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Nonlinear coupling between shear horizontal surface solitons and Rayleigh modes on elastic structures

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Based on a previous work [G. A. Maugin and H. Hadouaj, Phys. Rev. B **44**, 1266 (1991)], it is shown that a generalized Zakharov system (Zakharov's system of plasma physics with additional self-nonlinearity of the cubic type in the Schrödinger component equation) governs the (envelope) solitary-wave propagation of a primarily shear-horizontally (SH) polarized, and secondarily sagittally polarized, surface wave at the top of an elastic structure made of a nonlinear substrate on which is perfectly bonded an elastic film of a slower material. The proof relies on asymptotics. Also shown is the complication introduced by additional dispersive effects in the Rayleigh subsystem.

I. INTRODUCTION

In recent works[1,2] the possible propagation of envelope solitons having the nature of shear-horizontal (SH) elastic surface waves was unequivocally proved mathematically for an elastic structure likely to be experimentally tested. This phenomenon may occur along a composite structure made of a nonlinear elastic isotropic substrate coated with a thin "slow" linear elastic film. Couples of materials for which this in indeed realizable were also determined. However, a strong decoupling hypothesis lay dormant in that approach. Namely, it was assumed that the SH wave in question remains decoupled from the so-called Rayleigh component, i.e., that vectorial elastic component that is polarized parallel to the sagittal plane (plane spanned by the direction of propagation X_1 and the normal to the limiting surface **N**; see Fig. 1). This was considered in order to simplify the analysis, but it does not hold true rigorously. A simple way to realize this fact is to recall what happens for *bulk* waves in nonlinear isotropic *(a fortiori* anisotropic) elasticity (See Ref. 3, pp. 36–37). In that theory a *longitudinal* motion necessarily accompanies a transverse motion; e.g., one can write

$$\mathcal{D}_T v = 0\ ,\qquad \mathcal{D}_L u = \gamma v_x v_{xx}\ , \tag{1.1}$$

where $\mathcal{D}_T$ and $\mathcal{D}_L$ are linear "transverse" and "longitudinal" (d'Alembertian) wave operators along x, v is the transverse component, u is the longitudinal component, a subscript x denotes partial differentiation with respect to x, and γ is a third-order elasticity coefficient. In an asymptotic analysis where $v = O(\epsilon)$ as ϵ goes to zero, $u = O(\epsilon^2) = O(v^2)$ makes the system (1.1) fully consistent. In particular, there is no feedback of u in the v equation, while the longitudinal component u is excited through the second harmonic of v. If this is true for bulk waves, then the situation should be worse for surface waves as boundary conditions to be applied at the limiting surface—although free of tractions—usually couple both the longitudinal component u and remaining transverse component w [so-called shear-vertical (SV) component] to produce what is commonly referred to as a *Rayleigh* surface wave[4] in the *linear* approximation. But then, according to (1.1), the SH component should couple with the full Rayleigh one in nonlinear elasticity. This establishes the frame of mind in which the present paper develops. From (1.1), however, we shall still keep the idea that the SH component is *primary,* for instance, being preferably entered in the system through a transducer designed to that effect, while the Rayleigh component is only *secondary,* being essentially generated by the former. Furthermore, only the SH component *a priori* carries a *dispersion* effect due to the built-in vertical layering; but a straightforward generalization so as to include dispersion due, e.g., to discreteness, in the longitudinal component can be proposed heuristically (see Sec. V).

In all, the main result of the present analysis, which is essentially asymptotic in the manner of Whitham,[5] Benney and Newell,[6] and Newell,[7] is that the complex amplitude a of the slowly varying envelope of the SH component and real amplitude gradients (in the propagation direction) $n_1 = u_x$ and $n_2 = w_x$ of the Rayleigh com-

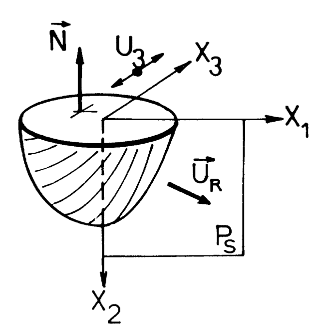

FIG. 1. Setting of the surface-wave problem: u_3, shear-horizontal elastic displacement; $\mathbf{u}_R$, Rayleigh component polarized parallel to the sagittal plane P_S.

ponents are in general governed by the following system (in the absence of dispersion in the Rayleigh subsystem) after normalization:

$$\begin{aligned} &ia_t+a_{xx}+2\lambda|a|^2a+(\alpha_L n_1+\alpha_T n_2)a=0 \ , \\ &(n_1)_{tt}-C_L^2(n_1)_{xx}=-\mu_L(|a|^2)_{xx} \ , \\ &(n_2)_{tt}-C_T^2(n_2)_{xx}=-\mu_T(|a|^2)_{xx} \ , \end{aligned} \tag{1.2}$$

where λ is the real coefficient of self-nonlinearity (of the SH mode) and α_L, α_T, μ_L, and μ_T are real coupling coefficients. For $\lambda=\alpha_T=\mu_T=0$, this system is none other than Zakharov's[8,9] system for Langmuir-ion acoustic waves in plasmas, but with different physical interpretations. We coined the name "generalized Zakharov's systems" for systems such as (1.2) and their associates. Two recent papers[10,11] are devoted to the analytical and numerical study of one- and two-soliton solutions of system (1.2). Thus the present work aims at providing the missing link between these works and the initial, simplified, ones (Ref. 2 in particular). Because it involves three-dimensional finite-strain elasticity, complicated boundary conditions relating to the presence of the thin film, and, in addition, two space dimensions to start with, the formulation is somewhat cumbersome. Only the outline of the deduction and most pertinent steps could be reported in a text of reasonable length. We shall *not* repeat in detail those steps that mimic the proof given in Ref. 2 for the simpler system (pure SH case). Lengthy intermediate expressions can be found in a thesis by one of the authors[12] (available on request).

