\tilde{t}|)\right]^2 \times L^{-2(d-d')/(2-d')}, \tag{58}$$

and

$$\xi(T;L)\sim\left[\frac{\tilde{A}_1|\tilde{t}|}{\tilde{A}_3}\right]^{1/(2-d')}L^{(d-d')/(2-d')}. \tag{59}$$

We note that these results are precisely the same as the ones for $2<d<4$, provided that the ratios $\tilde{A}_1/\tilde{A}_3$ and $\tilde{A}_2/\tilde{A}_3^{1/2}$ are identified, respectively, with the parameters $\tilde{C}_1$ and $\tilde{C}_2$ introduced previously.[9,12] Clearly, the parameter $\tilde{A}_3$ does not play any independent role in the region of the first-order phase transition, with the result that the pattern of behavior followed by the system in this region is formally the same—irrespective of whether d is less than or greater than 4. There is no reason why $d=4$ would be an exception to this rule, which is precisely what we found in Case 1 of Sec. II; cf. the actual results (14a), (15a), and (16a) with the scaling predictions (57)–(59), remembering that for the spherical model

$$\tilde{A}_1/\tilde{A}_3=K_c/a^{d-2},\quad \tilde{A}_2/\tilde{A}_3^{1/2}=1/(Ka^{d+2})^{1/2}, \tag{60}$$

and that $d=4$ here.

Case 2. For $T\simeq T_c$ and $L\to\infty$, our results depend crucially on whether d is less than, equal to, or greater than 4; this arises from the fact that in this region, and for $d\ge 4$, the scale factor $\tilde{A}_3$ plays a role independently of the combinations that govern the region $T<T_c$. Focusing our attention on the bulk critical point ($\tilde{t}=0$), the scaling formulas (52)–(55) now give (by letting $v\to 0$)

$$\chi(T_c;L)\sim(A_2^2/T_c)A_3^{-(6-d')/(4-d')}L^{2(d-d')/(4-d')}, \tag{61}$$

$$c^{(s)}(T_c;L)\sim-A_1^2/A_3, \tag{62}$$

while

$$\xi(T_c;L)\sim A_3^{-1/(4-d')}L^{(d-d')/(4-d')}. \tag{63}$$

Here, A_i denote the limiting values of the scale factors $\tilde{A}_i$ as $T\to T_c$; for the spherical model,

$$A_1=K_c/a^2w,\quad A_2=1/(K_ca^6w)^{1/2},\quad A_3=a^{d-4}/w, \tag{64}$$

where w is the number defined in (46). We note right away that the foregoing results for $d>4$ are significantly different from the corresponding ones for $2<d<4$, namely,[16]

$$\chi(T_c;L)\sim L^{\gamma/\nu},\quad c^{(s)}(T_c;L)\sim L^{\alpha/\nu},\quad \xi(T_c;L)\sim L^1. \tag{65}$$

To examine the situation as one approaches the borderline dimensionality 4 from above, we confine our attention to the spherical model for which we know that the parameter w, in this limit, is effectively replaced by the quantity $(1/8\pi^2)[\ln(L/a)+\text{const}]$; cf. Eqs. (44) and (47). The corresponding scaling hypothesis may then be written in the same form as in (49), except that now

$$\tilde{A}_1 \sim \frac{K_c}{a^2 \ln(L/a)}, \quad \tilde{A}_2 \sim \frac{1}{[Ka^6 \ln(L/a)]^{1/2}}, \quad \tilde{A}_3 \sim \frac{1}{\ln(L/a)} . \tag{66}$$

It will be noted that the logarithmic factors appearing here leave the combinations $\tilde{A}_1/\tilde{A}_3$ and $\tilde{A}_2/\tilde{A}_3^{1/2}$ unchanged, see (60), so that the behavior of the system in the region of the first-order phase transition is formally the same for $d=4$ as for d other than 4. In the region of the second-order phase transition, however, significant modifications result; for instance, Eqs. (61)–(63) now give

$$\chi(T_c;L) \sim \frac{L^2}{Ja^6}[\ln(L/a)]^{2/(4-d')} , \tag{67}$$

$$c^{(s)}(T_c;L) \sim -K_c^2/a^4 \ln(L/a) , \tag{68}$$

and

$$\xi(T_c;L) \sim L[\ln(L/a)]^{1/(4-d')} , \tag{69}$$

in perfect agreement with the actual results, (19a), (20a), and (21), for $d=4$ and $d'<2$. The case of the slab geometry ($d'=2$) is somewhat problematic but it can also be handled in the manner shown in Ref. 13. In closing, we would like to mention that the scaling formulation for the spherical model in the block geometry $L^4 \times \infty^0$ has already been given by Shapiro and Rudnick;[7] their formulation is in complete agreement with the special case $d'=0$ of ours, except that the quantity $\omega(L)$ of their treatment should be $\sim(\ln L+\text{const})^1$, rather than $(\ln L+\text{const})^{-1}$.

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