Wave propagation in one-dimensional disordered structures

Stephen B. Haley* and Paul Erdös
Institute of Theoretical Physics, University of Lausanne, CH-1015 Lausanne, Switzerland
(Received 23 August 1991)

Wave propagation in one-dimensional disordered structures is studied via a variable R, which represents the quantum-mechanical resistance to electronic transport, or the ratio of total-energy density to power flux for electromagnetic propagation. R is formulated by three distinct methods: (1) differential equation, (2) perturbation series, (3) state-variable representation. An explicit solution for R is obtained in the limit of small fluctuations of the potential (or dielectric constant). The configurational-average state vector is determined in the second cumulant approximation for stationary processes, and an explicit expression is given for the average $\langle R \rangle$ for any potential distribution. The electron localization length exhibits a minimum as a function of the correlation length of the random potential. The *exact* expression for $\langle R \rangle$ is obtained for a white-noise distribution of potentials (or dielectric constants), both from the perturbation series, and from the state-variable formulation.

I. INTRODUCTION

Electronic conduction in one-dimensional ordered and disordered solids commands great interest due to current advances in microstructure technology[1] and because it forms the basis for the investigation of complex phenomena, such as asymmetric magnetoresistance,[2] quantum Hall effect in two-dimensional systems,[3–5] universal conductance fluctuations in small metallic structures,[6–8] and electron localization. The Schrödinger equation, which appears in the treatment of the above-mentioned problems is mathematically equivalent to the wave equation which is used to describe light propagation and photon localization in dielectric media with a random, especially layered, refractive index.[9,10]

A number of theoretical papers dealing with these topics have appeared, using seemingly unrelated mathematical methods, such as invariant imbedding, stochastic differential equations, etc., deriving "exact" results in different approximations. It is the purpose of this work to present the interrelation between the different mathematical methods, and to derive additional results. To this end, we introduce a variable R, which plays the role of normalized resistance to electronic conduction, or the ratio of the energy density to power flux in the analogous electromagnetic wave propagation problem in a dielectric. We shall use different approaches: (1) Derive a linear, third-order differential equation for R when the potential (dielectric constant) $V(x)$ is arbitrary. (2) Develop a perturbation-theoretical series for R using the strength of the random part of $V(x)$ as the perturbation parameter. (3) Study the spatial evolution of a three-component state vector formed by bilinear combinations of the wave function. The state variable formulation is particularly well suited for one-dimensional scattering of both particle and electromagnetic waves. Following these developments we will use the theory of stochastic differential equations to calculate the averages of R in those cases where this is possible by analytic—as contrasted with numerical—methods.

Several explicit solutions are found for R. In particular, the exact expression for the average value $\langle R \rangle$ is developed for a white-noise distribution of $V(x)$. Comparison of our expression for $\langle R \rangle$ with previously published results clearly shows that approximations are, in fact, always present in prior work, contrary to claims of exactness. We will take the liberty of commenting on the results of various authors, in particular with respect to the physical meaning of their approximations, and concerning the validity of their results. Since these comments are extensive, they are reserved for Sec. VII. Finally, a relationship is found between the electron localization length and the correlation length of the random potential.

The starting point of the electronic conduction calculations is the single-channel Landauer resistance formula for R.[11,12] Single channel means that we have a one-dimensional potential $V(x)$, across which electrons are transmitted. This formula has been derived in many different ways, be it for electrons of a fixed wave number k_0, or for a Fermi distribution of electrons at zero and finite temperatures.[13] Generalizations include multiple channels[14] and conductors of variable cross section.[15] Calculations have also included an external electric field along with the one-dimensional potential $V(x)$.[16–18] In this case a deviation from the Landauer formula arises. However, this effect is very small for the usual range of parameters, and therefore does not invalidate the Landauer formula. The resistance formula is the same for monoenergetic electrons with energy equal to the Fermi energy E_F, and for a Fermi distribution of electrons. Therefore, we consider only the former case: k_0 will denote the Fermi wave vector.

II. BASIC SCATTERING FORMULAS

Electronic conduction

In a stationary state, the electron wave function $\psi(x)$ in a potential $V(x)$ satisfies the Schrödinger equation

$$\psi''(x)+k_0^2[1-U(x)]\psi(x)=0, \quad \psi''\equiv\frac{d^2\psi}{dx^2}, \tag{2.1}$$

where

$$U(x)=V(x)/E, \quad E=\frac{k_0^2\hbar^2}{2m}.$$

Since in papers dealing with invariant imbedding[19] the electrons are always impinging from the right on $V(x)$, we adopt the same convention, as illustrated in Fig. 1. Noting from Fig. 1 that $V(x)=0$ outside $(0,l)$ and that the incoming beam has amplitude 1, the solution of (2.1) is

$$\psi(x)=t(l)e^{-ik_0x}, \quad x<0 \tag{2.2a}$$

and

$$\psi(x)=e^{-ik_0(x-l)}+r(l)e^{ik_0(x-l)}, \quad x>l, \tag{2.2b}$$

where $r(l)$ and $t(l)$ are the complex reflection and transmission amplitudes. They are functionals of $U(x)$, but for simplicity we indicate only their dependence on l.

The electronic current density J_e is

$$J_e=e\hbar m^{-1}J,$$

where the particle current density J is defined by

$$J=\mathrm{Im}[\psi^*(x)\psi'(x)]. \tag{2.3}$$

Since $U(x)$ is real, it follows from (2.1) and (2.3) that $dJ/dx=0$, i.e., J is independent of x. Evaluating (2.3) at $x=0$ yields

$$J(l)\equiv J=-k_0|\psi(0)|^2=-k_0|t(l)|^2 \text{ for all } x. \tag{2.4}$$

Although J is independent of x, the reader is reminded by (2.4) that it depends on l. This point plays an important role in defining appropriate state variables to handle stochastic problems in Sec. VI.

From (2.2) it follows that the complex reflection coefficient is given by

$$r(l)=[k_0-iQ(l)][k_0+iQ(l)]^{-1}, \quad Q=\frac{d\ln\psi}{dx}. \tag{2.5}$$

Since J is conserved, one may use (2.3)–(2.5) to show that

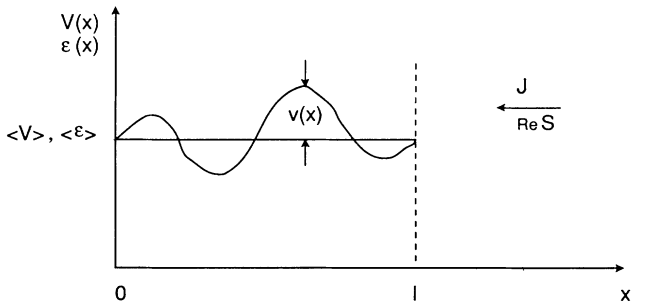

FIG. 1. The random potential $V(x)$ or dielectric constant $\epsilon(x)$ plotted as a function of x in the region $(0,l)$. V_0 and ϵ_0 are the averages. $v(x)$ is the deviation of the potential from its average. The arrow indicates the direction of the incident current **J** of electrons, or of the power flux **ReS** of the electromagnetic field.

$|r|^2+|t|^2=1$. As shown by Landauer, the physical resistance is $R_p(l)=(2e^2)^{-1}hR(l)$, with the unitless resistance $R(l)$ given by

$$R(l)=\left|\frac{r}{t}\right|^2=\frac{|r(l)|^2}{1-|r(l)|^2}=\frac{1}{|t(l)|^2}-1. \tag{2.6}$$

It should be noted that some authors define a normalized conductance $G=|t|^2=(R+1)^{-1}$. A detailed discussion of this point is given by Landauer.[12] Since for continuous potentials of zero length one has $r(0)=0$, the boundary condition for R is

$$R(0)=0. \tag{2.7}$$

Using (2.3), (2.5), and (2.6) one obtains

$$R(l)=\frac{|k_0\psi(l)-i\psi'(l)|^2}{4k_0|J(l)|}, \tag{2.8}$$

or, using (2.4),

$$R(l)=\tfrac{1}{2}\rho(l)-\tfrac{1}{2}, \tag{2.9}$$

where

$$2\rho(l)=\left|\frac{\psi(l)}{\psi(0)}\right|^2+\left|\frac{\psi'(l)}{k_0\psi(0)}\right|^2, \quad \rho(0)=1.$$

The expression (2.8) will be used in Sec. IV to formulate a perturbation series for R; whereas the latter form (2.9) is better suited for the state variable representation in Sec. V.

For a constant potential U_0, the resistance is

$$R_0(l)=\frac{1}{4}\frac{U_0^2}{1-U_0}\sin^2(k_0l\sqrt{1-U_0}), \tag{2.10}$$

which is also valid for $U_0>1$, where the argument of sin becomes imaginary.

Electromagnetic wave propagation

The mathematical formulation of electromagnetic energy transport through an inhomogeneous dielectric medium with dielectric constant $\epsilon(x)$ is analogous to the formulation of electronic transport through a spatially varying potential $V(x)$. However, the physical meaning of $R(l)$, in particular the electromagnetic equivalent of (2.8), requires investigation.

