Superconducting transition temperature of proximity-contact superconducting-normal double layers

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The transition temperature of a proximity-contact superconducting-normal finite double layer is discussed in the clean limit. The finite reflection coefficient R at the interface is explicitly taken into account. The thickness dependence and the R dependence of T_c are discussed. It is shown that the transition temperature of a very thin double layer cannot be well described by Cooper's formula. The present results qualitatively agree with experiments.

I. INTRODUCTION

A number of studies have been reported in proximity-contact superconducting-normal (S-N) double-layer or multilayer systems.[1,2] One of the fundamental properties in such systems is the reduction of the superconducting transition temperature T_c as a function of the layer thickness. An early theoretical work on this subject was reported by Cooper.[3] He expressed the transition temperature rather intuitively in terms of a spatially averaged pairing interaction. In clean-layer systems, electrons always remember the effect by the S-N interface. The presence of the interface much complicates the problem and no satisfactory theory that can be quantitatively compared with experiments has been developed since Cooper's work. On the other hand, in dirty-layer systems the problem is much simplified by the impurity average that smears the effects by the sharp S-N interface. Theories of T_c in dirty-layer systems have been proposed by de Gennes and Guyon,[4] Werthamer[5] and have been applied to various systems.[6-8] Many experiments have been analyzed using the de Gennes–Werthamer theory,[4-6] which is valid when the layer width is sufficiently longer than the mean free path. However, recent experimental work[9,10] has been done sometimes in thinner layers and the result does not always agree with the de Gennes–Werthamer theory.

In a previous paper,[11] we proposed a formulation for the Green's function in a pure superconducting-normal double-layer system. In this paper, we apply the formulation to the superconducting transition temperature. Our model system is illustrated in Fig. 1. The N side is a pure normal metal with the width L_N and the S side is a superconducting metal with the width L_S. We assume that the boundaries at $z=-L_N, L_S$ are completely specular. The S-N interface is also assumed to have translational symmetry in the x-y plane and is characterized by the reflection coefficient R for the electrons at the Fermi level. Here, the Fermi momentum p_F is taken to have the same value in both the sides, but the Fermi velocity v_F^S in the S side is not always equal to v_F^N in the N side. This yields a finite reflection coefficient

$$R=\left|\frac{v_F^N-v_F^S}{v_F^N+v_F^S}\right|^2$$

even in the case of ideal contact. In actual systems, R is considered to be a parameter to be determined by experiments. Throughout this paper we use the units $\hbar=k_B=1$ and we use the scaled layer width and temperature

$$\hat{L}_{S,N}\equiv\frac{2\pi T_{cB}}{v_F^{S,N}}L_{S,N}\ ,$$

$$t_c\equiv T_c/T_{cB}\ ,$$

where T_c and T_{cB} are the transition temperature in the double-layer system and in the bulk superconductor, respectively. Now, our system is completely specified by three dimensionless parameters $\hat{L}_S$, $\hat{L}_N$, and R. We consider the superconducting transition temperature t_c as a function of these three parameters.

Using the general formulation for a double-layer system proposed by Ref. 11, we derive the gap equation to determine the transition temperature. The gap equation is a rather complicated integral equation which is difficult to have analytic solution in general cases. We solve it numerically by expanding the pair function in a Fourier series. In some limiting cases, however, we can treat it analytically. When $R\simeq 1$, we can use perturbation theory to obtain an explicit expression for t_c. We can show for general R that, when $\hat{L}_N\to\infty$, there exists a critical length $\hat{L}_{Sc}$ of the S side below which there is no superconducting phase transition (see Appendix A). This fact has been already suggested by Bar-Sagi[12] in the case of $R=0$ and by Ohmi *et al.*[13] We can also show that, when $\hat{L}_S$ is large, the pair function of the S side is well approximated by a single cosine function (see Appendix B).

When $\hat{L}_S$ is not so large, the single cosine function is not a good approximation for the pair function itself. As long as the transition temperature t_c is concerned, however, we find that the single cosine pair function gives sa-

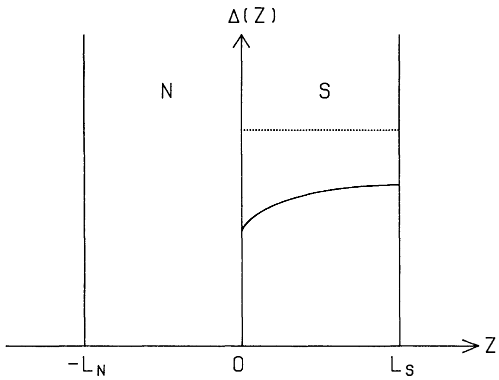

FIG. 1. Typical S-N geometry. A self-consistent pair function in the S-side is plotted by the solid curve when $\hat{L}_S=\hat{L}_N=1$, $R=0.5$. The dashed line represents a constant pair function of the corresponding bulk superconductor.

tisfactory results which agree very well with the numerical calculation. Using this approximate pair function, we propose some efficient formulas to calculate the transition temperature. In particular, we study sufficiently thin film double-layers ($\hat{L}_N,\hat{L}_S \ll 1$) and show that the transition temperature is not well described by Cooper's formula.[3] Bar-Sagi[12] has already treated the model of $\hat{L}_N\to\infty$ and $R=0$ by assuming the single cosine pair function. However, this gap equation does not keep the symmetry that the system should have and his results are different from ours in many respects.

In Sec. II, we derive the gap equation near the transition temperature. The transition temperature when $R\simeq 1$ is also discussed in this section. In Sec. III, we calculate the transition temperature numerically by expanding the pair function in a Fourier series. Typical behaviors of t_c as a function of $\hat{L}_S$, $\hat{L}_N$, and R are shown. In later sections, we assume that the pair function is given by a cosine function of z and derive approximate expressions of the transition temperature. In Sec. IV, we consider the case $\hat{L}_S t_c \gg 1$ and we discuss the transition temperature when $\hat{L}_N t_c \gg 1$ and $\hat{L}_N t_c \ll 1$. In Sec. V, we consider the case $\hat{L}_N\to\infty$. In this case the superconducting phase becomes unstable when $\hat{L}_S$ is shorter than some critical length $\hat{L}_{Sc}$. Here we discuss the R dependence of $\hat{L}_{Sc}$ and we also derive the expression of the transition temperature when $\hat{L}_S\simeq\hat{L}_{Sc}$. In Sec. VI, a very thin double layer ($\hat{L}_S t_c \ll 1$, $\hat{L}_N t_c \ll 1$) is treated. The transition temperature is shown to depend not only on the thickness ratio $\hat{L}_N/\hat{L}_S$ but also on $\hat{L}_S$, in contrast to the so-called Cooper limit[3] in which the transition temperature depends only on the thickness ratio. The final section is devoted to discussion.