II. SETTING OF THE PROBLEM

The starting equations are the three-dimensional equations of motion (for $X_2>0$) and boundary conditions (at $X_2=0$) written as Eqs. (2.29) in Ref. 2:

$$\rho_0\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial T_{ij}}{\partial X_j}, \quad X_2>0, \quad i,j=1,2,3 \ , \tag{2.1a}$$

$$\hat\rho_0\frac{\partial^2 \hat u_i}{\partial t^2}=\frac{\partial \hat T_{ik}}{\partial \hat X_k}-T_{ij}N_j \ , \qquad X_2=0, \quad i,j=1,2,3, \quad k=1,3 \ , \tag{2.1b}$$

respectively. This uses the index notation of Cartesian tensors (we refer to Ref. 2 for all notations of continuum mechanics). Here X_j, $j=1,2,3$, are Cartesian *material* coordinates in three dimensions; ρ_0, u_i, and T_{ij} are the mass density, displacement vector, and first Piola-Kirchhoff stress tensor pertaining to the substrate ($X_2>0$), a *nonlinear* elastic, isotropic, homogenous material with constitutive equation

$$T_{ij}=\frac{\partial W}{\partial E_{pj}}\left[\delta_{ip}+\frac{\partial u_i}{\partial X_p}\right] \tag{2.2}$$

and strain energy function (per unit volume) W such that (tr = trace)

$$W=\frac{\lambda}{2}I_1^2+\mu I_2+\alpha I_3^2+\bar\beta I_1 I_2+\gamma I_3+\xi I_1^4 +\eta I_1^2 I_2+\nu I_1 I_3+\delta I_2^2 \ , \tag{2.3}$$

$$I_1=\mathrm{tr}\mathbf{E}, \quad I_2=\mathrm{tr}\mathbf{E}_2, \quad I_3=\mathrm{tr}\mathbf{E}^3 \ , \tag{2.4}$$

$$E_{pj}=\frac{1}{2}\left[\frac{\partial u_p}{\partial X_j}+\frac{\partial u_j}{\partial X_p}+\frac{\partial u_m}{\partial X_j}\frac{\partial u_m}{\partial X_p}\right] . \tag{2.5}$$

The last quantity is the Lagrangian finite-strain tensor, δ_{ip} is Kronecker's symbol, and the Einstein summation convention on repeated (dummy) indices applies; λ and μ are elasticity coefficients of the second order; α, $\bar\beta$, and γ are elasticity coefficients of the third order; and ξ, η, ν, and δ are elasticity coefficients of the fourth order. The expression (2.3) follows Bland[13] and Kalyanasundaram.[14]

Similarly, $\hat X_k$, $k=1,3$, are Cartesian material coordinates in the two-dimensional plane corresponding to $X_2=0$, with $\hat X_1$ and $\hat X_3$ coinciding with X_1 and X_3, respectively; $\hat\rho_0$, $\hat u_i$, and $\hat T_{ik}$ are the mass density (per unit area), displacement vector, and first Piola-Kirchhoff stress tensor pertaining to the thin film ($X_2=0$), a *linear* elastic, isotropic, homogeneous material with constitutive equation (here nonlinearities,whether physical or geometrical, are *not* needed)

$$\hat T_{ik}=\frac{\partial \hat W}{\partial \hat E_{ik}} \ , \tag{2.6}$$

$$\hat W=\frac{\hat\lambda}{2}\hat I_1^2+\hat\mu I_2, \quad \hat I_1=\mathrm{tr}\hat{\mathbf{E}}, \quad \hat I_2=\mathrm{tr}\hat{\mathbf{E}}^2 \ , \tag{2.7}$$

$$\hat E_{ik}\simeq\frac{1}{2}\left[\frac{\partial \hat u_i}{\partial \hat X_k}+\frac{\partial \hat u_k}{\partial \hat X_i}\right], \quad i,k=1,3 \ , \tag{2.8}$$

where $\hat W$ is a strain energy per unit area, $\hat\lambda$ and $\hat\mu$ being the in-plane Lamé elasticity coefficients.