In Fig. 1, assume that an electromagnetic wave is propagating from right to left, as indicated by the arrow which denotes the direction of the Poynting vector. In the outside regions $x<0$ and $x>l$, the dielectric constant, magnetic susceptibility, and the speed of light are denoted by ϵ_0, μ_0, and c, respectively. The region $(0,l)$ is an inhomogeneous dielectric characterized by $\epsilon(x)$, and μ_0. The electric field is assumed to be plane polarized in the z direction, with sinusoidal time dependence of radian frequency ω. Using the complex phasor representation, the electric field is

$$\mathbf{E}(x,t)=\hat{\mathbf{z}}E(x)e^{-i\omega t}. \tag{2.11}$$

From Maxwell's equations it follows that the correspond-

ing magnetic field is

$$\mathbf{H}(x,t)=\hat{\mathbf{y}}H(x)e^{-i\omega t}\ , \tag{2.12}$$

where

$$E'(x)=-i\mu_0\omega H(x),\quad H'(x)=-i\omega\epsilon(x)E(x)\ . \tag{2.13}$$

Here, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ denote unit vectors in the y and z direction, respectively.

The wave equation for $E(x)$ is

$$E''(x)+k_0^2[1-U(x)]E(x)=0\ , \tag{2.14}$$

with $k_0=\sqrt{\mu_0\epsilon_0}\omega=\omega/c$, and $U(x)=[\epsilon_0-\epsilon(x)]/\epsilon_0$.

The complex Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}^*$ is given by

$$\mathbf{S}=S(x)\hat{\mathbf{x}}\ \text{ with }\ S(x)=-E(x)H^*(x)\ . \tag{2.15}$$

Comparing (2.14) with the Schrödinger equation (2.1) it is evident that electromagnetic propagation may be characterized by the transformation $\psi\rightarrow E$ in expression (2.8) for R. Using (2.13) and (2.15) leads to the form

$$R(l)=\frac{c}{|S_r(l)|}|\sqrt{\epsilon_0}E(l)-\sqrt{\mu_0}H(l)|^2\ , \tag{2.16}$$

where $S_r(l)=\mathrm{Re}S(x)<0$ is the power flux in the negative x direction. It follows from (2.16) and the continuity of E and H that $\rho(l)=2R(l)+1$ is related to the total energy density $\mathcal{E}(x)$ at $x=l^+$ by

$$\rho(l)=\frac{c}{|S_r(l)|}\mathcal{E}(l^+)\ . \tag{2.17}$$

Although the power flux $S_r(l)$ is independent of x, it is a functional of $\epsilon(x)$ in the region $(0,l)$.

III. INVARIANT IMBEDDING: DIFFERENTIAL EQUATIONS

The main objectives in this section are the development of a linear differential equation for the resistance $R(l)$, and the subsequent study of approximate solutions. A nonlinear Riccati equation for the complex reflection coefficient $r(l)$ has been studied by several authors.[20–23] The method used to develop these differential equations is referred to as "invariant imbedding."[19]

Since the reflection coefficient (or the resistance) is a global property of the sample, it is not defined for interior points x in $(0,l)$, but only for the whole length l. The essence of the invariance imbedding method is to consider not x but l as the new *independent* variable. Hence, even though the potential $U(x)$ is defined in an interval $(0,L)$, in the invariant imbedding method one adopts the convention that $U(x)=0$ for $x>l$, with $0<l<L$. Using this convention, either $r(l)$ or $R(l)$ can be introduced as a *dependent* variable, which, by virtue of (2.5) or (2.9) is related to $\psi(l)$, $\psi'(l)$, and $\psi(0)$. The advantage of such a variable is that it obeys only one boundary condition, namely, at $l=0$. This boundary condition follows from the physical requirement that $r(0)=R(0)=0$, and fixes the logarithmic derivative $Q(0)=-ik_0$.

From (2.1) and (2.5) it follows that the reflection coefficient satisfies the Riccati equation

$$2ik_0^{-1}r'(l)=U(l)[1+r^2(l)]-2[2-U(l)]r(l)\ . \tag{3.1a}$$

Since (3.1a) has been studied extensively in Refs. 18–20 and 31 we will not attempt to find solutions here. However, we do point out that a Riccati equation of simpler form is obtained by the transformation (2.5), yielding

$$Q'(l)+Q^2(l)=-k_0^2[1-U(l)]\ , \tag{3.1b}$$

with boundary condition $Q(0)=-ik_0$.

A homogeneous linear differential equation for $\rho(l)=2R(l)+1$ is obtained by repeated differentiation of (2.9), using (2.1) to eliminate $\psi''(l)$. The result is

$$\rho'''-\frac{4-3U}{2-U}w\rho''+\left[4k_0^2(1-U)+2\left[\frac{1-U}{2-U}\right]w^2-w'\right]\rho'-\frac{2k_0^2U^2}{2-U}w\rho=0\ , \tag{3.2}$$

with $\rho=\rho(l)$, $U=U(l)$, and $w=U'(l)/U(l)$. The boundary conditions which follow directly from (2.1) and (2.7) are

$$\rho(0)=1,\quad \rho'(0)=0,\quad \rho''(0)=k_0^2U^2(0^+)\ , \tag{3.3}$$

where $U(0^+)$ means the limiting value of the potential approaching zero from the positive side.[24] We exclude the case where a δ-function potential is placed at $l=0$, since that would contradict the condition $R(0)=0$.

We write the potential as the sum of a constant part U_0 and a fluctuating part $u(x)$. To obtain a solution for $|u(x)|\ll 1$ with $U_0=0$, we set

$$\rho'(l)=u(l)g(l) \tag{3.4}$$

and neglect $U(x)=u(x)$ compared to 1 in (3.2). Most terms containing derivatives of $U(x)$ then cancel out and we obtain for $g(l)$ the integro-differential equation

$$g''(l)+4k_0^2g(l)=k_0^2u'(l)\left[\int_0^l u(x)g(x)dx+1\right]\ . \tag{3.5}$$

Neglecting the integral, which is justified a *posteriori* by the smallness of $g(x)$, the solution is

$$g(l)\simeq k_0^2\int_0^l u(x)\cos[2k_0(l-x)]dx\ . \tag{3.6}$$

Using (3.6) in (3.4) we obtain for $R(l)$ the expression

$$R(l)=\tfrac{1}{4}k_0^2\int_0^l dx\int_0^l dy\ u(x)u(y)\cos 2k_0(x-y)\ , \tag{3.7}$$

which is the same as the result obtained from the first-order approximation of the perturbation series developed in Sec. IV. Any approximation based on the smallness of the fluctuations of the potential must reduce to (3.7), which will be used in Sec. VII as a benchmark to test results of different authors.

The case when the potential nearly equals twice the electron kinetic energy also lends itself to an approximate solution, because the coefficients of (3.2) become singular for $U=2$. Setting $U(x)=2+u(x)$ and neglecting $u(x)$ compared to unity [except, of course in $U'(x)$], we obtain

$$\rho''+\tfrac{1}{2}u'\rho'-4k_0^2\rho=0\ . \tag{3.8}$$

The solution of this equation yields

$$R(l)=\sinh^2\tfrac{1}{2}k_0 l[2+\bar{u}(l)]\ ,$$
$$\bar{u}(l)=\frac{1}{l}\int_0^l dx\, u(x)\ll 2\ , \tag{3.9}$$

in agreement with Ref. 20. To understand why $U_0=2$ is special, consider the parameter $K(U_0,u)\equiv k_0^2[1-U_0-u(x)]$, which characterizes solutions of the Schrödinger equation. Since $K(2,u)=-K(0,-u)$, the special solution for the case $U_0=0$ and the fluctuating part equal to $-u(x)$ is the same as the solution for $U_0=2$ and $u(x)$ with k_0 replaced by ik_0.

Equations (3.7) and (3.9) are the only approximate analytical solutions of (3.2) that we could find. Next, we consider a perturbation series solution for the resistance.

IV. PERTURBATION SERIES

We assume in this section that the normalized potential $U(x)=V(x)/E$ consists of a constant part U_0, plus an arbitrarily varying part $\mu u(x)$, such that $|\mu u(x)|<|U_0|$, where μ is introduced to identify the order of the perturbation. The Schrödinger equation (2.1) then may be written as

$$\psi''(x)+k^2\psi(x)=\mu k_0^2 u(x)\psi(x),\quad k^2=k_0^2(1-U_0)\ . \tag{4.1}$$

We seek a series solution of (4.1) in the form

$$\psi(x)=\sum_{n=0}^{\infty}\mu^n\psi_n(x)\ . \tag{4.2}$$

Substitution of (4.2) into (4.1) leads to

$$\psi_n(x)=\left[\frac{\mu k_0^2}{k}\right]^n\hat{P}_{n+1}(x,x_1)\psi_0(x_1)\ . \tag{4.3}$$

The function $\psi_0(x)$ is the general solution of the homogeneous part of (4.1), i.e., a linear combination of $\sin kx$ and $\cos kx$, and $\hat{P}_n(x,x_1)$ is an iterated integral operator defined by

$$\hat{P}_n(x,x_1)=:\prod_{i=1}^{n-1}\hat{I}_i(x_{i+1},x_i):\ ,\quad \hat{P}_0=1\ . \tag{4.4}$$

The symbol :: indicates ordering with i increasing from right to left, and $\hat{I}(x,y)$ is an integral operator defined for any function $f(y)$ by

$$\hat{I}(x,y)f(y)=\int_0^x dy\,\sin k(x-y)u(y)f(y)\ . \tag{4.5}$$

Resistance

Having obtained the solution $\psi(x)$ in series form, we turn now to the expression for the resistance $R(l)$ as a power series in μ. For this purpose we substitute (4.2) into (2.8), apply the boundary condition $R(0)=0$, and obtain the exact series expression

$$R(l)=\tfrac{1}{4}(1-U_0)^{-1}|U_0\sin kl+W(l)|^2 \tag{4.6}$$

with

$$W(l)=\sum_{n=1}^{\infty}(\mu k_0^2/k)^n\int_0^l dx_n\phi(x_n-l)u(x_n)\times\hat{P}_n(x_n,x_1)\phi(x_1)\ ,$$

where

$$\phi(x)=(k/k_0)\cos kx-i\sin kx\ .$$

When the potential is constant, $W(l)=0$ and (4.6) reduces to (2.10). Equation (4.6) allows, in principle, the calculation of the resistance by perturbation theory to any order in μ.