II. FORMULATION

In this section, we derive the gap equation in the double-layer system by using the formulation proposed in a previous paper.[11] According to Ref. 11, the gap equation for the pair function Δ is given by

$$\Delta(\tilde{z})=vN(0)\pi T\langle g_1(\tilde{z})\rangle\ , \tag{2.1}$$

where we have used the scaled coordinate $\tilde{z}=z/L_S$ and the bracket denotes

$$\langle\ \cdots\ \rangle\equiv 2\sum_{\omega_n>0}\int_0^{\pi/2}d\theta\,\sin\theta(\ \cdots\ )\ .$$

Here v is a pairing coupling constant in the S side, $N(0)$ is the density of states of the S side at the Fermi surface, θ is the polar angle of the Fermi momentum and $g_1(\tilde{z})$ is the off-diagonal element of the quasiclassical Green's function. The above gap equation is valid at arbitrary temperatures. To study the transition temperature, we approximate all the quantities up to first order in $\Delta(z)$. In this approximation, the quasiclassical Green's function $g_1(\tilde{z})$ is given by

$$\begin{aligned} g_1(\tilde{z})=&\left[\coth\lambda-\frac{Q}{\cosh\lambda\,\sinh\lambda}\right]\cosh\lambda(\tilde{z}-1)\,\lambda\\ &\times\int_0^1 d\tilde{z}'\tilde{\Delta}\cosh\lambda(\tilde{z}'-1)\\ &+\sinh\lambda(\tilde{z}-1)\,\lambda\int_{\tilde{z}}^1 d\tilde{z}'\tilde{\Delta}\cosh\lambda(\tilde{z}'-1)\\ &+\cosh\lambda(\tilde{z}-1)\,\lambda\int_0^{\tilde{z}} d\tilde{z}'\tilde{\Delta}\sinh\lambda(\tilde{z}'-1)\ , \end{aligned} \tag{2.2}$$

where the factor Q is defined by

$$Q=\frac{(1-R)\tanh\kappa}{\sqrt{(1-R)^2(\tanh^2\lambda+\tanh^2\kappa)+2(1-R^2)\tanh\lambda\,\tanh\kappa+4R\,\tanh^2\lambda\,\tanh^2\kappa}} \tag{2.3}$$

with tanh the hyperbolic tangent function and

$$\tilde{\Delta}=\frac{\Delta(\tilde{z})}{\omega_n}\ ,\quad \lambda=\frac{2\omega_n L_S}{v_F^S\cos\theta}\ ,\quad \kappa=\frac{\omega_n 2L_N}{v_F^N\cos\theta}\ . \tag{2.4}$$

Here R is the reflection coefficient at the S-N interface.[11]

Except just at the S-N boundary $z=0$, the divergence in the ω_n sum on the right-hand side can be removed by using the conventional replacement

$$2\pi T\sum_{\omega_n>0}\frac{1}{\omega_n}\to\ln\left[\frac{2\gamma'\omega_D}{\pi T}\right]\ ,$$

where $\gamma'=e^{\gamma}=1.78\cdots$, γ is Euler's constant, and ω_D is the Debye cutoff frequency. Then the gap equation is rewritten as

$$\frac{\Delta(\tilde{z})}{\pi T_c}\ln t_c = -\langle \delta g_1(\tilde{z})\rangle , \tag{2.5}$$

where $t_c = T_c/T_{cB}$ is the scaled transition temperature and

$$\begin{aligned}\delta g_1(\tilde{z}) &= \frac{\Delta(\tilde{z})}{\omega_n} - g_1(\tilde{z}) \\ &= \int_0^{\tilde{z}} d\tilde{z}'\tilde{\Delta}'\cosh\lambda(\tilde{z}'-\tilde{z}) + \frac{\cosh\lambda\tilde{z}}{\sinh\lambda}\int_0^1 d\tilde{z}'\tilde{\Delta}'\sinh\lambda(\tilde{z}'-1) + \frac{Q}{\cosh\lambda\sinh\lambda}\cosh\lambda(\tilde{z}-1)\,\lambda\int_0^1 d\tilde{z}'\tilde{\Delta}\cosh\lambda(\tilde{z}'-1)\end{aligned} \tag{2.6}$$

with

$$\tilde{\Delta}' = \frac{d\tilde{\Delta}}{d\tilde{z}} . \tag{2.7}$$

When $R \simeq 1$, consequently, $t_c \simeq 1$, we can easily estimate t_c. Let us take the spatial average of Eq. (2.5):

$$\ln t_c = -\left\langle \frac{Q}{\varepsilon_n \cosh\lambda}\int_0^1 d\tilde{z}\frac{\Delta}{\overline{\Delta}}\cosh\lambda(\tilde{z}-1)\right\rangle , \tag{2.8}$$

where $\varepsilon_n = 2n+1$ and $\overline{\Delta}$ is the spatial average of $\Delta(\tilde{z})$

$$\overline{\Delta} = \int_0^1 d\tilde{z}\Delta(\tilde{z}) . \tag{2.9}$$

Up to first order in $1-R$, Q is given by

$$Q = \frac{1-R}{2}\frac{1}{\tanh\lambda} \tag{2.10}$$

and we can replace $\Delta(\tilde{z})$ by $\overline{\Delta}$ in the right-hand side of Eq. (2.8). It follows that

$$\ln t_c \simeq t_c - 1 = -\frac{1-R}{2}\left\langle \frac{1}{\varepsilon_n\lambda}\right\rangle = -\frac{\pi^2}{16}\frac{1-R}{\hat{L}_S t_c} . \tag{2.11}$$

Note that t_c for $R \simeq 1$ is a linear function of R and it does not depend on the normal metal thickness $\hat{L}_N$.

For general R, the integral equation (2.5) is too complicated to solve analytically. We solve it numerically in the next section and discuss several special limits in later sections.

III. NUMERICAL RESULTS

In this section, we solve Eq. (2.5) numerically and show the behaviors of the transition temperature as a function of $\hat{L}_S$, $\hat{L}_N$, and R. We expand the pair function in a Fourier series

$$\frac{\Delta(\tilde{z})}{\pi T_c} = a_0 + \sum_{p=1}^{p_{\max}} a_p\sqrt{2}\cos k_p(\tilde{z}-1) , \tag{3.1}$$

where $k_p = p\pi$. Then, the gap equation is reduced to the following eigenvalue equation:

$$a_p \ln t_c = -\sum_{q=0}^{p_{\max}} \langle b_{pq}\rangle a_q , \tag{3.2}$$

where

$$b_{00} = \frac{Q\tanh\lambda}{\varepsilon_n\lambda} , \tag{3.3}$$

$$b_{0p} = b_{p0} = (-1)^p\sqrt{2}\frac{Q\tanh\lambda}{\varepsilon_n}\frac{\lambda}{\lambda^2+k_p^2} \quad \text{for } p \geq 1 , \tag{3.4}$$

$$b_{pq} = (-1)^{p+q}2\frac{Q}{\varepsilon_n}\frac{\lambda^3\tanh\lambda}{(\lambda^2+k_p^2)(\lambda^2+k_q^2)} + \frac{1}{\varepsilon_n}\frac{k_p^2}{\lambda^2+k_p^2}\delta_{p,q} \quad \text{for } p,q \geq 1 . \tag{3.5}$$

Typical numerical results with $p_{\max}=20$ are shown in Figs. 2 and 3. When we take larger $p_{\max}$, we find that the pair function is modified only near $z=0$ to a more rapidly increasing function but that the change in the transition temperature itself is negligibly small. It implies that the transition temperature is governed by the global profile of the pair function.

In Fig. 2(a), we show the $\hat{L}_S$ dependence of t_c with $\hat{L}_N = 0.5$, 1, 2, ∞ for $R=0$ by solid curves and for $R=0.5$ by dashed curves. For comparison, a transition temperature obtained from Eq. (2.5) with spatially uniform pair function is also shown in Fig. 2(b). We see that when the spatial variation of Δ is neglected, the transition temperature is reduced by about 10% or more.

The lack of data at $\hat{L}_S t_c \lesssim 1$ does not mean the absence of superconducting state. Sufficiently long-time iterations give a nonzero solution of t_c, except for the case $\hat{L}_N \to \infty$. When $\hat{L}_N \to \infty$, the superconducting state is unstable if $\hat{L}_S$ is shorter than some critical length $\hat{L}_{Sc}$. The existence of $\hat{L}_{Sc}$ is discussed in more detail in Appendix A. When $\hat{L}_N \gtrsim 1$, the transition temperature is not sensitive to $\hat{L}_N$ because $\hat{L}_N$ appears only through $\tanh\kappa$ in Q. In Fig. 3, we show the R dependence of t_c for $\hat{L}_S = 1$. When $R \simeq 1$, t_c increases linearly and does not depend on $\hat{L}_N$ in accordance with Eq. (2.11).