Equations (2.1) are essentially the simplified form of the bulk and jump-boundary conditions of the theory of elastic interfaces sandwiched between two half spaces.[15] The flat (in the Lagrangian description) interface separates here a nonlinear elastic substrate ($X_2>0$) from vacuum. There is *no* applied traction on the exterior face of the interface. In the modeling the influence of a superimposed thin film has been accounted for through an inertial term (containing $\hat\rho$) and a surface elasticity (or capillarity) term in the boundary condition for T_{ij} [Eq. (2.1b)]. Perfect bonding between the interface and substrate occurs at $X_2=0$, i.e.,

$$u_i=\hat u_i, \quad i=1,2,3, \quad \text{at } X_2=0 \ . \tag{2.9}$$

We shall consider a *wave motion* that takes place in the X_1 direction and is of the *surface-wave* type in the sense that

$$u_i(X_1,X_2\to\infty,X_3,t)=0 \ . \tag{2.10}$$

There is no loss of generality in assuming that the problem in fact does not depend on X_3 (but it *does* involve u_3 and $\hat u_3$) so that, symbolically,

$$\frac{\partial}{\partial X_3}=0\ . \tag{2.11}$$

Accounting for the remark about system (1.1) in the Introduction, we consider that strains developed through the displacements u_3 and $\hat{u}_3$ are of order ϵ (where ϵ is infinitesimally small), while those occurring from u_1 and u_2 or $\hat{u}_1$ will be of order ϵ^2 as ϵ goes to zero. In spite of the considered simplifying hypotheses, the equations recalled in this section reflect the whole complexity of, and perhaps, more, the mere size of algebraic manipulations at hand in, the present problem.

III. REDUCED WAVE EQUATIONS

On substituting from (2.2)–(2.5) into (2.1a) and (2.6)–(2.8) and (2.2) in (2.1b) and keeping in mind that $|u_{3,j}|=O(\epsilon)$, whereas $|u_{1,j}|$ and $|u_{2,j}|$ are $O(\epsilon^2)$, so that we keep at most terms which are $O(\epsilon^2)$ in $X_1=X$ and $X_2=Y$ components and $O(\epsilon^3)$ for the SH ($X_3=Z$) component, after some lengthy algebra we obtain the following equations.

In the substrate ($X_2=Y>0$),

$$\rho_0\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)u_{XX}+\mu u_{YY}+(\lambda+\mu)W_{XY}+A_e v_X v_{XX}+A_f v_Y v_{XY}+A_g v_X v_{YY}\ , \tag{3.1}$$

$$\rho_0\frac{\partial^2 w}{\partial t^2}=(\lambda+2\mu)w_{YY}+\mu w_{XX}+(\lambda+\mu)u_{XY}+A_e v_Y v_{YY}+A_f v_X v_{XY}+A_g v_Y v_{XX}\ , \tag{3.2}$$

$$\begin{aligned}\rho_0\frac{\partial^2 v}{\partial t^2}=&\mu(v_{XX}+v_{YY})+\bar{\beta}(v_{XX}u_x+v_X u_{XX}+v_{XX}w_Y+v_X w_{XY}+v_{YY}u_X+v_Y u_{XY}+v_{YY}w_Y+v_Y w_{YY})\\&+\frac{3\gamma}{4}(2v_{XX}u_X+2v_X u_{XX}+2v_{XY}u_Y+v_Y u_{XY}+v_{XY}w_X+v_Y w_{XX}+v_X u_{YY}+v_{YY}w_X+v_Y w_{XY}+4v_{YY}w_Y)\\&+\lambda(v_X u_{XX}+v_X w_{XY}+v_Y u_{XY}+v_Y w_{YY}+\mu(v_X u_{YY}+v_x w_{XY}+v_Y u_{XY}+v_Y w_{XX}+2v_X u_{XX}+2v_Y w_{YY})\\&+\delta^{\text{eff}}\{[v_X(v_X^2+v_Y^2)]_X+[v_Y(v_X^2+v_Y^2)]_Y\}\ ,\end{aligned} \tag{3.3}$$

in which we have set

$$u_1=u,\quad u_2=w,\quad u_3=v \tag{3.4}$$

and

$$A_e=\lambda+2\mu+\bar{\beta}+\tfrac{3}{2}\gamma,\quad A_f=\mu+\lambda+\bar{\beta}+\tfrac{3}{4}\gamma,\quad A_g=\tfrac{3}{4}\gamma+\mu,\quad \delta^{\text{eff}}=\delta+\bar{\beta}+\tfrac{3}{2}\gamma+\mu+\frac{\lambda}{2}\ . \tag{3.5}$$

At the interface ($X_2=Y=0$),

$$\begin{aligned}\hat{\rho}_0\frac{\partial^2\hat{u}}{\partial t^2}=&(\hat{\lambda}+2\hat{\mu})\hat{u}_{XX}+\mu(u_Y+w_X)+\mu(u_X u_Y+w_X w_Y+v_X w_Y)+\bar{\beta}(u_Y+w_X)(u_X+w_Y)\\&+\tfrac{3}{2}\gamma[u_X(u_Y+w_X)+w_Y(u_Y+w_X)+\tfrac{1}{2}v_X v_y]+\lambda u_Y(u_X+w_Y)+\mu(u_X u_Y+u_X w_X+2u_Y w_Y)\ ,\end{aligned} \tag{3.6}$$