For the case $U_0=0$, $R(l)$ reduces to the simpler form

$$R(l)=\tfrac{1}{4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}R_{mn}(l)\ ,$$
$$R_{mn}(l)=(\mu k_0)^{m+n}\int_0^l dx_n\int_0^l dx_m' u(x_n)\hat{P}_n(x_n,x_1)u(x_m')\hat{P}_m(x_m',x_1')\cos k_0(x_n-x_m'+x_1-x_1')\ . \tag{4.7}$$

For an arbitrary potential, the calculation of (4.7) to first order yields (3.7), which was obtained from the differential equation for $\rho(l)$.

Of special interest are ensemble averaged results which may be obtained for random potentials. To calculate the average resistance $\langle R(l)\rangle$ for an ensemble of potentials, Eqs. (4.6) or (4.7) with (4.4) shows that one has to calculate certain integrals of the correlation functions of the potential. The correlation functions are defined by

$$C_{nm}=\langle u(x_n)u(x_{n-1})\cdots u(x_0)u(x_m')\times u(x_{m-1}')\cdots u(x_0')\rangle\ , \tag{4.8}$$

where $\langle\cdots\rangle$ denotes the average over the probability distribution of the potentials in the ensemble, and $x_0,\ldots,x_n,x_0',\ldots,x_m'$ are a set of $n+m+2$ points on the x axis.

For an arbitrary probability distribution of potentials the average resistance $\langle R\rangle$ cannot be found exactly, and a reasonable truncation of (4.6) is sought. After considerable algebra, the complete second-order approximation for $\langle R\rangle$ is found to be

$$\langle R(l)\rangle=R_0(l)+\frac{1}{8}\left[\frac{k_0U_0}{1-U_0}\right]^2\times\int_0^l dx\int_0^x dy\,\langle u(x)u(x-y)\rangle g(x,y)\ , \tag{4.9}$$

where $R_0(l)$ is given by (2.10), and

$$g(x,y)=\cos 2kl+\left[1-\frac{2}{U_0}\right]\cos 2kx+\left[1-\frac{2}{U_0}\right]^2\cos 2ky+\left[1-\frac{2}{U_0}\right]\cos 2k(l-x+y)+\tfrac{1}{2}\cos 2k(l-2x+y)-\tfrac{1}{2}\cos 2k(l-y)\ .$$

Equation (4.9) is valid for any pair-correlation function, and will be used in a subsequent paper to compare the accuracy of several methods of finding $\langle R \rangle$.

Gaussian white noise

An exact expression for $\langle R \rangle$ can be obtained for a "Gaussian white noise" distribution of potentials. In this case the correlation functions c_{nm} are easily evaluated. A Gaussian process is characterized by the property

$$\langle u(x_1)u(x_2)\cdots u(x_n)\rangle = \sum_{i,j} \langle u(x_i)u(x_j)\rangle \cdots \times \langle u(x_m)u(x_n)\rangle ,$$

where the sum is over all partitions of the set $x_1,\ldots,x_n$ into pairs. The white-noise attribute further specifies that

$$\langle u(x_i)u(x_j)\rangle = \zeta\delta(x_i - x_j) \quad \text{for all } i,j . \tag{4.10}$$

The detailed calculation shows that the argument-ordering property of the operators $\hat{P}_n$ and the asymmetric factor $\sin k(x_{i+1}-x_i)$ occurring in $\hat{P}_n$ eliminate all pairing in (4.7), except between $u(x_i)$ and $u(x_i')$, for any i. Thus the only contribution to c_{mn} arises from

$$C_{nn} = (2\zeta)^n \prod_{i=1}^{n} \delta(x_i - x_i') . \tag{4.11}$$

It follows from (4.7) and (4.11) that

$$\langle R(l)\rangle = \tfrac{1}{2} \sum_{n=1}^{\infty} (k_0^2\zeta)^n \int_0^l dx_n h_n(x_n) , \tag{4.12}$$

where $h_n(x_n)$ is given by the recursion relation

$$h_n(x_n) = \int_0^{x_n} dx_{n-1}[1-\cos 2k_0(x_n - x_{n-1})]h_{n-1}(x_{n-1}) ,$$

with initial value $h_1(x)=1$. When the average resistance $\langle R(l)\rangle$, given by (4.12), is differentiated with respect to l, the following differential equation is obtained:

$$\langle R(l)\rangle''' + 4k_0^2\langle R(l)\rangle' - 4k_0^4\zeta\langle R(l)\rangle = 2k_0^4\zeta . \tag{4.13}$$

The boundary conditions are

$$\langle R(0)\rangle = 0, \quad \langle R(0)\rangle' = \tfrac{1}{2}k_0^2\zeta, \quad \langle R(0)\rangle'' = 0 . \tag{4.14}$$

Note that these boundary conditions for $\langle R \rangle$, which follow directly from (4.12), are different from those for R, developed in Sec. III. This is a consequence of the white-noise condition, (4.10), which places a potential at $l=0$, in contrast to the commonly used probability distributions of the position of potentials, which are such that the probability of finding a potential at $l=0$ is zero. The solution of the differential equation (4.13) may be obtained in closed form, but we delay its presentation until Sec. VI where the complete solution of the white-noise problem is given for the general case with $U_0 \neq 0$.

V. THE STATE-VARIABLE METHOD

Starting from an nth-order differential equation one may consider the original function and its derivatives as the components of an n-dimensional vector. The resulting first-order vector differential equation is equivalent to the original scalar differential equation. In linear system theory the n components of the vector are called the state variables. The introduction of state variables, in the present context, will allow a systematic treatment of stochastic processes.

As state variables one could use the complex functions ψ and ψ'; however, to determine a physical characteristic of the system, such as R, it is more direct to introduce real, bilinear products of these primary state variables and their complex conjugates ψ^*, ψ'^*. These will represent the components of vectors in the direct product space of the primary state variables and their complex conjugates. An elegant geometrical description of a particular state-variable representation of the Schrödinger equation, and a subsequent analysis of moments of R in terms of state transition matrices belonging to different representations of the SO(2,1) group is given by Peres.[25,26] State variables have also been introduced by van Kampen in the analysis of the classical harmonic oscillator.[27]

For reasons that will become apparent in the following, we introduce the real, dimensionless state variables

$$z_1(x) = k_0\frac{|\psi|^2}{|J(l)|} , \quad z_2(x) = \frac{-\mathrm{Im}(\psi^*\hat{p}\psi)}{|J(l)|} , \quad z_3(x) = \frac{|\hat{p}\psi|^2}{k_0|J(l)|} , \tag{5.1}$$

where $\hat{p} = -i\partial/\partial x$. The current density $J(l) = \mathrm{Re}(\psi^*\hat{p}\psi) < 0$, also given by (2.3), is independent of x. However, it is important to note that $J(l)$ is a functional of the potential $V(x)$. Thus, $J(l)$ cannot be normalized to unity for each member of an ensemble when $V(x)$ is a stochastic variable, as assumed by Peres in Ref. 25. It is evident from (5.1) that $z_1(x)$ and $z_3(x)$ are proportional to the particle and kinetic-energy densities, respectively. Writing ψ in the modulus-phase form it follows that $z_2(x) = n'(x)/|2J(l)|$ is proportional to a "diffusion" current density.

The corresponding state variables for electromagnetic propagation are obtained from (5.1) by the transformation $\psi \to E$ and $|J(l)| \to \mu_0\omega S_r(ll)$, with $k_0 = \omega/c$. Explicitly, they are

$$z_1(x) = \frac{c}{|S_r(l)|}\epsilon_0|E(x)|^2, \quad z_2(x) = \frac{\mathrm{Im}S(x)}{|S_r(l)|} , \quad z_3(x) = \frac{c}{|S_r(l)|}\mu_0|H(x)|^2 , \tag{5.2}$$

where $S_r(l)$, the real power flux, is independent of x. Physically, z_1 and z_3 are, respectively, proportional to the electric- and magnetic-field energy densities, and z_2 is proportional to the reactive power that supplies the changes in the stored energy.