IV. APPROXIMATE FORMULA WHEN $\hat{L}_S t_c \gg 1$

In this section, we consider a system with a sufficiently large width superconductor layer, i.e., $\hat{L}_S t_c \gg 1$, and we present approximate methods to determine t_c when $\hat{L}_N t_c \ll 1$ and when $\hat{L}_N t_c \gg 1$. When $\hat{L}_S t_c \gg 1$, the pair function is shown to be well described by a single cosine function (see Appendix B)

$$\Delta(\tilde{z}) = \Delta(1)\cos k(\tilde{z}-1) , \tag{4.1}$$

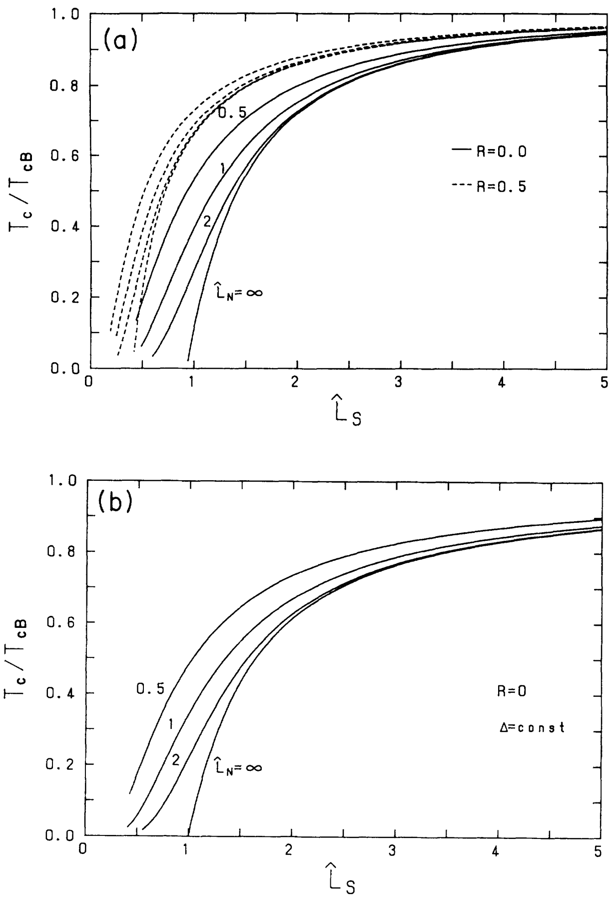

FIG. 2. Reduced transition temperature vs reduced layer thickness of the S side when $\hat{L}_N=0.5$, 1, 2, ∞ with (a) a self-consistent pair function, and (b) a constant pair function. Solid curves are for $R=0$ and dashed curves are for $R=0.5$.

where $\tilde{z}=z/L_S$. To determine t_c and k, we use the gap equation (2.5) at $\tilde{z}=1$ and the spatial average of the gap equation given by Eq. (2.8). Substituting Eq. (4.1) into Eqs. (2.5)–(2.8), we obtain

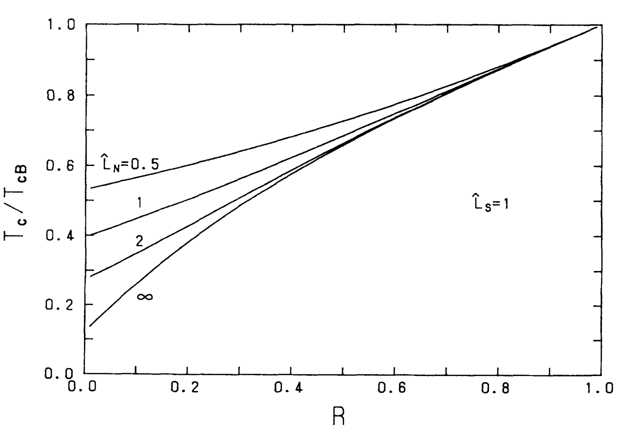

FIG. 3. Reduced transition temperature vs reflection coefficient R with $\hat{L}_S=1$, $\hat{L}_N=0.5$, 1, 2, ∞.

$$-\ln t_c=\left\langle \frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\left[\frac{k^2}{\lambda^2}-\frac{k\sin k}{\lambda\sinh\lambda}\right]\right\rangle +\left\langle \frac{Q}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\left[\frac{\cos k}{\cosh\lambda}+\frac{k}{\lambda}\frac{\sin k}{\sinh\lambda}\right]\right\rangle , \tag{4.2}$$

$$-\ln t_c=\left\langle \frac{Q}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\left[\frac{k}{\lambda}\tanh\lambda\cot k+\frac{k^2}{\lambda^2}\right]\right\rangle . \tag{4.3}$$

This set of equations is the starting point of the discussion in the rest of this article. When $\hat{L}_S t_c$ is not so large, the pair function no longer has a simple analytic form. Even when $\hat{L}_S t_c \sim 0$, however, the numerical results obtained from Eqs. (4.2) and (4.3) show very good agreement with the corresponding solutions by the Fourier series expansion discussed in Sec. III. It should be emphasized, however, that the pair function itself cannot be always given by a single cosine function.

Now, we proceed to further approximation. From the condition $\hat{L}_S t_c \gg 1$, the factor $\exp(-\lambda)$ in Eqs. (4.2) and (4.3) is negligibly small. Therefore, the function Q is reduced to

$$Q=\frac{1-R}{2}\frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} \tag{4.4}$$

and Eqs. (4.2) and (4.3) are rewritten as

$$-\ln t_c=f_2(\tilde{k}) , \tag{4.5}$$

$$-\ln t_c=\frac{1-R}{2}[F_1(\hat{L}_N t_c,R,\tilde{k})\cot k+F_2(\hat{L}_N t_c,R,\tilde{k})] \tag{4.6}$$

with

$$\tilde{k}=k/\hat{L}_S t_c . \tag{4.7}$$

In the above expressions, we have defined f_2, F_1, and F_2 by

$$\begin{aligned} f_2(\tilde{k})&=\left\langle \frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\frac{k^2}{\lambda^2}\right\rangle \\ &=\frac{1}{i\tilde{k}}\ln\left[\frac{\Gamma(1/2+i\tilde{k}/2)}{\Gamma(1/2-i\tilde{k}/2)}\right]-\psi(\tfrac{1}{2}) \\ &=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}\frac{2^{2n+1}-1}{2^{2n+1}}\zeta(2n+1)\tilde{k}^{2n} , \end{aligned} \tag{4.8}$$

$$F_1(\hat{L}_N t_c,R,\tilde{k})=\left\langle \frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}}\frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\frac{k}{\lambda}\right\rangle , \tag{4.9}$$

$$F_2(\hat{L}_N t_c,R,\tilde{k})=\left\langle \frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}}\frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\frac{k^2}{\lambda^2}\right\rangle . \tag{4.10}$$

If we know the behaviors of functions F_1 and F_2, we can estimate t_c and k from Eqs. (4.5) and (4.6). We consider separately the two special cases $\hat{L}_N t_c \gg 1$ and $\hat{L}_N t_c \ll 1$.