$$\begin{aligned}0=&(\lambda+2\mu)w_Y+\lambda u_X+\frac{\lambda}{2}[u_X^2+w_X^2+v_X^2+u_Y^2+w_Y^2+v_Y^2+2w_Y(u_X+w_Y)]+3\alpha(u_X+w_Y)^2\\&+\bar{\beta}[u_X^2+\tfrac{1}{2}(u_Y+w_X)^2+\tfrac{1}{2}(v_X^2+v_Y^2)+w_Y^2+2w_Y(u_X+w_Y)]+\mu(u_Y^2+3w_Y^2+v_Y^2+w_X^2+u_Y w_X)\\&+3\gamma[\tfrac{1}{4}(u_Y+w_X)^2+w_Y^2+\tfrac{1}{4}v_Y^2]\ ,\end{aligned} \tag{3.7}$$

$$\hat{\rho}_0\frac{\partial^2\hat{v}}{\partial t^2}=\hat{\mu}\hat{v}_{XX}+\mu v_Y+(\lambda+\bar{\beta})v_Y u_X+A_e v_Y w_Y+A_g(v_X u_Y+v_X w_X)+\delta^{\text{eff}}v_Y(v_X^2+v_Y^2)\ . \tag{3.8}$$

For the sake of comparison, had we discarded the coupling of the u and w components with v in Eqs. (3.3) and (3.8), we would have reached the system

$$\rho_0\frac{\partial^2 v}{\partial t^2}=\mu(v_{XX}+v_{YY})+\delta^{\text{eff}}\{[v_X(v_X^2+v_Y^2)]_X+[v_Y(v_X^2+v_Y^2)]_Y\}\ , \tag{3.9a}$$

$$\hat{\rho}_0\frac{\partial^2\hat{v}}{\partial t^2}=\hat{\mu}\hat{v}_{XX}+\mu v_Y+\delta^{\text{eff}}v_Y(v_X^2+v_Y^2)\ , \tag{3.9b}$$

which is the couple of equations considered in previous works (Refs. 1 and 2). Note that this decoupling hypothesis makes that the first nonlinearities to manifest themselves in the simplified system (3.9) be of the *third* order—i.e., the right-hand sides in Eqs. (3.9) would generate the *third* harmonic of an initially monochromatic signal—while the full coupled system (3.1)–(3.3) and (3.6)–(3.8) includes second-order nonlinearities yielding, apparently (see below) *second-*

harmonic generation in Eqs. (3.1)–(3.8). It is easy to recognize in the linear version of Eqs. (3.9) the system of Murdoch[16] for SH motion, and in the linear version of Eqs. (3.1)–(3.2) and (3.6)–(3.7) the celebrated system of Rayleigh[17] for sagitally polarized waves.

The above-obtained system is now made nondimensional in the following manner. Let k_a and w_a be typical wave number and frequency. Then new nondimensional independent and dependent variables (τ,x,y) and (U,V,W) and $(\hat{U},\hat{V})$ are introduced by

$$\tau=\omega_a t,\quad x=k_a X,\quad y=k_a Y,\quad U=k_a u,\quad V=k_a v,\quad W=k_a W,\quad \hat{U}=k_a\hat{u},\quad \hat{V}=k_a\hat{v}\ , \tag{3.10}$$

together with

$$\begin{gathered} k_a=\mu/\hat{\mu},\quad C_S^2=\hat{\mu}/\hat{\rho}_0=\omega_a^2/k_a^2,\quad C_T^2=\mu/\rho_0,\quad C=C_T/C_S,\quad \beta^2=1/C^2\ , \\ C_{L1}^2=(\lambda+2\mu)/\rho_0,\quad \hat{C}_{L1}^2=(\hat{\lambda}+2\hat{\mu})/\hat{\rho}_0,\quad C_L^2=C_{L1}^2/C_S^2,\quad \hat{C}_L^2=C_S^2/\hat{C}_{L1}^2\ , \end{gathered} \tag{3.11}$$

and the nondimensional material coefficients

$$c_\alpha=(\bar{\beta}+\tfrac{3}{2}\gamma)/\mu,\quad c_\beta=A_e/\mu,\quad c_\gamma=\bar{\beta}/\mu,\quad c_\eta=(\lambda+\mu+\bar{\beta})/\mu,\quad c_\delta=A_f/\mu,\quad c_\nu=A_g/\mu,\quad c_\kappa=3\gamma/4\mu\ , \tag{3.12}$$

so that

$$c_\delta=c_\eta+c_\kappa,\quad c_\nu=c_\kappa+1,\quad c_\delta=c_\nu+c_\eta-1\ . \tag{3.13}$$

On account of these, some simple rearrangement allows one to deduce the following nondimensional equation.

