Slip of normal-phase liquid ^{3}He on a rough boundary

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We consider the flow of normal-phase liquid ^{3}He at low temperatures in the presence of a rough boundary, with a roughness smaller than the particle mean free path and larger than the particle wavelength. The theoretical description of fluid slip, based on a Boltzmann-type transport equation, is extended to a semimicroscopic treatment of rough boundaries. Results for the slip length as a function of a Gaussian-shaped roughness are presented. Further, we find a pressure dependence of the slip length due to Fermi-liquid interactions, not included in former theoretical approaches.

The description of transport phenomena in dilute systems like classical rarefied gases or the elementary excitations of quantum liquids requires a very careful treatment of the boundary conditions due to the long mean free paths of the (quasi)particles characterizing these systems. A general problem in this context is the roughness of the fluid-solid interface. We regard the case of degenerate normal-phase liquid ^{3}He at temperatures of some mK; in this regime there are discrepancies between several viscosity experiments performed by Parpia and Rhodes,[1] Eisenstein, Swift, and Packard,[2] Ritchie, Saunders, and Brewer[3] and the theoretical results from Jaffe[4] and Jensen *et al.*,[5] who assumed plane boundaries. The influence of the roughness of the interface was studied by Einzel, Panzer, and Liu[6] in the framework of hydrodynamics. However, this is not valid for mean free paths λ comparable to or greater than typical dimensions of the system like the size of the roughness itself.

In this paper we approach this subject on a level beyond hydrodynamics by a semimicroscopic treatment starting from a Boltzmann-type transport equation for the distribution function f of quasiparticles. Other approaches[4–6] based on a kinetic equation deal with a phenomenological boundary condition on a plane wall assuming a mixture of specular and diffuse quasiparticle scattering, where the latter accounts for diffraction effects due to a microscopic roughness on an atomic scale and/or accommodation. We regard additional features of the boundary, given by the roughness on a much larger length scale than the atomic ones. In particular we consider a roughness larger than the particle de Broglie wavelength $2\pi/k_F$ on the Fermi surface and smaller than the mean free path λ. The above-mentioned boundary conditions, specular reflection and diffuse scattering (dominated by adsorption and desorption) are now *local* ones on the rough structure. The effects induced by this roughness are shown by calculating the slip length, which represents a measure for the ballisticity of quasiparticle flow near boundaries. The slip length yields a first-order correction (in the ratio of the mean free path λ and the characteristic size d) to the usual hydrodynamics. In particular it is an important quantity in the description of different types of flow experiments (Couette and Poiseuille flows,[1,2] surface impedance measurements[3]) and directly related to measured quantities like the effective viscosity.

In the following we briefly report on dilute fluid flow and slip theory,[5,7] restricting ourselves for mathematical convenience to the simple case of a stationary Couette flow in the x direction in the half space ($z > 0$). Then we extend it, under the above-mentioned conditions, to include rough boundaries by introducing an averaged distribution function and an effective scattering law. Finally, we present our results for the slip length as a function of a Gaussian-distributed roughness.

In the description of the flow of dilute systems one has to take into account the slip at the boundary. This can be done by using a hydrodynamical slip boundary condition on the solution of Navier-Stokes equation for the fluid velocity $u_{\rm HD}$ in the bulk. For Couette flow with a boundary at $z = 0$ the slip-corrected hydrodynamic velocity is $u_{\rm HD}(z) = a(z+\zeta)$. Here the slip length ζ and the slope $a = \partial(\mathbf{u}\cdot\mathbf{e}_x)/\partial z$ are obtained by a linear extrapolation of the fluid velocity $\mathbf{u} = \int d^3p\,\mathbf{v}\,f_{\mathbf{p}}$ in the bulk through the boundary, as shown in Fig. 1.

We first consider a stationary Couette flow with a smooth wall at $z = 0$. Then the problem is only one dimensional in ordinary space, so $f(\mathbf{p},\mathbf{r},t) = f_{\mathbf{p}}(z)$ and $\mathbf{u} = u(z)\mathbf{e}_x$. The fluid velocity u and hence the slip length ζ are obtained from an integral equation based on the linearized Boltzmann equation in the relaxation-time approximation

$$v_z\,\frac{\partial g_{\mathbf{p}}(z)}{\partial z} - v_z p_x \frac{\partial f^0}{\partial \varepsilon}\frac{\partial u(z)}{\partial z} = -\frac{g_{\mathbf{p}}(z)}{\tau}$$

for the deviation $g_{\mathbf{p}}(z) = f_{\mathbf{p}}(z) - f^0(\varepsilon_{\mathbf{p}} - p_x u(z))$ from local equilibrium. Here