A. Case A: $\hat{L}_N t_c \gg 1$

When $\hat{L}_N t_c \gg 1$, the $\exp[-2\kappa]$ in F_1 and F_2 is also negligibly small, so F_1 and F_2 are reduced to

$$F_1 = f_1(\tilde{k}) \equiv \left\langle \frac{1}{\varepsilon_n} \frac{1}{1+k^2/\lambda^2} \frac{k}{\lambda} \right\rangle = \frac{1}{\tilde{k}} \ln\cosh\left[\frac{\pi}{2}\tilde{k}\right] , \tag{4.11}$$

$$F_2 = f_2(\tilde{k}) , \tag{4.12}$$

where f_2 is already given by Eq. (4.8). From Eqs. (4.5) and (4.6) together with Eqs. (4.11) and (4.12), we find a set of equations

$$\tan k = \tan(\hat{L}_S \tilde{k} e^{-f_2(\tilde{k})}) = \frac{1-R}{1+R} \frac{f_1(\tilde{k})}{f_2(\tilde{k})} , \tag{4.13}$$

$$t_c = e^{-f_2(\tilde{k})} \tag{4.14}$$

to determine the parameter $\tilde{k}$ and t_c. The transition temperature calculated using Eqs. (4.13) and (4.14) is shown in Fig. 4 by dashed curves. Solid curves are numerical results obtained by the Fourier series expansion in Sec. III. They show good agreement even when $\hat{L}_S t_c$ is not so large. The dash-dotted curves shall be discussed in a later section.

When $R \simeq 1$ and $\hat{L}_S$ is finite, we can expect $k \simeq 0$ and $\tilde{k} \simeq 0$, therefore, Eq. (4.13) is easily solved to give

$$f_2(\tilde{k}) = \tfrac{7}{12}\zeta(3)\tilde{k}^2 = \frac{\pi^2}{16} \frac{1-R}{\hat{L}_S t_c} \tag{4.15}$$

and the transition temperature from Eq. (4.14) agrees with Eq. (2.11) as is expected. On the other hand, when $\hat{L}_S$ is sufficiently large and $1-R$ is not so small, we expect that $\tilde{k} \simeq 0$ but $k \simeq \pi/2$. We find in this case that

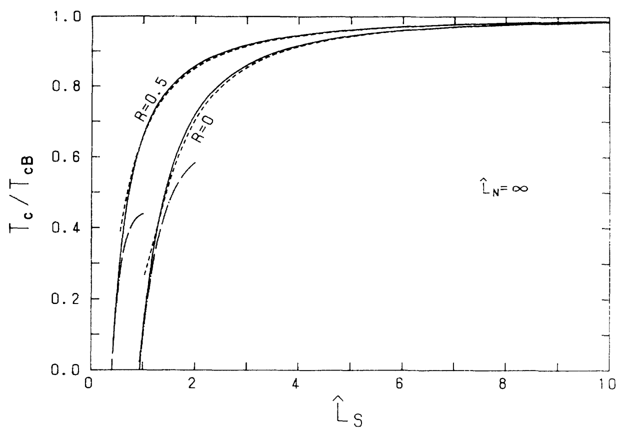

FIG. 4. Reduced transition temperature vs S-side layer thickness in the half-infinite system $\hat{L}_N = \infty$ with $R = 0$ and 0.5. Solid curves are obtained by Fourier series expansion method in Sec. III. Dashed curves represent approximate solutions given by Eq. (4.14). Dash-dotted curves are calculated from Eq. (5.18).

$$t_c - 1 = -\tfrac{7}{12}\zeta(3)\left[\frac{\pi}{2}\frac{1}{\hat{L}_S t_c}\right]^2 . \tag{4.16}$$

This agrees with Bar-Sagi[12] except for the numerical factor on the right-hand side. As can be seen in Fig. 4, however, this R independent asymptotic behavior can be observed when $\hat{L}_S \gtrsim 10$.

B. Case B: $\hat{L}_N t_c \ll 1$

Next we consider the case $\hat{L}_N t_c \ll 1$ and approximate F_1 and F_2 up to first order in $\hat{L}_N t_c$. Since, in this case, the parameter κ varies from very small value to infinity, we cannot simply expand $\exp[-2\kappa]$ in power series of κ. Let us first consider F_2. We rewrite F_2 as

$$\begin{aligned} F_2(\hat{L}_N t_c, R, \tilde{k}) &= \left\langle \frac{2\kappa}{1-R} \frac{1}{\varepsilon_n} \frac{1}{1+k^2/\lambda^2} \frac{k^2}{\lambda^2} \right\rangle + \left\langle \left[\frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} - \frac{2\kappa}{1-R} \right] \frac{1}{\varepsilon_n} \frac{1}{1+k^2/\lambda^2} \frac{k^2}{\lambda^2} \right\rangle \\ &= \frac{2lk}{1-R} f_1(\tilde{k}) + l^2 k^2 \left\langle \left[\frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} - \frac{2\kappa}{1-R} \right] \frac{1}{\varepsilon_n} \frac{1}{\kappa^2 + l^2 k^2} \right\rangle , \end{aligned} \tag{4.17}$$

where l is the thickness ratio defined by $l = \hat{L}_N / \hat{L}_S$. We evaluate the second term by using the trapezoidal formula of the integral (see Appendix C) and find that it gives higher-order terms, i.e.,

$$k^2 l^2 \int_0^\infty d\kappa \left[\frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} - \frac{2\kappa}{1-R} \right] \frac{1}{\kappa} \frac{1}{\kappa^2 + l^2 k^2} = k^2 O(l^2 \ln l) .$$

Thus, up to first order in $\hat{L}_N t_c$

$$F_2(\hat{L}_N t_c, R, \tilde{k}) = \frac{2lk}{1-R} f_1(\tilde{k}) . \tag{4.18}$$

The function F_1 is rewritten as

$$\begin{aligned} F_1(\hat{L}_N t_c, R, \tilde{k}) &= \left\langle \frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} \frac{1}{\varepsilon_n} \frac{k}{\lambda} \right\rangle \\ &\quad - \left\langle \frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} \frac{1}{\varepsilon_n} \frac{1}{1+k^2/\lambda^2} \frac{k^3}{\lambda^3} \right\rangle \\ &= lk \left\langle \frac{1-e^{-2\kappa}}{1-Re^{-2\kappa}} \frac{1}{\varepsilon_n \kappa} \right\rangle - \frac{2lk}{1-R} f_2(\tilde{k}) . \end{aligned} \tag{4.19}$$

The first term is that which appears when we discuss the critical length (see Appendix A). Putting $\Delta = \overline{\Delta}$ and replacing $\hat{L}_S$ by $\hat{L}_N$ in Eq. (A2), we get

$$F_1(\hat{L}_N t_c, R, \tilde{k}) = \frac{2lk}{1-R}\left[\ln\frac{\hat{L}_{Sc0}}{\hat{L}_N} - \ln t_c - f_2(\tilde{k})\right] = \frac{2lk}{1-R}\ln\frac{\hat{L}_{Sc0}}{\hat{L}_N} . \quad (4.20)$$

In the above, the last expression is obtained by using Eq. (4.5). Here $\hat{L}_{Sc0}$ is the critical length with the constant pair function (see Appendix A).

Substituting Eqs. (4.18) and (4.20) into Eq. (4.6) and using Eq. (4.5), we obtain an equation

$$k \tan k = \frac{12}{7\zeta(3)}\hat{L}_S\hat{L}_N \ln\frac{\hat{L}_{Sc0}}{\hat{L}_N} . \quad (4.21)$$

In the case when $\hat{L}_S$ is fixed and $\hat{L}_N \to 0$, the wave number and the transition temperature are given by

$$k^2 = \frac{12}{7\zeta(3)}\hat{L}_S\hat{L}_N \ln\frac{\hat{L}_{Sc0}}{\hat{L}_N} , \quad (4.22)$$

$$t_c = 1 - l\ln\frac{\hat{L}_{Sc0}}{\hat{L}_N} . \quad (4.23)$$

On the other hand, when $\hat{L}_S \to \infty$ and $\hat{L}_N$ is fixed, the wave number k goes to $\pi/2$ and the transition temperature is given by Eq. (4.16).