In the bulk $(Y>0)$,

$$\zeta(C_L^{-2}U_{\tau\tau}-U_{xx})=(\zeta-1)W_{xy}+U_{yy}+c_\beta V_x V_{xx}+c_\delta V_y V_{xy}+c_\nu V_x V_{yy}\ , \tag{3.14}$$

$$\zeta(C_L^{-2}W_{\tau\tau}-W_{yy})=W_{xx}+(c_\eta-c_\gamma)U_{xy}+c_\beta V_y V_{yy}+c_\delta V_x V_{xy}+c_\nu V_y V_{xx}\ , \tag{3.15}$$

$$\begin{aligned} \beta^2 V_{\tau\tau}-(V_{xx}+V_{yy})=&\,c_\alpha(U_x V_{xx}+V_{yy}W_y)+c_\beta(V_x U_{xx}+V_y W_{yy})+c_\gamma(V_{xx}W_y+U_x V_{yy})+c_\delta V_y U_{xy} \\ &+c_\eta V_x W_{xy}+c_\nu(V_x U_{yy}+V_y W_{xx})+c_\kappa(2U_y V_{xy}+W_x V_{xy}+V_{yy}W_x+V_y W_{xy}) \\ &+\beta^2\Delta\{[V_x(V_x^2+V_y^2)]_x+[V_y(V_x^2+V_y^2)]_y\}\ . \end{aligned} \tag{3.16}$$

At the interface $(y=0)$,

$$\hat{U}_{\tau\tau}-\hat{C}_L^2\hat{U}_{xx}=U_y+W_x+c_\nu V_x V_y\ , \tag{3.17}$$

$$0=\zeta W_y+(c_\eta-c_\gamma-1)U_x+\tfrac{1}{2}(c_\eta-1)V_x^2+\tfrac{1}{2}c_\beta V_y^2\ , \tag{3.18}$$

$$\hat{V}_{\tau\tau}-\hat{V}_{xx}=(c_\eta-1)V_y U_x+c_\beta V_y W_y+c_\nu(U_y V_x+V_x W_x)+V_y[1+\beta^2\Delta(V_x^2+V_y^2)]\ , \tag{3.19}$$

in which we have set

$$\Delta=\delta^{\text{eff}}/\mu,\quad \zeta=\frac{\lambda+2\mu}{\mu}=C_{L1}^2/C_T^2\ , \tag{3.20}$$

and have purged Eqs. (3.6) and (3.7) of some terms of order four which still spoiled them. The coefficients β and Δ are dispersion and nonlinearity parameters for the pure SH problem.

IV. TOWARD THE GENERALIZED ZAKHAROV SYSTEM

A. Transformation of the equations for the Rayleigh component

The *surface-wave* solutions that we shall consider will vanish sufficiently rapidly in depth in the substrate, say, exponentially, and the material particles that belong to the superimposed film are not distinguishable from those of the substrate that they match at this interface [Eq. (2.9)]. Then, if practical (this is approximately true after the study of the pure SH case in Ref. 2), we set

$$(U,V,W)(x,y,\tau)=(\hat{U}(x,\tau),\hat{V}(x,\tau),\hat{W}(x,\tau))\exp(-\chi y)\ , \tag{4.1}$$

where χ is a positive real number; we can use this property to bring the whole above-obtained system *to the interface,* at least here for the Rayleigh component. This naive way of getting rid of the behavior in depth (along y) goes as follows. For instance, integrate (3.14) from $y=+\infty$ to the interface $y=0$. Accounting for the vanishing condition at infinity, we obtain thus

$$\begin{aligned} \int_{+\infty}^0 \zeta(C_L^{-2}U_{\tau\tau}-U_{xx})dy \\ =&[(\zeta-1)W_x+U_y]_{y=0} \\ &+\int_{+\infty}^0[c_\beta V_x V_{xx}+(c_\eta-1)V_y V_{xy}]dy \\ &+[c_\nu V_x V_y]_{y=0}\ , \end{aligned} \tag{4.2}$$

where we used the last of (3.13). Now subtract side by side Eq. (3.17) from this to get

$$\int_{+\infty}^{0}\zeta(C_L^{-2}U_{\tau\tau}-U_{xx})dy-(\hat{U}_{\tau\tau}-\hat{C}_L^2\hat{U}_{xx})$$
$$=(\lambda/\mu)[W_x]_{y=0}+\int_{+\infty}^{0}[c_\beta V_x V_{xx}+(c_\eta-1)V_y V_{xy}]dy \ . \quad (4.3)$$

For solutions which can be approximated by (4.1), this last result can also be rewritten as

$$\zeta(C_L^{-2}\hat{U}_{\tau\tau}-\hat{U}_{xx})+\chi(\hat{U}_{\tau\tau}-\hat{C}_L^2\hat{U}_{xx})$$
$$=-(\chi\lambda/\mu)\hat{W}_x+\tfrac{1}{2}c_\beta\hat{V}_x\hat{V}_{xx}+\frac{\chi^2}{2}(c_\eta-1)\hat{V}\hat{V}_x \ . \quad (4.4)$$

Proceeding in a like manner for Eq. (3.15) and associated boundary condition (3.18), we are led to a relation valid at the interface:

$$\zeta C_L^{-2}\hat{W}_{\tau\tau}-\hat{W}_{xx}=-\chi[\hat{U}_x+\tfrac{1}{2}(1+c_\kappa)\hat{V}_x^2+\tfrac{1}{2}c_\nu\hat{V}\hat{V}_{xx}] \ . \quad (4.5)$$