Differentiation of the variables in (5.1) and use of the Schrödinger equation (2.1) with $U(x) = U_0 + u(x)$ to eliminate second derivatives, yields the coupled set of linear, first-order differential equations

$$\mathbf{z}'(x) = k_0[\mathbf{A}_0 + u(x)\mathbf{A}_1]\mathbf{z}(x) , \tag{5.3}$$

where $\mathbf{z}(x)=[z_1(x),z_2(x),z_3(x)]^T$ is the state vector. The symbol T indicates transposition. The unitless, traceless, matrices $\mathbf{A}_0$ and $\mathbf{A}_1$ are defined by

$$\mathbf{A}_0=\begin{bmatrix}0 & 2 & 0\\ -q^2 & 0 & 1\\ 0 & -2q^2 & 0\end{bmatrix},\quad \mathbf{A}_1=\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}, \tag{5.4}$$

where $q=k/k_0=\sqrt{1-U_0}$.

Equation (5.3) is equivalent to the Schrödinger equation (2.1), or to the electromagnetic wave Eq. (2.14) if the state variables are defined by (5.2). It follows directly from the definitions (5.1) that the state variables satisfy the constraints

$$z_2^2(x)-z_1(x)z_3(x)+1=0\ ,\quad z_1(x)z_3(x)\geq 1 \text{ for all } x\ . \tag{5.5}$$

At an extremum point x_0 of $z_1(x)$, defined by $z_1'(x_0)=0$, it follows from the state-variable equation (5.3) that $z_2(x_0)=0$ and $z_3'(x_0)=0$, i.e., x_0 is also an extremum point for $z_3(x)$. At $x=x_0$, the equality holds in (5.5); thus $z_1(x_0)$ and $z_3(x_0)$ are opposite extrema. For electromagnetic wave propagation $z_1(x)z_3(x)\geq 1$ may be written as $2c(x)\sqrt{\mathcal{E}_E(x)\mathcal{E}_H(x)}\geq|S_r(l)|$, where $c(x)$ is the velocity of light in the dielectric, and $\mathcal{E}_E(x)$ and $\mathcal{E}_H(x)$ are the electric- and magnetic-energy densities, and $S_r(l)$ is the power flux.

For the following, it is advantageous to transform to an "interaction representation" by introducing a new state vector $\mathbf{v}$ whose derivative with respect to x is zero when $u(x)\equiv 0$. Accordingly, we define

$$\mathbf{v}(x)=\mathbf{M}(x)\mathbf{z}(x)\ , \tag{5.6}$$

subject to the requirement that

$$\mathbf{v}_0'(x)=\frac{d}{dx}[\mathbf{M}(x)\mathbf{z}_0(x)]=0\ , \tag{5.7}$$

where $\mathbf{z}_0(x)$ is the solution of (5.3) when $u(x)=0$. Using (5.3) in $\mathbf{v}_0'=0$, it follows that for $\mathbf{z}_0$ arbitrary, $\mathbf{M}(x)$ satisfies the differential equation $\mathbf{M}'(x)=-k_0\mathbf{M}(x)\mathbf{A}_0$, which has the solution

$$\mathbf{M}(x)=\exp(-k_0\mathbf{A}_0x)=\frac{1}{2}\begin{bmatrix}1+\cos 2kx & -2q^{-1}\sin 2kx & q^{-2}(1-\cos 2kx)\\ q\sin 2kx & 2\cos 2kx & -q^{-1}\sin 2kx\\ q^2(1-\cos 2kx) & 2q\sin 2kx & 1+\cos 2kx\end{bmatrix}. \tag{5.8}$$

Note the relation $\mathbf{M}^{-1}(x)=\mathbf{M}(-x)$. Taking the derivative of (5.6), and using (5.3) and (5.8) one obtains

$$\mathbf{v}'(x)=k_0u(x)\mathbf{M}(x)\mathbf{A}_1\mathbf{M}(-x)\mathbf{v}(x)\ . \tag{5.9}$$

The formal iterated solution of (5.9) is

$$\mathbf{v}(x)=\mathbf{\Phi}(x,x_0)\mathbf{v}(x_0)\ , \tag{5.10}$$

where the matricant $\mathbf{\Phi}$ is the ordered evolution operator [see explanation after (4.4)]

$$\mathbf{\Phi}(x,x_0)=:\exp\left[k_0\int_{x_0}^{x}dy\,u(y)\mathbf{M}(y)\mathbf{A}_1\mathbf{M}(-y)\right]:\ .$$

The solution for the state vector may now be expressed as

$$\mathbf{z}(x)=\mathbf{M}(-x)\mathbf{\Phi}(x,x_0)\mathbf{M}(x_0)\mathbf{z}(x_0)\ . \tag{5.11}$$

Scattering

The complete determination of the state vector requires the specification of the initial vector $\mathbf{z}(x_0)$. For the scattering problem, illustrated in Fig. 1, the boundary condition at $x_0=0$ follows directly from the definitions of z_1, z_2, and z_3 in (5.1). Using (2.2) and (2.4) one obtains the initial state vector

$$\mathbf{z}(0)=[1,0,1]^T\ . \tag{5.12}$$

The "resistance" R assumes a particularly simple form in terms of the state variables. Using (5.1) and (2.9), it follows

$$R(l)=\tfrac{1}{2}\rho(l)-\tfrac{1}{2}\ \text{ with }\ \rho(l)=\tfrac{1}{2}[z_1(l)+z_3(l)]\ . \tag{5.13}$$

It is evident that the boundary condition $R(0)=0$ is consistent with (5.12). An expression of the same form as (5.13) was obtained using the transfer-matrix method,[28] and a two-dimensional harmonic-oscillator analogy.[29,30]

In Sec. III we derived a linear third-order differential equation for $\rho(l)$ with the boundary conditions (3.3). Since $\rho(l)$ is a linear combination of state variables it is of interest to examine the compatibility of the boundary conditions (3.3) and those for the derivatives of $\mathbf{z}$. Differentiation of (5.3) allows decoupling of the state variables and leads third-order differential equations for each variable. Taking the derivative of (5.3) and using (5.12) yields the following boundary conditions for these differential equations:

$$\begin{aligned}&\mathbf{z}'(0)=k_0U(0)[0,1,0]^T\ ,\\ &\mathbf{z}''(0)=k_0U(0)[2k_0,U'(0)/U(0),2k_0[U(0)-1]]^T\ .\end{aligned} \tag{5.14}$$

The boundary conditions (3.3) follow immediately from (5.12) and (5.14).

VI. STOCHASTIC PROCESSES

We wish to treat a stochastic problem in which $U(x,\epsilon)$ is a random process. This means that there exists a class of potentials $U(x,\epsilon)$, one for each value of a random variable ϵ. The symbol $\langle\cdots\rangle$ will denote averaging over ϵ. We then consider the stochastic differential equation

$$\mathbf{z}'(x,\epsilon)=k_0[\mathbf{A}_0+u(x,\epsilon)\mathbf{A}_1]\mathbf{z}(x,\epsilon) \tag{6.1}$$

with $\mathbf{A}_0$ and $\mathbf{A}_1$ given by (5.4), and the state vector $\mathbf{z}$

defined by (5.1). In the corresponding Schrödinger equation the wave function $\psi(x)$ is replaced by $\psi(x,\epsilon)$, since it depends on the particular potential chosen from the class $U(x,\epsilon)$. It is known in the theory of stochastic differential equations that their treatment may be greatly simplified if the boundary condition, i.e., $\mathbf{z}(x_0,\epsilon)$, can be made statistically independent of ϵ. The solution of (6.1) for a particular choice of $U(x,\epsilon)$ is given by (5.11). Since $\mathbf{M}(x)$ is independent of the random part of the potential, averaging $\mathbf{z}(x,\epsilon)$ over ϵ in (5.11) and assuming $\mathbf{z}(x_0)$ independent of ϵ, yields

$$\langle \mathbf{z}(x)\rangle = \mathbf{M}(-x)\langle \mathbf{\Phi}(x,x_0)\rangle \mathbf{M}(x_0)\mathbf{z}(x_0) . \tag{6.2}$$

The particular form of the state vector of Sec. V was chosen so that $\mathbf{z}(0)$ would be independent of ϵ for the scattering problem of Fig. 1. In fact, the boundary condition (5.12) is $\mathbf{z}(x_0,\epsilon)=[1,0,1]^T$. Examination of (5.10) shows that, in general, the average of the matricant $\mathbf{\Phi}(x,x_0)$ cannot be evaluated in closed form. However, as shown by Kubo[31] and van Kampen,[27] a reasonable approximation can be obtained from a cumulant expansion of $\ln\langle \mathbf{\Phi}\rangle$.