V. HALF-INFINITE SYSTEM $\hat{L}_N = \infty$, $\hat{L}_S \simeq \hat{L}_{Sc}$

Since the transition temperature depends on $\hat{L}_N$ only through $\tanh\kappa$, the t_c when $\hat{L}_N t_c \gg 1$ is very close to that in the half-infinite system $\hat{L}_N = \infty$. In this section we neglect such a small correction by $\exp(-2\hat{L}_N)$ and consider the half-infinite system $\hat{L}_N = \infty$. In this system, we can rigorously show that there exists a critical length $\hat{L}_{Sc}$ (see Appendix A). Here, we consider the case $\hat{L}_S \simeq \hat{L}_{Sc}$, consequently, $t_c \simeq 0$.

We start again from Eqs. (4.2) and (4.3). When $\hat{L}_N \to \infty$, the function Q becomes

$$Q = \frac{1-R}{2}\frac{1+e^{-2\lambda}}{1-Re^{-2\lambda}} , \quad (5.1)$$

and Eqs. (4.2) and (4.3) are rewritten as

$$-\ln t_c = f_2(\tilde{k}) + \frac{1-R}{2}G_1(\hat{L}_S t_c, R, \tilde{k})\cos k - \frac{1+R}{2}G_2(\hat{L}_S t_c, R, \tilde{k})\sin k , \quad (5.2)$$

$$-\ln t_c = f_2(\tilde{k}) + \frac{1-R}{2}F_1(\hat{L}_S t_c, R, \tilde{k})\cot k - \frac{1+R}{2}F_2(\hat{L}_S t_c, R, \tilde{k}) , \quad (5.3)$$

where the functions f_2, F_1, and F_2 are already defined by Eqs. (4.8), (4.9), and (4.10). The functions G_1 and G_2 are defined by

$$G_1(\hat{L}_S t_c, R, \tilde{k}) = \left\langle \frac{2e^{-\lambda}}{1-Re^{-2\lambda}}\frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\right\rangle \quad (5.4)$$

and

$$G_2(\hat{L}_S t_c, R, \tilde{k}) = \left\langle \frac{2e^{-\lambda}}{1-Re^{-2\lambda}}\frac{1}{\varepsilon_n}\frac{1}{1+k^2/\lambda^2}\frac{k}{\lambda}\right\rangle . \quad (5.5)$$

When $\hat{L}_S \simeq \hat{L}_{Sc}$, consequently, $t_c \simeq 0$, we can neglect the higher-order terms in $\hat{L}_S t_c$. From Eq. (4.8), we obtain

$$f_2(\tilde{k}) = -\ln t_c + \ln\frac{k}{2\hat{L}_S} - 1 - \psi(\tfrac{1}{2}) . \quad (5.6)$$

Using the trapezoidal formula in Appendix C and neglecting terms of order $(\hat{L}_S t_c)^2$, we find that F_1, F_2, G_1, and G_2 are reduced to

$$F_1(\hat{L}_S t_c, R, \tilde{k}) = \int_0^\infty d\lambda \frac{1-e^{-2\lambda}}{1-Re^{-2\lambda}}\frac{k}{\lambda^2+k^2} \equiv F_1^{(0)}(R,k) , \quad (5.7)$$

$$F_2(\hat{L}_S t_c, R, \tilde{k}) = \frac{2k}{1-R}f_1(\tilde{k}) + \left\langle \left[\frac{1-e^{-2\lambda}}{1-Re^{-2\lambda}} - \frac{2\lambda}{1-R}\right]\frac{1}{\varepsilon_n}\frac{k^2}{\lambda^2+k^2}\right\rangle = F_2^{(0)}(R,k) - \frac{2\hat{L}_S t_c}{1-R}\ln 2 , \quad (5.8)$$

$$G_1(\hat{L}_S t_c, R, \tilde{k}) = \int_0^\infty d\lambda \frac{2e^{-\lambda}}{1-Re^{-2\lambda}}\frac{\lambda}{\lambda^2+k^2} \equiv G_1^{(0)}(R,k) , \quad (5.9)$$

$$G_2(\hat{L}_S t_c, R, \tilde{k}) = \frac{2}{1-R}f_1(\tilde{k}) + \left\langle \left[\frac{2\lambda}{1-Re^{-2\lambda}} - \frac{2\lambda}{1-R}\right]\frac{1}{\varepsilon_n}\frac{k}{\lambda^2+k^2}\right\rangle = G_2^{(0)}(R,k) - \frac{2}{1-R}\frac{\hat{L}_S t_c \ln 2}{k} . \quad (5.10)$$

Here, the functions $F_2^{(0)}$ and $G_2^{(0)}$ are given by

$$F_2^{(0)}(R,k) = \int_0^\infty d\lambda \frac{1-e^{-2\lambda}}{1-Re^{-2\lambda}}\frac{1}{\lambda}\frac{k^2}{\lambda^2+k^2} , \quad (5.11)$$

$$G_2^{(0)}(R,k) = \int_0^\infty d\lambda \frac{2e^{-\lambda}}{1-Re^{-2\lambda}}\frac{k}{\lambda^2+k^2} . \quad (5.12)$$

Using Eqs. (5.6)–(5.12), we rewrite Eqs. (5.2) and (5.3) as

$$-\ln t_c = f_2(\tilde{k}) + h_1(R,k) + \hat{L}_S t_c \frac{1+R}{1-R}(\ln 2)\frac{\sin k}{k} , \quad (5.13)$$

$$-\ln t_c = f_2(\tilde{k}) + h_2(R,k) + \hat{L}_S t_c \frac{1+R}{1-R}\ln 2 , \quad (5.14)$$

where h_1 and h_2 are given by

$$h_1(R,k) = \frac{1-R}{2}G_1^{(0)}(R,k)\cos k - \frac{1+R}{2}G_2^{(0)}(R,k)\sin k , \quad (5.15)$$

$$h_2(R,k)=\frac{1-R}{2}F_1^{(0)}(R,k)\cot k-\frac{1+R}{2}F_2^{(0)}(R,k)\ . \tag{5.16}$$

From Eqs. (5.13) and (5.14), we can determine the wave number k through

$$\left[\ln\frac{\hat{L}_S}{2k}+1-\gamma-h_1(R,k)\right]\left[\frac{\sin k}{k}-1\right]+\frac{\sin k}{k}[h_1(R,k)-h_2(R,k)]=0 \tag{5.17}$$

and using this k, the transition temperature is given by

$$t_c=\frac{1-R}{1+R}\frac{1}{\ln 2}\frac{1}{\hat{L}_S}\ln\left[\frac{\hat{L}_S}{2k}e^{1-\gamma-h_2(R,k)}\right]. \tag{5.18}$$

We plot this approximate transition temperature in Fig. 4 by dash-dotted curves. Since the result agrees very well with the numerical result in Sec. III, we use Eq. (5.18) to estimate the critical length. Putting $t_c=0$ in Eqs. (5.17) and (5.18), we obtain

$$\hat{L}_{Sc}=2k_c e^{-1+\gamma+h_2(R,k_c)}\ , \tag{5.19}$$

where k_c is the wave number at $t_c=0$ determined by

$$h_1(R,k_c)=h_2(R,k_c)\ . \tag{5.20}$$

We plot the critical length obtained from Eqs. (5.19) and (5.20) in Fig. 5 by a solid curve. The dashed curve in this figure is $\hat{L}_{Sc0}$, the critical length with the constant pair function (see Appendix A). When $R\simeq 1$, we expect that the pair function is almost uniform in the system, therefore the critical length is well approximated by $\hat{L}_{Sc0}$. In fact, two curves in Fig. 5 show good agreement when $R\simeq 1$.