B. Equation governing the SH component at the interface

We now have to deal with Eqs. (3.16) and (3.19). These can be rewritten in the suggestive form

$$\mathcal{D}_B V-\beta^2\Delta T_{\rm NL}^B-P_{\rm NL}^B=0, \quad y>0 \ , \quad (4.6)$$

$$\mathcal{D}_S \hat{V}-V_y-\beta^2\Delta T_{\rm NL}^S-P_{\rm NL}^S=0, \quad y=0 \ , \quad (4.7)$$

where we have set

$$\mathcal{D}_B=\beta^2\frac{\partial^2}{\partial\tau^2}-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right], \quad \mathcal{D}_S=\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2} \ , \quad (4.8)$$

and

$$T_{\rm NL}^B=[V_x(V_x^2+V_y^2)]_x+[V_y(V_x^2+V_y^2)]_y \ ,$$
$$T_{\rm NL}^S=V_y(V_x^2+V_y^2) \ . \quad (4.9)$$

Here $P_{\rm NL}^B$ and $P_{\rm NL}^S$ are given by the remainders in Eqs. (3.16) and (3.19). These last two terms are made of linear combinations of contributions which always are products of a spatial derivative of V and a spatial derivative of either U or W. Equations (4.6) and (4.7) would be identical to Eqs. (3.9) if it were not for these additional terms. We can, therefore, proceed as in Ref. 2. All we need to do is to evaluate the influence of these additional contributions while referring the reader to Ref. 2 for the detail of the deduction of the other terms. To that effect, we recall that the V component shall be a spatially (along x) slowly modulated carrier wave (whose frequency ω_0 and wave number k_0 practically satisfy the linear dispersion relation of Murdoch's surface waves [Eq. (3.4) in Ref. 2]). We expect then that the U and W components which are driven by the V one will also vary slowly, so that we can write the inequalities

$$|U_{xx}|\ll|U_x|, \quad |U_{yy}|\ll|U_{xx}|, \quad |U_y|\ll|U_x| \ ,$$
$$|W_{xx}|\ll|W_x|, \quad |W_{yy}|\ll|W_{xx}|, \quad |W_y|\ll|W_x| \ , \quad (4.10)$$

as the y variation is even more slow. On account of these and setting

$$U_x=N_1, \quad W_x=N_2 \ , \quad (4.11)$$

we can approximate $P_{\rm NL}^B$ and $P_{\rm NL}^S$ by

$$P_{\rm NL}^B\simeq c_\alpha N_1V_{xx}+c_\gamma N_1V_{yy}+c_\kappa N_2(V_{xy}+V_{yy}) \ ,$$
$$P_{\rm NL}^S\simeq(c_\eta-1)N_1V_y+c_\nu N_2V_x \ . \quad (4.12)$$

To get some idea of the influence of those terms, we may momentarily consider a V solution in the form

$$V=A\exp(-\chi y)\cos\theta \ , \quad (4.13)$$

with $\theta=kx-\omega\tau$, as in the search for harmonic generation (see Ref. 2, Sec. IV A). In that case the contributions (4.12) will produce predominant nonlinearities in the form (terms in factor of $\cos\theta$)

$$(-k^2c_\alpha N_1+\chi^2c_\gamma N_1+\chi^2c_\kappa N_2)A\exp(-\chi y) \ , \quad (4.14a)$$

for $y>0$ in Eq. (4.6), and

$$-\chi N_1(c_\eta-1)A \ , \quad (4.14b)$$

at $y=0$ in Eq. (4.7). At the interface we have

$$N_1(y=0)=n_1\equiv\hat{U}_x, \quad N_2(y=0)=n_2\equiv\hat{W}_x \ . \quad (4.15)$$

Following the guideline provided by Appendix B in Ref. 2, we then know that the correction due to n_1 and n_2 in the nonlinear Schrödinger equation governing the complex amplitude $a=A\exp(i\theta)$ at $y=0$ is simply obtained by combining (4.14a) for $y=0$ with 2χ times the surface contribution (4.14b), i.e., the contribution

$$\alpha_{L1}n_1A+\alpha_{T1}n_2A \ , \quad (4.16)$$

with

$$\alpha_{L1}=-k^2c_\alpha+\chi^2c_\gamma+2\chi^2(c_\eta-1) \ ,$$
$$\alpha_{T1}=\chi^2c_\kappa \ , \quad (4.17)$$

where, we emphasize, k and χ have to belong to the linear dispersion relation of Murdoch so that they should be noted k_0 and χ_0, with χ_0 given by Eq. (4.24b) of Ref. 2 in terms of k_0 and ω_0. Proceeding then as in Ref. 2, on account of the additional terms (4.16) we shall find that the complex amplitude a is governed by the nonlinear Schrödinger (NS) equation

$$ia_\tau+\frac{\omega_0''}{2}a_{\xi\xi}+q(\omega_0,k_0;\Delta)|a|^2a+(\alpha_Ln_1+\alpha_Tn_2)a=0 \ , \quad (4.18)$$

at $y=0$, where $\xi=x-\omega_0'\tau$, ω_0' and ω_0'' are the slope and curvature of Murdoch's dispersion relation at (ω_0,k_0), and we have set

$$q(\omega_0,k_0;\Delta)=\frac{3\beta^4\omega_0\Delta(\beta^2\omega_0^2-2k_0^2)}{8[\beta^2+2(\omega_0^2-k_0^2)]} \ , \quad (4.19)$$

$$\alpha_L(\omega_0,k_0)=\frac{-k_0^2c_\alpha+\chi_0^2(c_\gamma+2c_\eta-2)}{2\omega_0[\beta^2+2(\omega_0^2-k_0^2)]} \ , \quad (4.20a)$$

$$\alpha_T(\omega_0,k_0)=\frac{\chi_0^2 c_\kappa}{2\omega_0[\beta^2+2(\omega_0^2-k_0^2)]} . \quad (4.20b)$$