Stationary processes in the second-order cumulant approximation

We consider the random process $U(x,\epsilon)$ with the properties $U(x,\epsilon)=U_0+u(x,\epsilon)$, $U_0=\langle U(x)\rangle$, i.e., $\langle u\rangle=0$. Stationary processes are defined by the property that their correlation function may be expressed as

$$\langle u(x)u(x-y)\rangle = \langle u^2\rangle f(y) \quad \text{for all } x \text{ and } y . \tag{6.3}$$

In most applications, $f(y)$ depends on a correlation length l_c, defined such that the value of the random potential at x_1 has negligible influence on its value at x_2 if $|x_1-x_2| \gg l_c$. For example, for dichotomic Markov and Ornstein-Uhlenbeck processes $f(y)$ has the form

$$f(y)=e^{-y/l_c} . \tag{6.4}$$

The advantage of a cumulant expansion is that successive terms in the expansion of $\ln\langle \mathbf{\Phi}(x,x_0)\rangle$ are all proportional to $(x-x_0)l_c^{n-1}$, whereas the nth term of the expansion of $\langle \mathbf{\Phi}(x,x_0)\rangle$ is proportional to $(x-x_0)^n$. This property renders the latter expansion useless for $x-x_0 > l_c$. In contrast, one may expect to obtain a reasonable approximation to $\langle \mathbf{z}(x)\rangle$ using only a second-order cumulant expansion. Since we assume $\langle u(x)\rangle=0$, the second cumulant equals the second moment of $U(x)$. For a Gaussian distribution of potentials all moments of $U(x)$ higher than the second moment vanish, in which case the second cumulant expansion is exact.[27] To second order in $u(x)$ the cumulant expansion is

$$\ln\langle \mathbf{\Phi}(x,x_0)\rangle = k_0^2\langle u^2\rangle \int_{x_0}^{x} dy \int_{x_0}^{y} dy' f(y-y')\mathbf{M}(y)\mathbf{A}_1\mathbf{M}(-y)\mathbf{M}(y')\mathbf{A}_1\mathbf{M}(-y') .$$

Using the group property of $\mathbf{M}(x)$, $\mathbf{M}(x)\mathbf{M}(y)=\mathbf{M}(y)\mathbf{M}(x)=\mathbf{M}(x+y)$, it follows that

$$\ln\langle \mathbf{\Phi}(x,x_0)\rangle = k_0\langle u^2\rangle \int_{x_0}^{x} dy\, \mathbf{M}(y)\mathbf{A}_1\mathbf{Q}(y-x_0)\mathbf{M}(-y) , \tag{6.5}$$

where the matrix $\underline{Q}(y-x_0)$ is defined by

$$\mathbf{Q}(y-x_0)=k_0\int_0^{y-x_0} d\tau f(\tau)\mathbf{M}(-\tau)\mathbf{A}_1\mathbf{M}(\tau) . \tag{6.6}$$

Substituting (6.5) into (6.2) yields the approximate average state vector

$$\langle \mathbf{z}(x)\rangle = \mathbf{M}(-x)\exp\left[k_0\langle u^2\rangle \int_{x_0}^{x} dy\, \mathbf{M}(y)\mathbf{A}_1\mathbf{Q}(y-x_0)\mathbf{M}(-y)\right]\mathbf{M}(x_0)\mathbf{z}(x_0) . \tag{6.7}$$

As noted by Kubo,[31] for many applications it is convenient to transform (6.7) into a differential equation. Differentiating (6.7), using (5.8) and the properties of $\mathbf{M}(x)$, one finds

$$\frac{d}{dx}\langle \mathbf{z}(x)\rangle = \mathbf{K}(x-x_0)\mathbf{z}(x_0) , \tag{6.8}$$

with

$$\mathbf{K}(x-x_0)=k_0[\,\mathbf{A}_0+\langle u^2\rangle \mathbf{A}_1\mathbf{Q}(x-x_0)] .$$

Here we invoke the assumption that the state vector $\mathbf{z}(x)$ is calculated at a point x such that $x-x_0 \gg l_c$. Since the largest contribution to the integral over y in (6.5) comes from the region $y-x_0 < l_c$, the upper limit in $\mathbf{Q}(x-x_0)$ may be replaced by infinity. Using (5.3) and (5.8) it follows from (6.6) that $\mathbf{Q}\equiv\mathbf{Q}(\infty)$ is the constant matrix

$$\mathbf{Q}=\frac{1}{2}\begin{bmatrix} q^{-1}S & -q^2(F-C) & 0 \\ \frac{1}{2}(F+C) & 0 & -\frac{1}{2}q^{-2}(F-C) \\ 0 & F+C & -q^{-1}S \end{bmatrix} , \tag{6.9}$$

where

$$q=k/k_0=\sqrt{1-U_0}$$

and

$$F=k_0\int_0^\infty d\tau f(\tau)\ ,\quad S=k_0\int_0^\infty d\tau f(\tau)\sin 2k\tau\ ,$$
$$C=k_0\int_0^\infty d\tau f(\tau)\cos 2k\tau\ . \tag{6.10}$$

Equation (6.9) is also valid for $U_0>1$.

Setting $\mathbf{K}=\mathbf{K}(\infty)$ constant, the solution of (6.8) is

$$\langle \mathbf{z}(x)\rangle = e^{\mathbf{K}(x-x_0)}\mathbf{z}(x_0)\ , \tag{6.11}$$

where

$$\mathbf{K}=k_0\begin{bmatrix} 0 & 2 & 0 \\ q^{-1}\langle u^2\rangle S-q^2 & -q^2\langle u^2\rangle(F-C) & 1 \\ \langle u^2\rangle(F+C) & -2q^2 & -q^{-2}\langle u^2\rangle(F-C)\end{bmatrix}.$$

The behavior of $\langle \mathbf{z}(x)\rangle$ is determined by the eigenvalues of $\mathbf{K}$, which are the roots λ_i of the characteristic equation

$$\left[\frac{\lambda}{k_0}\right]^3+2q^{-2}\langle u^2\rangle(F-C)\left[\frac{\lambda}{k_0}\right]^2+[q^{-4}\langle u^2\rangle^2(F-C)^2+4q^2-2q^{-1}\langle u^2\rangle S]\left[\frac{\lambda}{k_0}\right]$$
$$-2\langle u^2\rangle[2C+q^{-3}\langle u^2\rangle S(F-C)]=0\ . \tag{6.12}$$

Since (6.12) is the characteristic equation for $\langle \mathbf{z}\rangle$, it is also the characteristic equation for the average of any physical quantity which is a linear combination of the components of $\langle \mathbf{z}\rangle$, such as $\langle\rho\rangle$.

An important question is whether the state vector $\langle \mathbf{z}(x)\rangle$ is asymptotically stable for large x. For stability, i.e., no exponential growth of $\langle \mathbf{z}\rangle$ as $x\to\infty$, it is required that $\mathrm{Re}\lambda<0$ for $i=1,2,3$. The parameters which enter these stability requirements are the average squared amplitude $\langle u^2\rangle$ of the random part of the normalized potential $U(x)$, the average $U_0=\langle U(x)\rangle$ of the potential, and the sine and cosine transforms of the two-point correlation function $f(\tau)$ at the wave numbers 0, and $2k$ [see Eq. (6.10)]. General stability conditions, based on the Routh-Hurwitz criteria, are developed in Appendix A.

To linear order in $\langle u^2\rangle$ the roots of (6.12) are given by

$$\lambda_1=k_0(1-U_0)^{-1}\langle u^2\rangle C\ ,$$
$$\lambda_{2,3}=-\frac{k_0}{2}(1-U_0)^{-1}\langle u^2\rangle(2F-C) \tag{6.13}$$
$$\pm ik_0[2\sqrt{1-U_0}-\tfrac{1}{2}(1-U_0)^{-1}\langle u^2\rangle S]\ .$$

The state vector is asymptotically unstable for all $U_0=\langle V\rangle/E$. For $U_0<1$, the positive real root is $\lambda_1\sim\langle u^2\rangle$. For $U_0>1$, all roots are real and $\lambda_3\sim 2\sqrt{U_0-1}>0$. Thus the exponential behavior of $\langle \mathbf{z}\rangle$ is determined by fluctuations in the potential for $E>\langle V\rangle$ and by the average potential for $E<\langle V\rangle$.

Average resistance

We now develop a general explicit expression for the average state vector $\langle \mathbf{z}\rangle$ in terms of the eigenvalues of (6.12). From matrix function theory

$$e^{\mathbf{K}x}=g_1(x)\mathbf{I}+g_2(x)\mathbf{K}+g_3(x)\mathbf{K}^2\ , \tag{6.14}$$

where $g_i(x)$ $(i=1,2,3)$ are the components of a vector $g(x)$ defined by

$$\mathbf{\Lambda}\mathbf{g}(x)=\mathbf{h}(x)\ . \tag{6.15}$$

The rows of $\mathbf{\Lambda}$ are $\Lambda_i=[1,\lambda_i,\lambda_i^2]$, $i=1,2,3$ and $\mathbf{h}(x)=[e^{\lambda_1 x},e^{\lambda_2 x},e^{\lambda_3 x}]^T$. Substituting (6.14) into (6.11), and applying the scattering boundary condition (5.12), it follows that

$$\langle z_1(x)\rangle=g_1(x)+2(U_0+\alpha)g_3(x)\ ,$$
$$\langle z_2(x)\rangle=2(U_0+\alpha)g_2(x)+2[\gamma-(1+U_0+\alpha)\beta]g_3(x)\ , \tag{6.16}$$
$$\langle z_3(x)\rangle=g_1(x)+(\gamma-\beta)g_2(x)$$
$$+[(\beta-\gamma)\beta-2(1-U_0)(U_0+\alpha)]g_3(x)\ ,$$

where

$$\alpha=(1-U_0)^{-1/2}\langle u^2\rangle S\ ,$$
$$\beta=(1-U_0)^{-1}\langle u^2\rangle(F-C)\ , \tag{6.17}$$
$$\gamma=\langle u^2\rangle(F+C)\ .$$

Combining these results with (5.13), the average resistance is given by

$$\langle R(l)\rangle=c_{32}e^{\lambda_1 l}+c_{13}e^{\lambda_2 l}+c_{21}e^{\lambda_3 l}-\tfrac{1}{2}\ , \tag{6.18}$$

where

$$c_{mn}=\frac{1}{4D}(\lambda_m-\lambda_n)\{2\lambda_m\lambda_n+k_0(\beta-\gamma)(\lambda_m+\lambda_n)$$
$$+k_0^2[(\beta-\gamma)\beta+2U_0(U_0+\alpha)]\}\ , \tag{6.19}$$
$$D=(\lambda_3-\lambda_2)\lambda_3\lambda_2+(\lambda_1-\lambda_3)\lambda_1\lambda_3+(\lambda_2-\lambda_1)\lambda_1\lambda_2\ .$$

Equation (6.18) shows that the evaluation of $\langle R(l)\rangle$ requires only a knowledge of the pair-correlation functions of the potential (or dielectric constant) which occur in (6.17), and the roots of the third-order polynomial (6.12).