The existence of the critical length in the case of $R=0$ was first pointed out by Bar-Sagi[12] and numerically discussed by Ohmi *et al.*[13] who also treated a finite R case where a superfluid ^{3}He film is adsorbed on a bulk ^{3}He-^{4}He mixture. When $R\to 0$, our result agrees with Ohmi

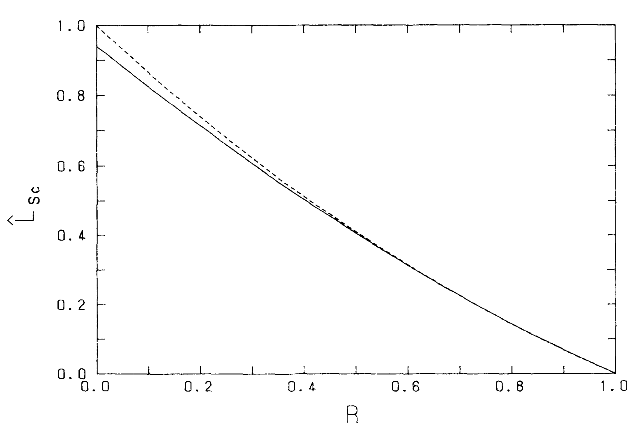

FIG. 5. Reduced critical length vs reflection coefficient. Dashed curve represents $\hat{L}_{Sc}$ with a uniform pair function, and the solid curve is obtained with a cosine pair function.

et al. As can be found in Fig. 4 and was also shown by Ohmi *et al.*, the transition temperature t_c is linearly proportional to $\hat{L}_S-\hat{L}_{Sc}$ when $\hat{L}_S\sim\hat{L}_{Sc}$ in contrast to Bar-Sagi[12] who suggested that $t_c\propto(\hat{L}_S-\hat{L}_{Sc})^{1/2}$.

VI. SUFFICIENTLY THIN FILMS $\hat{L}_S t_c \ll 1$, $\hat{L}_N t_c \ll 1$

Finally, we consider the case $\hat{L}_S t_c\ll 1$, $\hat{L}_N t_c\ll 1$. We start again from Eqs. (4.2) and (4.3). For small $\hat{L}_S t_c$ and $\hat{L}_N t_c$, we can approximate them by logarithmically divergent terms and constant terms in t_c. We also approximate them up to second order in k. By a similar method to that used in Sec. IV (see also Appendix C), we divide Eq. (4.3) into three terms, a logarithmically divergent term when $t_c\to 0$, a convergent term with the uniform Δ, and the correction term from the spatial variation of Δ:

$$\ln t_c=Q_0\{A_1+A_2+A_3k^2\}\ , \tag{6.1}$$

where A_1 is the logarithmically divergent term given by

$$\begin{aligned}A_1&=-\left\langle\frac{1}{\varepsilon_n}\frac{1}{1+(Q_0/Q_\infty)\lambda}\right\rangle\\&=\psi\left[\frac{1}{2}\right]-\frac{Q_0}{Q_\infty}2\hat{L}_S t_c\ln\frac{\Gamma(1/2+Q_\infty/Q_0 2\hat{L}_S t_c)}{\Gamma(1/2)}\\&=\ln t_c+\ln\left[\hat{L}_S\frac{Q_0}{Q_\infty}\right]-\ln 2-\gamma+1+\cdots\ ,\end{aligned} \tag{6.2}$$

A_2 is the convergent term with the uniform Δ

$$\begin{aligned}A_2&=-\left\langle\frac{1}{\varepsilon_n\lambda}\left[\frac{Q}{Q_0}\tanh\lambda-\frac{\lambda}{1+(Q_0/Q_\infty)\lambda}\right]\right\rangle\\&\simeq-\int_0^\infty d\lambda\frac{1}{\lambda^2}\left[\frac{Q}{Q_0}\tanh\lambda-\frac{\lambda}{1+(Q_0/Q_\infty)\lambda}\right]\end{aligned} \tag{6.3}$$

and A_3k^2 is the correction term given by

$$\begin{aligned}A_3&=\left\langle\frac{Q}{Q_0}\frac{1}{\varepsilon_n\lambda^3}\left[\left[1+\frac{\lambda^2}{3}\right]\tanh\lambda-\lambda\right]\right\rangle\\&\simeq\int_0^\infty d\lambda\frac{Q}{Q_0}\frac{1}{\lambda^4}\left[\left[1+\frac{\lambda^2}{3}\right]\tanh\lambda-\lambda\right].\end{aligned} \tag{6.4}$$

In the above, $l=\hat{L}_N/\hat{L}_S$, Q is given by Eq. (2.3) with κ replaced by $l\lambda$, and we have introduced Q_0 and Q_∞ defined by

$$Q_0=\frac{l}{\sqrt{1+l^2+2l(1+R)/(1-R)}}\ , \tag{6.5}$$

$$Q_\infty=\frac{1-R}{2}\ . \tag{6.6}$$

From Eqs. (6.1) and (6.2), we find that

$$t_c=(C\hat{L}_S)^{Q_0/(1-Q_0)}\ , \tag{6.7}$$

where

$$\ln C=\ln\left[\frac{Q_0}{Q_\infty}\right]-\ln 2-\gamma+1+A_2+A_3k^2\ . \tag{6.8}$$

The term A_3k^2 expresses the effect of the spatial variation of $\Delta(\tilde{z})$. The wave number k is determined by Eqs. (4.2) and (4.3). Subtracting Eq. (4.2) from Eq. (4.3) and expanding the result in k up to k^2, we find

$$k^2=\frac{B_1}{B_2+B_3}\ , \tag{6.9}$$

where

$$B_1=\left\langle \frac{Q}{\varepsilon_n}\tanh\lambda\left[\frac{1}{\lambda}-\frac{1}{\sinh\lambda}\right]\right\rangle\ , \tag{6.10}$$

$$B_2=\left\langle \frac{1-Q}{\varepsilon_n\lambda}\left[\frac{1}{\lambda}-\frac{1}{\sinh\lambda}\right]\right\rangle\ , \tag{6.11}$$

and

$$B_3=\left\langle \frac{Q}{\varepsilon_n\lambda^3\cosh\lambda}\left[\left[1+\frac{\lambda^2}{3}\right]\sinh\lambda-\lambda\left[1+\frac{\lambda^2}{2}\right]\right]\right\rangle\ . \tag{6.12}$$

We evaluate B_1, B_2, and B_3 up to logarithmically divergent terms and constant terms in t_c:

$$B_1\simeq\int_0^\infty d\lambda\frac{Q}{\lambda}\tanh\lambda\left[\frac{1}{\lambda}-\frac{1}{\sinh\lambda}\right]\ , \tag{6.13}$$

$$\begin{aligned}B_2&=\frac{1-Q_0}{6}f_2\left[\frac{\sqrt{6}}{\hat{L}_St_c}\right]+\left\langle\frac{1-Q}{\varepsilon_n\lambda}\left[\frac{1}{\lambda}-\frac{1}{\sinh\lambda}\right]-\frac{1-Q_0}{\varepsilon_n}\frac{1}{\lambda^2+6}\right\rangle\\&\simeq-\frac{1}{6}\ln\left[\frac{\hat{L}_S}{2\sqrt{6}}\left[\sqrt{6}\frac{Q_0}{Q_\infty}\right]^{Q_0}\right]-\tfrac{1}{6}(1-\gamma+Q_0A_2)+\int_0^\infty d\lambda\left[\frac{1-Q}{\lambda^2}\left[\frac{1}{\lambda}-\frac{1}{\sinh\lambda}\right]-\frac{1-Q_0}{\lambda}\frac{1}{\lambda^2+6}\right]\ ,\end{aligned} \tag{6.14}$$

$$B_3\simeq\int_0^\infty d\lambda\frac{Q}{\lambda^4\cosh\lambda}\left[\left[1+\frac{\lambda^2}{3}\right]\sinh\lambda-\lambda\left[1+\frac{\lambda^2}{2}\right]\right]\ . \tag{6.15}$$

In Eq. (6.14), we have rearranged the terms so that the trapezoidal formula can be applied and have used the expression (6.7) for t_c with the higher-order terms in k neglected.