According to Ref. 2 [Eqs. (4.24b), (4.41a), and (4.41b)],

$$\chi_0=\omega_0^2-k_0^2 , \quad (4.21a)$$

$$\omega_0'=\frac{k_0[1+2(\omega_0^2-k_0^2)]}{\omega_0[\beta^2+2(\omega_0^2-k_0^2)]} , \quad (4.21b)$$

$$\omega_0''=\frac{(\omega_0^2-k_0^2)[2(\beta^4\omega_0^2-k_0^2)-\beta^2(\omega_0^2-k_0^2)]}{\omega_0^3[\beta^2+2(\omega_0^2-k_0^2)]^3} , \quad (4.21c)$$

so that we know all about Eq. (4.18).

C. Generalized Zakharov system

It remains to transform Eqs. (4.4) and (4.5) when the SH component is governed by Eq. (4.18). On the one hand, the solution should vary slowly in x space, so that we can write

$$|k_0^2 A| \gg |k_0 A_x| \gg |A_{xx}| , \quad (4.22)$$

for a typical amplitude. On the other hand, we assume that $p_0=\chi_0^{-1} \gg \lambda_0$ (wavelength) or $\chi_0 \ll k_0$, with, in our framework, $\partial/\partial y=-\chi_0$, $\partial^2/\partial y^2=\chi_0^2$. On account of these, (4.4) and (4.5) can be approximated by

$$\zeta(C_L^{-2}\hat{U}_{\tau\tau}-\hat{U}_{xx})+\chi_0(\hat{U}_{\tau\tau}-\hat{C}_L^2\hat{U}_{xx})=\tfrac{1}{2}c_\beta \hat{V}_x\hat{V}_{xx} \quad (4.23)$$

and

$$\zeta C_L^{-2}\hat{W}_{\tau\tau}-\hat{W}_{xx}=0 . \quad (4.24)$$

In this approximation the $\hat{W}$ component is no longer driven by $\hat{V}$. We still have to evaluate the $\hat{V}$ contribution in the right-hand side of Eq. (4.23). We write

$$\hat{V}=a\exp(i\theta)+a^*\exp(-i\theta) . \quad (4.25)$$

In the analysis yielding the NS equation in Ref. 2, we had $|a|=O(\epsilon^2)$. Then, according to the remark of Sec. I, we should take $\hat{U}=O(\epsilon^3)$. On computing $\hat{V}_x$ and $\hat{V}_{xx}$ from (4.25), evaluating their product, and rescaling by $\tau'=\epsilon\tau$ and $x'=\epsilon x$, we find that the first term to be conserved in the source contribution of Eq. (4.23) shall be of order ϵ^5. Discarding secular terms that may arise in the expression of $\hat{V}_x\hat{V}_{xx}$ and proceeding as in Newell (Ref. 7, p. 39), we find that Eq. (4.23) takes on the form (order ϵ^5)

$$C_n\hat{U}_{\tau\tau}-C_m\hat{U}_{xx}=\tfrac{1}{2}c_\beta k_0^2(aa^*)_{xx} , \quad (4.26)$$

wherein

$$C_n(\omega_0,k_0)\equiv(\zeta/C_L^2)+\chi_0 , \qquad C_m=\zeta+C_L^2 . \quad (4.27)$$

Taking finally the x derivative of (4.26) and (4.24) and accounting for (4.15), we obtain the equations governing $\hat{U}$ and $\hat{W}$ at $y=0$ as

$$(n_1)_{\tau\tau}-C_{S1}^2(n_1)_{\xi\xi}=R(|a|^2)_{\xi\xi} , \qquad (n_2)_{\tau\tau}-C_{S2}^2(n_2)_{\xi\xi}=0 , \quad (4.28)$$

where

$$R(\omega_0,k_0)=\frac{1}{2C_n}k_0^2c_\beta , \qquad C_{S1}^2=C_m/C_n, \quad C_{S2}^2=C_L^2/\zeta=C_T^2 . \quad (4.29)$$

The system formed by Eqs. (4.18) and (4.28) indeed is of the form of system (1.2), with $\mu_T=0$. Note that many of the coefficients in these equations still depend on the working regime (ω_0,k_0) of the carrier wave, the point (ω_0,k_0) having to belong to Murdoch's linear dispersion relation. However, an appropriate scaling does the trick to recover the universal form (1.2). Thus our surface-wave problem is governed by a so-called *generalized Zakharov system* at the thin film (interface).