Equations (6.11) and (6.18), obtained in the second-order cumulant approximation, are asymptotically exact for distributions of $U(x,\epsilon)$ in the limit when l is much larger than the correlation length l_c. The asymptotic limit corresponds to a Gaussian white-noise distribution of

potentials, for which l_c is zero. For this case

$$\langle u^2\rangle f(\tau)\to\zeta\delta(\tau)\ , \tag{6.20}$$

and the characteristic equation (6.12) reduces to

$$(\lambda/k_0)^3+4(1-U_0)(\lambda/k_0)-2p=0,\quad p=2k_0\zeta\ . \tag{6.21}$$

Comparing (6.20) with (4.13), where $U_0=0$, it is evident that (6.21) is the characteristic equation for the average resistance $\langle R\rangle$. If the condition

$$[4(1-U_0)/3]^3+p^2<0 \tag{6.22}$$

is satisfied, all roots of (6.21) are real, and they, together with the coefficients c_{mn}, are given in Appendix B.

Insertion of the real roots and c_{mn} into (6.18) yields the final formula for $R(l)$. If (6.22) is not fulfilled, the roots λ_i are complex, and the final result for $R(l)$ is conveniently written in the form

$$\langle R(l)\rangle = Ae^{ak_0l}+e^{-(1/2)ak_0l}[B\cos(\sqrt{3}bk_0l/2) + C\sin(\sqrt{3}bk_0l/2)]-\tfrac{1}{2}\ . \tag{6.23}$$

The quantities A,B,C and a,b are expressed in Appendix B in terms of the electron energy and parameters of the potential.

To our knowledge, Eqs. (6.21)–(6.23) plus Appendix B is the first complete, exact solution for the average resistance in the white-noise case. For $U_0=0$, (6.23) is the solution of the differential equation (4.13) obtained from the perturbation approach, and it satisfies the derivative boundary conditions (4.14). In the weak correlation limit, $p\ll 1$, Eq. (6.23), with $U_0=0$, reduces to the expression

$$\langle R(l)\rangle=\tfrac{1}{2}(e^{k_0^2\zeta l}-1)\ . \tag{6.24}$$

This relation can also be obtained directly from the series (4.12) by dropping the oscillatory terms in all Q_n. This is justified, if l is much larger than the electron wavelength divided by 2π, since then the cosine oscillates many times in the integration intervals.

VII. DISCUSSION

The three methods developed in the text allow, in principle, the calculation of any physical observable connected to matter- or electromagnetic-wave propagation in one-dimensional inhomogeneous media. The most generally applicable of these methods is the perturbation series expansion for the wave function, which has been applied to the resistance. The homogeneous linear differential equation (3.2) also allows one to obtain a solution to any order of accuracy by iteration.

In most applications the local properties of the inhomogeneous medium are not known in detail, but statistical information is available. An example is a lossless optical fiber with a nonuniform index of refraction. For this type of medium the state-variable method developed in Secs. V and VI is the most appropriate one if the correlation length of the relevant property of the medium is small compared with the length of the sample. This method is again illustrated for the example of the electrical resistance, which is found as a function of the average, the standard deviation, as well as the two-point correlation function of the random potential (or dielectric constant) of the medium.

We will now discuss some results obtained by other authors. First, we consider some work done on the white-noise case. Heinrichs[32] obtained the result that the average resistance is proportional to $\exp[(\frac{1}{2}k_0\zeta+U_0)k_0l]$, valid for $k_0\zeta\ll 1$. However, all derivations in his work are valid only in the very special case $U_0=2$, i.e., the potential is constant and equal to twice the electron energy. It is clear that in this case the exponential is completely dominated by the term U_0k_0l, and the fluctuations of the potential are negligible. Our result, valid for any $U_0\neq 1$, when $k_0\zeta\ll 1$, is proportional to $\exp(\lambda_i l)$, where λ_i is the positive root in (B6). For the special case $U_0=1$, the average resistance is proportional to $\exp[(4k_0\zeta)^{1/3}k_0l]$.

Heinrichs also obtained a probability distribution for the resistance, which is different from the results of Abrikosov,[33] or Kumar,[23] or Mel'nikov,[34] or Papanicolau and Keller,[22] but since it is also based on a constant potential $U_0=2$, the results cannot be directly compared to the results of the other authors which refer to $U_0=0$ [see discussion following (3.9)].

Comparison of our results for the white-noise potential with those of Kumar[23] is particularly interesting. Kumar assumes that the logarithmic derivative of the potential, and not the potential itself, is of the Gaussian white-noise type. This, however, does not seem to be the reason that his result for the resistance does not contain oscillatory terms which are present in the exact solution. It is more likely that his assumption of a "circular ensemble" is generally not justified. This assumption states that the angle θ in the complex plane between the real and imaginary parts of the reflected amplitude has a uniform distribution between 0 and 2π for the respective ensemble of potentials. Kumar obtains the correct exponential behavior (6.24) (for $U_0=0$), which may indicate that when $k_0\zeta\ll 1$ the distribution of θ does approach the circular ensemble. In fact, results obtained for the phase angle of the wave function for a two-step random potential by Erdös and Domanski[35] show that the distribution of θ is far from circular, except in the limit stated.

Other authors also obtained expressions for the resistance in the white-noise case. Mel'nikov[34] has the same results as Papanicolau and Keller[22] with a prefactor $l^{-3/2}$ of the exponential. As (6.24) shows, this result is not correct. A correct result, with a constant prefactor was obtained by Abrikosov,[33] although he did not find an explicit relation between the mean square fluctuation of the potential and the exponent in (6.24), as we did.

All the above-mentioned authors solve the white-noise problem in the limit $k_0\zeta\ll 1$ and therefore obtain pure exponential dependence of the resistance on the length of the sample. It should be pointed out, however, that the complete solution (6.23) also contains oscillatory terms, which die out when the sample length l is such that $k_0^2\zeta l>1$. Furthermore, since (6.23) depends on two independent parameters, k_0l and $k_0\zeta$, the exact average

resistance cannot be derived from a one-parameter scaling theory of localization,[36] except in the weak-correlation limit $k_0\zeta \ll 1$ where (6.24) is valid.

Figures 2 and 3 illustrate the results. In Fig. 2 the average resistance $\langle R \rangle$ is plotted as a function of $k_0 l$. The electron wave number is k_0, and the length of the region of definition of the potential is l. The ratio of the average potential to the incident electron energy is $U_0 = 0.5$. Superimposed on this average potential is the Gaussian white noise with zero mean, of amplitude ζ as defined in (6.20). The curves are calculated for different values of the parameter $p = 2k_0\zeta$. For $p = 0$ the potential is constant, and the resistance is a periodic function of $k_0 l$, with a period of $\sqrt{2}\pi$. Complete transmission is observed when $k_0 l$ is an integer multiple of this period, as well known from the quantum mechanics of a square potential of width l and height $E/2$. As p increases, the resistance becomes more and more exponential as a function of $k_0 l$, and the period becomes shorter.

In Fig. 3, $\langle R \rangle$ is plotted as a function of $k_0 l$ for $p = 0.1$, and different ratios U_0. In accordance with (6.22), when $U_0 < 1 + 0.75p^{2/3} = 1.162$, the average resistance, given by (6.23), exhibits oscillation. In the pure white-noise limit $U_0 = 0$, the amplitude of the oscillations is indiscernible in the figure. As U_0 increases, the effect of the constant background potential is manifested by the appearance of oscillations superimposed on the sum of exponentials. The boundary between exponential-oscillatory behavior and pure exponential behavior is indicated by the dashed line.