The transition temperature obtained from Eqs. (6.7) and (6.9) is plotted in Fig. 6 by dashed curves. The solid curves are obtained by the Fourier series expansion method in Sec. III. They agree quite well when $\hat{L}_S \lesssim \hat{L}_{Sc}$. In Fig. 7, we show the transition temperature of double layers with the fixed ratio l as a function of $\hat{L}_S$. The dashed curves are calculated from Eqs. (6.7) and (6.9) and the solid curves are obtained by the Fourier series expansion.

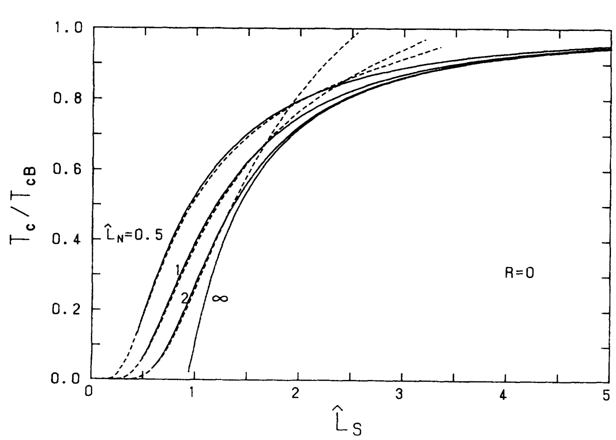

FIG. 6. Reduced transition temperature vs S-side layer thickness when $R=0$, $\hat{L}_N=0.5$, 1, 2, ∞. Solid curves are obtained by the Fourier series expansion method in Sec. III. Dashed curves represent approximate solutions given by Eq. (6.7).

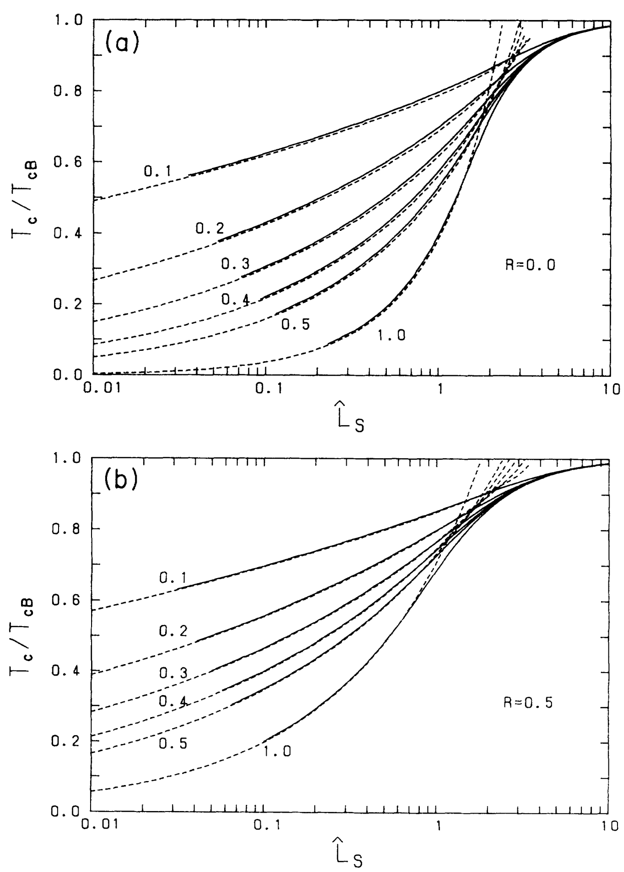

FIG. 7. Reduced transition temperature vs S-side layer thickness with a fixed thickness ratio $l=\hat{L}_N/\hat{L}_S=0.1$, 0.2, 0.3, 0.4, 0.5, and 1.0 when (a) $R=0$ and (b) $R=0.5$. Solid curves are obtained by Fourier series expansion with the self-consistent pair function. Dashed curves represent approximate solutions given by Eq. (6.7) with the cosine pair function.

Note that t_c thus obtained depends not only on l but also on $\hat{L}_S$. It is in contrast to Cooper's theory,[3] in which the transition temperature is determined only by the thickness ratio l, i.e.,

$$t_c = \exp\left[-\frac{l}{vN(0)}\right] . \tag{6.16}$$

Banerjee *et al.*[9] measured the thickness dependence of the transition temperature in Nb/Cu superlattices, in which both the layers have the same width. Their transition temperature monotonously decreases when the layer width decreases. Our result qualitatively agrees with the experiment. A more detailed comparison between our theory and experiments needs the information on the Fermi velocity $v_F^{S,N}$ and the reflection coefficient R, since the transition temperature is sensitive to the scaled layer widths and R as shown in Fig. 7.

VII. SUMMARY AND DISCUSSION

We obtained the gap equation to determine the transition temperature of S-N double-layer system with the finite reflection coefficient R in the clean limit. First, we solved it numerically by use of the Fourier series expansion. Then we derived some approximate formulas according to the range of the parameters $\hat{L}_S$, $\hat{L}_N$ and R. Every formula, in its proper parameter range satisfactorily reproduces the result by the Fourier series expansion. We studied a very thin-film double layer and showed that the transition temperature cannot be explained by Cooper's formula. The present theory may be applied to multilayer systems with the layer width $l_{S,N}$ by replacing our length $L_{S,N}$ by $L_{S,N} = l_{S,N}/2$.[11]

When $\hat{L}_N \to \infty$, there exists a critical length $\hat{L}_{Sc}$. When $\hat{L}_S < \hat{L}_{Sc}$, the superconducting state is unstable at any temperature. We discussed the R dependence of the critical length. We have shown that the pair function is well described by a single cosine function when $\hat{L}_S t_c \gg 1$. When $\hat{L}_S t_c \simeq 0$, the pair function is no longer a cosine function. Nevertheless, the transition temperature can be well estimated by using a cosine pair function. It implies that the transition temperature is not sensitive to the fine profile of $\Delta(\bar{z})$ but dominated by the global profile of $\Delta(\bar{z})$.

It was shown that the transition temperature is sensitive to the layer width $L_{S,N}$, the Fermi velocity $v_F^{S,N}$ and the reflection coefficient R. More precise information on the parameters are necessary to compare the present theory with experiments. The reflection coefficient R may be determined from the measurement of the contact resistance of the double layer in the normal state.

APPENDIX A: CRITICAL LENGTH $\hat{L}_{Sc}$

In this appendix, we show the existence of critical length $\hat{L}_{Sc}$ when $\hat{L}_N \to \infty$. The critical length can be derived from the spatial average of the gap equation Eq. (2.8). When $\hat{L}_N \to \infty$, Q is reduced to

$$Q = \frac{1-R}{(1+R)\tanh\lambda + (1-R)} \tag{A1}$$

and Eq. (2.8) is rewritten as

$$\begin{aligned} \ln t_c &= -\left\langle \frac{1-R}{2\varepsilon_n} \frac{1}{1-Re^{-2\lambda}} \int_0^1 d\bar{z} \frac{\Delta}{\bar{\Delta}} (e^{\lambda(\bar{z}-2)} + e^{-\lambda \bar{z}}) \right\rangle \\ &= -\frac{1-R}{2} \int_0^1 d\bar{z} \frac{\Delta}{\bar{\Delta}} \sum_{p=0}^{\infty} R^p \hat{L}_S t_c \int_0^{1/\hat{L}_S t_c} dx \left[\ln \frac{1+e^{-(2+2p-\bar{z})/x}}{1-e^{-(2+2p-\bar{z})/x}} + \ln \frac{1+e^{-(2p+\bar{z})/x}}{1-e^{-(2p+\bar{z})/x}} \right] . \end{aligned} \tag{A2}$$