V. ZAKHAROV SYSTEM AND BEYOND

A. One-soliton solution

There is no loss of generality in the nature of the problem in setting $\alpha_T=\mu_T=0$ in system (1.2), so that we extract the system with two degrees of freedom ($n_1=n$ and a slightly changed but obvious notation):

$$ia_t+a_{xx}+2\lambda|a|^2a+2na=0 , \quad (5.1a)$$

$$n_{tt}-C^2n_{xx}=-\mu(|a|^2)_{xx} . \quad (5.1b)$$

Note that the present coefficients λ and μ bear no relationship whatever to elasticity moduli introduced in previous sections. Just like the pure NS equation (with $\lambda>0$), the system (5.1) possesses *one*-soliton solutions. This is simply seen by noting that for such types of solutions a and n functions of $\xi=x-Vt$, the second of (5.1) yields

$$n_{\xi\xi}(V^2-C^2)=-\mu(a^2)_{\xi\xi} . \quad (5.2)$$

Upon integration, for conditions of nonresonance ($V\neq C$), n is substituted from (5.2) into (5.1), providing thus the new (pure) NS equation

$$ia_{tt}+a_{xx}+2\lambda_{\text{eff}}|a|^2a=0 , \quad (5.3)$$

wherein

$$\lambda_{\text{eff}}=\lambda+\mu(C^2-V^2)^{-1} . \quad (5.4)$$

And this admits (envelope) soliton solutions of the form

$$a_{\text{sol}}(x,t)=2i\eta\,\text{sech}[2\eta(x-Vt)] \times\exp[(i/2)Vx+i(4\eta^2-V^2/4)t] , \quad (5.5)$$

so that (5.2) yields

$$n_{\text{sol}}(x,t)=4\mu\eta^2\lambda_{\text{eff}}^{-1}(C^2-V^2)^{-1}\text{sech}^2(2\eta x) . \quad (5.6)$$

With $\lambda>0$ (the existence condition for solitons of the pure NS equation), the solution (5.5) and (5.6) exists only for $\lambda_{\text{eff}}>0$; thus either in the *subsonic* range,

$$V^2<C^2 , \quad (5.7)$$

or in the *supersonic range* (also called the transsonic range),

$$V^2 > C^2 + (\mu/\lambda) \ . \tag{5.8}$$

But on account of (5.4), soliton solutions can also exist for $\lambda < 0$ on the condition that

$$(\mu/|\lambda|) < V^2 < C^2 \ , \tag{5.9}$$

provided that $\mu/|\lambda| < C^2$ at all. If $\mu/|\lambda| > C^2$, then this solution exists for the whole subsonic range. For *two-soliton* solutions of the system (5.1), we refer the reader to Ref. 11.

B. Influence of dispersion effects in the Rayleigh subsystem

According to the physical interpretation which motivated this paper, system (5.1) couples the slowly varying envelope of the *dispersive* SH component of the surface-wave problem at $y=0$ to the *nondispersive,* longitudinal component of the Rayleigh mode. According to this model, then, both dispersion and self-nonlinearity are built in the SH component, dispersion arising from the presence of the superimposed thin film, i.e., a vertical stratification. As reflected in the typical solution (5.5) and (5.6), n is of second order with respect to a. This was introduced initially as one of the working hypotheses. The above theory breaks down if $a^2 = O(n^2)$. In that case terms involving biquadratic nonlinearity and dispersive effects in the Rayleigh component become of the same order as the term retained in the right-hand side of Eq. (5.1b). These additional effects occur in the propagation space (x) and, therefore, will transform (5.1b) essentially in a *Boussinesq* equation coupled to the NS equation (5.1a) (for Boussinesq's model applied to the modeling of surface acoustic waves, see Bataille and Lund[18]). In these conditions, (5.1) should be replaced by the more involved system (with a new but trivial change in scaling)

$$\begin{aligned} &2ia_t + 3a_{xx} + 2\lambda|a|^2 a - na = 0 \ , \\ &n_{tt} - C^2 n_{xx} = (|a|^2 + n^2 + n_{xx})_{xx} \ . \end{aligned} \tag{5.10}$$

This is but a system known in plasma physics[19] with the additional self-nonlinearity in the NS equation. In the absence of this nonlinearity ($\lambda=0$), system (5.10) was shown by several authors to possess *one-soliton solutions.*[20,21] Typically (remember that $\lambda=0$),

$$\begin{aligned} a_{\rm sol}(x,t) &= -\frac{2n_0}{\sqrt{3}}(\text{sech}\xi)(\tanh\xi) \\ &\quad \times \exp[(i/3)Vx + (i/12)n_0 t] , \\ n_{\rm sol}(x,t) &= -n_0 \text{sech}^2\xi \ , \end{aligned} \tag{5.11}$$

with

$$\xi = \left[\frac{n_0}{18}\right]^{1/2}(x - Vt), \quad V = 1 - 5n_0/9 \ , \tag{5.12}$$

in which the last relation plays the role of a dispersion relation $D(V, n_0) = 0$. Equations (5.11) provide a one-parameter (n_0) family of solutions where the field a has a node at the center. These solutions are quite different from (5.5) and (5.6) because the orders are *not* the same. For the time being, we do not have soliton solutions for the more general system (5.10), with λ finite and positive or possibly negative [see remark preceding Eq. (5.9)]. Although the extension to the system (5.10) may find some practical interest (e.g., in the elastic surface problem, for a defective transducer exciting simultaneously both SH and Rayleigh components), our main concern remains the essentially SH-polarized solitons of systems such as (1.2) or (5.1).

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