We leave now the white-noise case and consider a general random potential. In this case there also appear os-

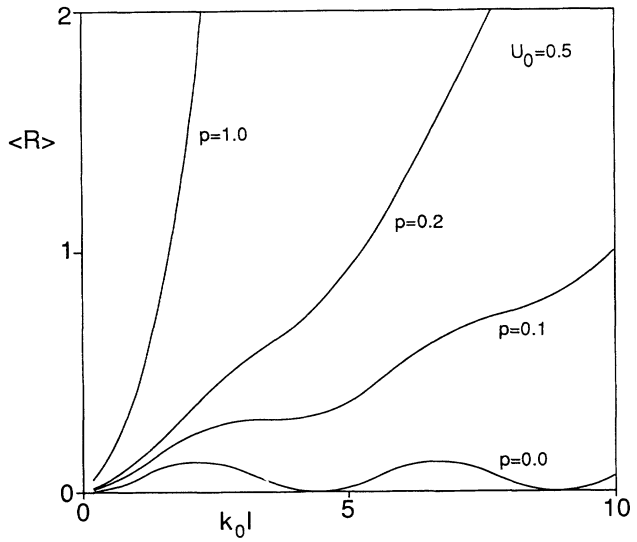

FIG. 2. The average resistance $\langle R \rangle$ of a Gaussian white-noise-type potential is plotted as a function of $k_0 l$ for different values of the relative strength of the potential fluctuations $p = 2k_0\zeta$, where ζ is the coefficient of white noise [cf. (6.20)]. The ratio of the average potential V_0 to the incident electron energy E is $U_0 = 0.5$. Here, l is the length of the region of definition of the potential, and k_0 is the wave number of the incident electron. The resistance is defined as the ratio of the intensities of the reflected and the transmitted beams.

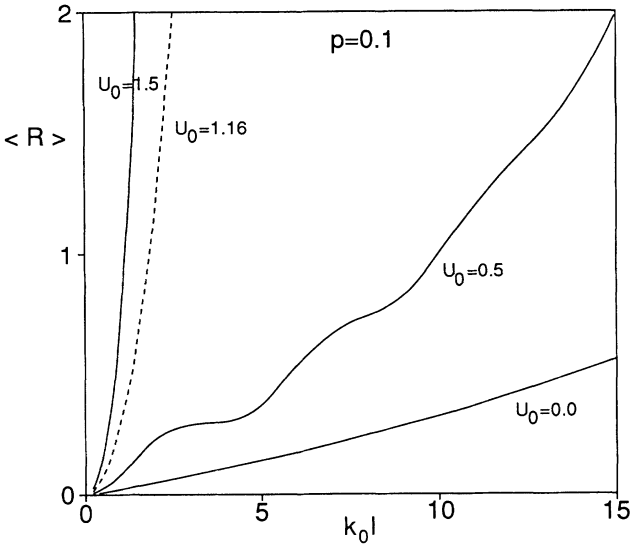

FIG. 3. $\langle R \rangle$ plotted as a function of U_0 for a fixed value of p. The parameters are defined as in Fig. 2. The dashed line is the boundary between the exponential-oscillatory region with $U_0 < 1.16$, and the pure exponential regions. Oscillations in the pure white-noise limit $U_0 = 0$ are indiscernible.

cillatory terms in the resistance, as a function of l, due to the phase coherence, as evident from (6.13). The wavelength of these oscillations is proportional to the electron wavelength λ. Authors[11,23,35] who use phase averaging do not find these oscillations. In two papers Papanicolaou[21] and Papanicolau and Keller[22] developed an approximate method of solution for a set of stochastic differential equations, and applied the results to obtain the mean power transmission coefficient $|T|^2$ of waves through a random medium. It is therefore of interest to compare their result with ours. For a potential of zero average, and under the condition $\langle u^2 \rangle k_0 l C \ll 1$, where C is the cosine transform of the correlation function $\langle u^2 \rangle f(\tau)$, they find the same result as follows from (3.7) of the differential equation approach, or from the perturbation result (4.7) approximated to first order,

$$|T|^2 = 1 - \tfrac{1}{2}\langle u^2 \rangle l k_0^2 \int_0^\infty d\tau f(\tau)\cos 2k\tau \ . \tag{7.1}$$

However, their asymptotic result for $\langle u^2 \rangle k_0 l C \gg 1$ disagrees both in the exponent and in the prefactor with our result. We find, in the same limit, using (6.13) with $U_0 = 0$

$$|T|^2 \sim \exp\left[-\langle u^2 \rangle l k_0^2 \int_0^\infty d\tau f(\tau)\cos 2k\tau \right] . \tag{7.2}$$

Our result (7.2) is corroborated by the result (6.23) for the white-noise potential, which has been obtained both by summing up the complete perturbation series (4.12), and by the cumulant expansion (6.5), which is exact for the white-noise case, because the higher-order terms vanish. It should be pointed out that Peres, Revzen, and Ron[25] derived, assuming $\langle u^2 \rangle \ll 1$, a result which one would obtain if one replaced $\cos 2k\tau$ by unity in (7.1). It is clear that this is justified only under the additional restriction $k_0 l_c \ll 1$.

Approximate analytic solutions for the state vector and observables, such as the resistance, can be obtained for a correlation length l_c less than the sample length l, i.e.,

$$l_c/l < 1 , \tag{7.3}$$

as discussed in Sec. VI.

To linear order in $\langle u^2 \rangle = \langle (v/E)^2 \rangle$, where $v(x)$ is the deviation of the potential from its average V_0, and E is the energy of the incident electrons, the average resistance grows exponentially with $\lambda_1 l$ for $E > V_0$, where

$$\lambda_1 = \frac{k_0^2 \langle (v/E)^2 \rangle}{1 - V_0/E} \int_0^\infty d\tau f(\tau) \cos 2k_0 \tau \sqrt{1 - V_0/E} . \tag{7.4}$$

The correlation function $\langle v(x)v(x-\tau) \rangle E^{-2} \equiv \langle (v/E)^2 \rangle f(\tau)$ is independent of x since we are dealing with a stationary random process. A sufficient condition for the validity of (7.4) is

$$(2\pi l_c/\lambda) \langle (v/E)^2 \rangle \ll 1 , \tag{7.5}$$

where λ is the electron wavelength. Comparing this expression with results obtained by scaling arguments,[36] we see that the quantity usually designated as the "localization length" is the inverse of λ_1. Hence the method of Sec. VI expresses the localization length in terms of the physical parameters of the problem.

The localization length $l_0 = \lambda_1^{-1}$ for stationary processes characterized by exponentially decaying correlation functions with $f(\tau) = \exp(-\tau/l_c)$ is

$$l_0 = \frac{2q^3 k_0^{-1}}{\langle (v/E)^2 \rangle} [2qk_0 l_c + (2qk_0 l_c)^{-1}] , \tag{7.6}$$

where

$$q = \sqrt{1 - V_0/E} .$$

The normalized localization length $k_0 l_0$ has a minimum value $4q^3/\langle (v/E)^2 \rangle$ at $2qk_0 l_c = 1$, which corresponds to a maximum of $\ln\langle R \rangle$ for a given value of l. The behavior of l_0 as a function of l_c and $\langle u^2 \rangle$ for $V_0 = 0$ is shown in Fig. 4. When the correlation length l_c is much larger than $k_0^{-1} = \lambda/2\pi$, the localization length l_0 increases linearly with l_c as the system becomes less random. In the opposite limit, when $\lambda/2\pi$ is much larger than l_c, the effect of the potential fluctuations is diminished because the electrons do not "sense" the fluctuations. In this case l_0 increases, inversely proportional to l_c. In the extreme limit $\langle u^2 \rangle \to \infty$ and $l_c \to 0$, with $\langle u^2 \rangle l_c = \zeta$ kept constant, (7.6) yields the white-noise result

$$k_0 l_0 = (1 - V_0/E)(k_0 \zeta)^{-1} . \tag{7.7}$$

For $V_0 = 0$, (7.7) agrees with (6.24).

There remains the question of the accuracy of the different methods for random processes other than white noise. This question is addressed in Ref. 38. Based on the methods developed in this paper, the probability distribution of the transmission characteristics of a dichotomic Markov filter is developed in the following publication.[39]

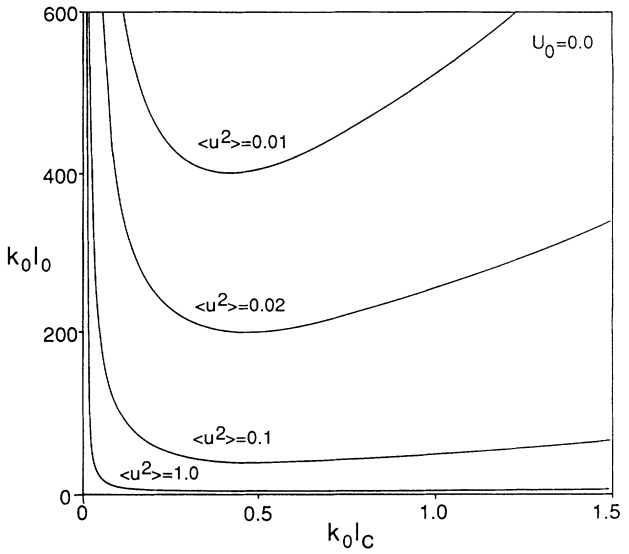

FIG. 4. The localization length l_0 plotted as a function of the correlation length l_c for exponentially decaying correlation functions for $U_0 = 0$. The scales are normalized by the electron wavelength divided by 2π. The different curves exhibit the dependence on the strength $\langle u^2 \rangle$ of the potential fluctuations. In the white-noise limit $\langle u^2 \rangle \to \infty$ and $l_c \to 0$ such that $\langle u^2 \rangle l_c = \zeta$. This limit would be represented by a point $k_0 l_0 = (k_0 \zeta)^{-1}$ on the vertical axis.