When $t_c \to 0$, the left-hand side of Eq. (A2) has a logarithmic divergence. We pick up a similar divergent term and constant terms from the right-hand side,

$$\ln t_c = \ln \left[\frac{e \hat{L}_S t_c}{2} \right] + \frac{1-R}{2} \int_0^1 d\bar{z} \frac{\Delta}{\bar{\Delta}} \sum_{p=0}^{\infty} R^p \ln[(2+2p-\bar{z})(2p+\bar{z})] . \tag{A3}$$

Subtracting the logarithmic divergence from both the sides, we obtain the critical length as

$$\ln \hat{L}_{Sc} = -1 + \ln 2 - \frac{1-R}{2} \int_0^1 d\bar{z} \frac{\Delta}{\bar{\Delta}} \sum_{p=0}^{\infty} R^p \ln[(2+2p-\bar{z})(2p+\bar{z})] . \tag{A4}$$

When $\hat{L}_N$ is finite the logarithmic divergence in Eq. (2.8) is not canceled, as a result the critical length does not exist. In other words, the superconducting state is stable for any $\hat{L}_S$ when $\hat{L}_N$ is finite. The explicit calculation of $\hat{L}_{Sc}$ needs the profile of $\Delta(\bar{z})$. In order to estimate $\hat{L}_{Sc}$, we rewrite the critical length as a sum of the critical length for a constant pair function and a correction term. Let us define the critical length when $\Delta(\tilde{z})=\text{const}=\bar{\Delta}$ by $\hat{L}_{Sc0}$. Then, the critical length is rewritten as

$$\ln\frac{\hat{L}_{Sc}}{\hat{L}_{Sc0}}=-\frac{1-R}{2}\int_0^1 d\tilde{z}\frac{\Delta-\bar{\Delta}}{\bar{\Delta}}\sum_{p=0}^{\infty}R^p\ln[(2+2p-\tilde{z})(2p+\tilde{z})] , \tag{A5}$$

where

$$\begin{aligned}\hat{L}_{Sc0}&=\exp\left[-(1-R)^2\sum_{p=2}^{\infty}R^{p-1}p\ln p\right]\\&=(1-R)\exp\left[-R(2\ln 2-1)-\sum_{p=2}^{\infty}R^p\left\{p\ln\left[1-\frac{1}{p^2}\right]+\ln\frac{p+1}{p-1}-\frac{1}{p}\right\}\right]\\&\simeq(1-R)\exp[-R(2\ln 2-1)-R^2(\ln\tfrac{27}{16}-\tfrac{1}{2})] .\end{aligned} \tag{A6}$$

The final expression of Eq. (A6) is obtained by retaining the terms up to $p=2$.

APPENDIX B: PAIR FUNCTION WHEN $\hat{L}_S t_c \gg 1$

In this appendix we consider a double layer with sufficiently large width superconductor, $\hat{L}_S t_c \gg 1$, and show that the pair function in this case can be well described by a cosine function. We start with Eq. (2.2). For convenience, let us extend the system symmetrically across $\tilde{z}=1$ and transform the variable $\tilde{z}$ to $y=\hat{L}_S t_c(\tilde{z}-1)$. Then Eq. (2.2) is rewritten as

$$\begin{aligned}g_1(y)=&\left[\frac{e^{-\lambda}}{\sinh\lambda}-\frac{Q}{\cosh\lambda\sinh\lambda}\right]\frac{\varepsilon_n}{2\cos\theta}\int_{-\hat{L}_S t_c}^{\hat{L}_S t_c}dy'\frac{\Delta(y')}{\omega_n}\cosh\left[\frac{\varepsilon_n}{\cos\theta}(y'-y)\right]\\&+\frac{\varepsilon_n}{2\cos\theta}\int_{-\hat{L}_S t_c}^{\hat{L}_S t_c}dy'\frac{\Delta(y')}{\omega_n}\exp\left[-\frac{\varepsilon_n}{\cos\theta}|y'-y|\right] .\end{aligned} \tag{B1}$$

When $\hat{L}_S t_c \gg 1$, except just at $y=\pm\hat{L}_S t_c$ and $y'=\mp\hat{L}_S t_c$, the first term is exponentially small due to the hyperbolic functions in the denominator. On the other hand, the second term always gives finite contribution when $y'\simeq y$. Thus, the first term can be neglected almost always, except when we need the pair function just at the S-N boundary. Apart from the S-N boundary, Eq. (B1) is reduced to

$$\begin{aligned}g_1(y)&=\frac{1}{2\pi t_c\cos\theta}\int_{-\infty}^{\infty}dy'\Delta(y')\exp\left[-\frac{\varepsilon_n}{\cos\theta}|y'-y|\right]\\&=\frac{1}{\pi t_c}\sum_{p=0}^{\infty}\frac{\cos^{2p}\theta}{\varepsilon_n^{2p+1}}\frac{d^{2p}}{dy^{2p}}\Delta(y) .\end{aligned} \tag{B2}$$

In the above expression, the integral range was extended to infinite range, which is allowed when $y\not\simeq\hat{L}_S t_c$. Substituting Eq. (B2) into Eq. (2.1), we get a gap equation

$$\Delta(y)\ln t_c=\sum_{p=1}^{\infty}\left\langle\frac{\cos^{2p}\theta}{\varepsilon_n^{2p+1}}\right\rangle\frac{d^{2p}}{dy^{2p}}\Delta(y) . \tag{B3}$$

We can easily find that the above equation has a solution

$$\Delta(y)=\Delta_1\cos\tilde{k}y=\Delta_1\cos k(\tilde{z}-1) , \tag{B4}$$

where $\tilde{k}$ is defined by (4.7). From Eq. (B3) the transition temperature t_c and the wave number $\tilde{k}$ satisfy the following relation:

$$-\ln t_c=\sum_{p=1}^{\infty}\left\langle\frac{\cos^{2p}\theta}{\varepsilon_n^{2p+1}}\right\rangle\tilde{k}^{2p}(-1)^{p-1}=f_2(\tilde{k}) , \tag{B5}$$

where the function f_2 agrees with Eq. (4.8).

APPENDIX C: TRAPEZOIDAL FORMULA

In this appendix, we estimate the following quantity up to first order in $\hat{L}_N t_c$:

$$I=\left\langle\frac{\kappa}{\varepsilon_n}f(\kappa)\right\rangle=2\sum_{n=0}^{\infty}\int_0^{\pi/2}d\theta\sin\theta\frac{\kappa}{\varepsilon_n}f(\kappa) , \tag{C1}$$

where $\kappa=\hat{L}_N t_c\varepsilon_n/\cos\theta$. We assume that the function $f(\kappa)$ goes to zero sufficiently rapidly when $\kappa\to 0$ and $\kappa\to\infty$. Integrating first over the angle θ, we obtain

$$I=\Delta x\sum_{n=0}^{\infty}F(x_n) , \tag{C2}$$

where $x_n=\Delta x(n+\frac{1}{2})$, $\Delta x=2\hat{L}_N t_c$, and the function F is given by

$$F(x)=\int_x^{\infty}dt\, f(t)/t . \tag{C3}$$

Note that the sum in Eq. (C2) can be written as

$$I=\frac{\Delta x}{2}F(x_0)+\frac{\Delta x}{2}\sum_{n=0}^{\infty}\{F(x_n)+F(x_{n+1})\} . \tag{C4}$$

When $\Delta x\to 0$, the second term is interpreted as the trapezoidal formula of the numerical integral; therefore, we can put

$$I=\frac{\Delta x}{2}F(x_0)+\int_{x_0}^{\infty}dx\,F(x)+O(\Delta x^2)\ . \tag{C5}$$

Integrating by parts, we finally obtain

$$I=\int_{x_0}^{\infty}dx\,f(x)+O(\Delta x^2)\ . \tag{C6}$$

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