ACKNOWLEDGMENTS

We thank the Fondation Herbette of the University of Lausanne for making possible the stay of one of the authors (S.B.H.) in Lausanne. We thank Dr. Z. Domanski for useful discussions. The support of the Swiss National Science Foundation through Grant No. 20-28846.90 is also gratefully acknowledged.

APPENDIX A

In general, the asymptotic stability of (6.11) can be determined by applying the Routh-Hurwitz criterion[37] directly to the matrix **K**. In order for the state vector $\langle \mathbf{z} \rangle$ to exhibit no exponential growth for large x, the Routh-Hurwitz criterion lead to the constraints

$$\begin{aligned} &\beta_1 \equiv q^{-2}(F - C) > 0 , \\ &\beta_2 \equiv -q^{-3} \langle u^2 \rangle S(F - C) - 2C > 0 , \\ &\beta_3 \equiv [3 - 2q^{-3} \langle u^2 \rangle S + q^{-6} \langle u^2 \rangle^2 (F - C)^2](F - C) \\ &\qquad + F + C > 0 . \end{aligned} \tag{A1}$$

If $\beta_i < 0$ for any $i = 1,2,3$, then at least one root of the characteristic equation (6.12) has a positive real part, and consequently the system is asymptotically unstable.

Consider stationary processes characterized by correlation functions of the form $\langle u(x)u(x+\tau) \rangle = \langle u^2 \rangle f(\tau)$ with

$$f(\tau) = e^{-\tau/l_C} . \tag{A2}$$

Using (A2) in the definitions of F, S, and C in (6.10), (A1)

yields

$$\beta_1 = k_0 l_c [1 - U_0 + (2k_0 l_c)^{-2}]^{-1} > 0$$

and

$$\beta_2 = -\tfrac{1}{2}\beta_1 (k_0 l_c)^{-2} (1 + k_0^2 l_c \langle u^2 \rangle \beta_1) > 0 \ . \tag{A3}$$

If $\beta_1 < 0$, the system is unstable, and if $\beta_1 > 0$, it is evident that $\beta_2 < 0$, hence the system is also instable. Therefore (6.11) is always an asymptotically unstable solution for stationary processes with exponentially decaying correlation functions, such as, e.g., a dichotomic Markov process, or the Ornstein-Uhlenbeck process.[27]

APPENDIX B

For the Gaussian white-noise-type potential, if (6.22) is satisfied, the characteristic Eq. (6.21) has real roots

$$\lambda_{n+1} = 2k_0 [\tfrac{4}{3}(U_0 - 1)]^{1/2} \cos(\theta + 2n\pi/3) \ , \quad n = 0, 1, 2 \ , \tag{B1}$$

where

$$\theta = \tfrac{1}{3}\cos^{-1}\left[\frac{p}{[4(U_0 - 1)/3]^{3/2}} \right] . \tag{B2}$$

The coefficients in (6.18) are

$$c_{mn} = (1/4D)(\lambda_m - \lambda_n)[2\lambda_m \lambda_n - p(\lambda_m + \lambda_n) + 2k_0^2 U_0^2] \ . \tag{B3}$$

When (6.22) is not satisfied the roots of (6.21) are

$$\lambda_1 = ak_0, \quad \lambda_{2,3} = \tfrac{1}{2}(-a \pm i\sqrt{3}b)k_0 \ , \tag{B4}$$

where

$$\left.\begin{matrix} a \\ b \end{matrix}\right\} = (w + p)^{1/3} \mp (w - p)^{1/3} \ ,$$
$$w = \{[\tfrac{4}{3}(1 - U_0)]^3 + p^2\}^{1/2} \ . \tag{B5}$$

In the weak correlation limit, to linear order in $p = 2k_0\zeta$, and for $U_0 \neq 1$, the roots are

$$\lambda_1 = (1 - U_0)^{-1} k_0^2 \zeta \ ,$$
$$\lambda_{2,3} = -\tfrac{1}{2}(1 - U_0)^{-1} k_0^2 \zeta \pm i 2\sqrt{1 - U_0} k_0 \ . \tag{B6}$$

For the special case $U_0 = 1$, the roots are easily obtained from (B4) and (B5). In the complex case, noting that $b^2 = a^2 + 16(1 - U_0)/3$, the average resistance assumes the form given by (6.23) in the main text, where

$$A = [2a^2 + ap + 2(2 - U_0)^2]D \ ,$$
$$D = 4[4(1 - U_0) + 3a^2] \ ,$$
$$B = (4a^2 - ap - 2U_0^2)/D \ , \tag{B7}$$
$$C = [3a^2 p + 8(1 - U_0)p$$
$$- 2a(8 - 8U_0 + 3U_0^2)]/(\sqrt{3}bD) \ .$$

*Permanent address: Department of Electrical and Computer Engineering, University of California, Davis, CA 95616.

[1] L. P. Kouwenhoven *et al.*, Phys. Rev. Lett. **65**, 361 (1990).
[2] M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986).
[3] J. K. Jain, Phys. Rev. Lett. **60**, 2074 (1988).
[4] C. W. J. Beenakker, Phys. Rev. Lett. **64**, 216 (1990).
[5] R. L. Schult, H. W. Wyld, and D. G. Ravenhall, Phys. Rev. B **41**, 12 760 (1990).
[6] H. U. Baranger, A. D. Stone, and D. P. DiVicenzo, Phys. Rev. B **37**, 6521 (1988).
[7] P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B **35**, 1039 (1987).
[8] S. Iida, H. A. Weidenmüller, and J. A. Zuk, Phys. Rev. Lett. **64**, 583 (1990).
[9] S. Dutta Gupta and D. Shankar Ray, Phys. Rev. B **41**, 8047 (1990).
[10] S. John, Phys. Rev. Lett. **53**, 2169 (1984).
[11] R. Landauer, Philos. Mag. **21**, 863 (1970).
[12] R. Landauer, J. Phys. Condens. Matter **1**, 8099 (1989).
[13] M. Y. Azbel, A. Hartstein, and D. P. Di Vencenzo, Phys. Rev. Lett. **52**, 1641 (1984).
[14] M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B **31**, 6207 (1985).
[15] M. Yosefin and M. Kaveh, Phys. Rev. Lett. **64**, 2819 (1990).
[16] G. V. Vijayagovindan, A. M. Jayannavar, and N. Kumar, Phys. Rev. B **35**, 2029 (1987).
[17] P. Hu, Phys. Rev. B **35**, 4078 (1987).
[18] J. C. Flores, H. R. Joustin, and C. P. Enz, J. Phys. Condens. Matter **1**,123 (1989).
[19] R. Bellman and G. M. Wing, *An Introduction to Invariant Imbedding* (Wiley, New York, 1976).
[20] J. Heinrichs, Phys. Rev. B **33**, 5261 (1986).
[21] G. C. Papanicolaou, SIAM J. Appl. Math. **21**, 13 (1971).
[22] G. C. Papanicolau and J. B. Keller, SIAM J. Appl. Math. **21**, 287 (1971).
[23] N. Kumar, Phys. Rev. B **31**, 5513 (1985).
[24] The requirement $\rho'(0) = 0$ is a strict consequence of the Schrödinger equation and of the boundary condition (2.7) [see (5.14)]. We disagree with any other choice, such as, e.g., with $\rho'(0) = 1/l_e$, with l_e some "mean free path," as used by Heinrichs (Ref. 20).
[25] A. Peres, M. Revzen, and A. Ron, Phys. Rev. B **24**, 7463 (1981).
[26] A. Peres, J. Math. Phys. **24**, 1110 (1983).
[27] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1984).
[28] Paul Erdös and R. C. Herndon, Adv. Phys. **31**, 63 (1982).
[29] B. U. Felderhof, J. Stat. Phys. **43**, 267 (1986).
[30] B. U. Felderhof and G. W. Ford, J. Stat. Phys. **45**, 695 (1986).
[31] Ryogo Kubo, J. Math. Phys. **4**, 174 (1963).
[32] J. Heinrichs, Phys. Rev. B **36**, 2867 (1987).
[33] A. A. Abrikosov, Solid State Commun. **37**, 997 (1981).

[34]V. I. Mel'nikov, Fiz. Tverd. Tela (Leningrad) **22**, 2404 (1980) [Sov. Phys. Solid State **22**, 1398 (1980)].

[35]P. Erdös, and Z. Domanski, in *Analogies in Optics and Micro Electronics,* edited by W. Van Haeringen and D. Lenstra (Kluwer Academic, Dordrecht, 1990).

[36]P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fischer, Phys. Rev. B **22**, 3519 (1980).

[37]F. R. Gantmacher, *The Theory of Matrices* (Chelsea, New York, 1960), Vol. II.

[38]P. Erdös, Z. Domanski, and S. B. Haley, Physica B **175**, 25 (1991).

[39]Zbigniew Domanski, Paul Erdös, and Stephen B. Haley, Phys. Rev. B **45**, 8585 (1992) (following paper).