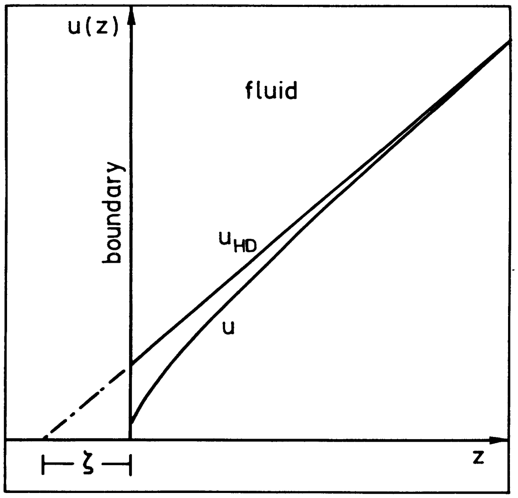

FIG. 1. Definition of the slip length ζ for Couette flow.

$$f^0(\varepsilon) = \left[\exp\left(\frac{\varepsilon-\mu}{k_BT}\right)+1\right]^{-1}$$

is the Fermi-Dirac distribution with the chemical potential μ, the temperature T, the quasiparticle energy $\varepsilon_{\mathbf p}$, the Boltzmann constant k_B and τ is the relaxation time and $v_z = \partial\varepsilon/\partial p_z$ is the quasiparticle velocity. The quasiparticle energy is $\varepsilon_{\mathbf p} = \varepsilon^0_{\mathbf p} + \delta\varepsilon_{\mathbf p}(z)$ with the global equilibrium energy $\varepsilon^0_{\mathbf p}$ and the change of energy $\delta\varepsilon_{\mathbf p}(z) = (2/h^3)\int d^3p' f_{\mathbf{pp}'}\delta f_{\mathbf p'}$ induced by the quasiparticle interaction $f_{\mathbf{pp}'}$, where $\delta f_{\mathbf p} = f_{\mathbf p} - f^0(\varepsilon^0_{\mathbf p})$ and h is Planck's constant. The boundary condition at $z=0$ is formulated in terms of a scattering law $w_{\mathbf p'\to\mathbf p}$ connecting the current densities at the wall

$$f^>_{\mathbf p}(z=0)\;|v_z| = \int_{v'_z<0} d^3p'\; w_{\mathbf p'\to\mathbf p} f^<_{\mathbf p'}(z=0)\;|v'_z|,$$

where $f_{\mathbf p} = f^>_{\mathbf p}\;\Theta(v_z) + f^<_{\mathbf p}\;\Theta(-v_z)$ has been split in the distributions for incoming and outgoing particles with the help of the Heaviside functions Θ. As the momentum current density $\Pi_{xz} = (2/h^3)\int d^3p\; p_x\, v_z\, f_{\mathbf p}$ is independent of z, one gets for u the equation

$$u(\alpha) = \frac{1}{2L_0(0)}\int_0^\infty d\tilde\alpha\, u(\tilde\alpha)\;[\;L_{-1}(|\alpha-\tilde\alpha|) - M_{0,0}(\alpha,\tilde\alpha)\;]. \tag{1}$$

Here the reduced quantities $\alpha = z/v_F\tau = z/\lambda$, $u(\alpha) = u(z)/a\lambda$ with the Fermi velocity v_F have been introduced. In the derivation of (1) we have disregarded a term $(\partial f^0/\partial\varepsilon)\delta\varepsilon(z=0)$, as was done by other authors.[5,7] This term represents a source for a pressure dependence of the flow velocity and the slip length through the quasiparticle interaction. We will come back to this point below. The integral kernels in Eq. (1) are defined by

$$L_n(\alpha) \equiv -\;\tilde N\int_{v_z>0} d^3p\,p_x^2\;\left(\frac{v_z}{v_F}\right)^n\frac{\partial f^0}{\partial\varepsilon}\;\exp\left(-\alpha\frac{v_F}{v_z}\right),$$

$$M_{n,m}(\alpha,\tilde\alpha) \equiv \tilde N\int_{v_z>0} d^3p\int_{v'_z<0} d^3p'\;w_{\mathbf p'\to\mathbf p}\;p_x\,p'_x\;\left(\frac{v_z}{v_F}\right)^{n-1}\left(-\frac{v'_z}{v_F}\right)^m\frac{\partial f'^0}{\partial\varepsilon}\;\exp\left(-\alpha\frac{v_F}{|v_z|}\right)\;\exp\left(-\tilde\alpha\frac{v_F}{|v'_z|}\right), \tag{2}$$

with the effective mass m^*, the particle density n, and $\tilde N = 2/h^3nm^*$. The kernels $M_{n,m}$ contain through $w_{\mathbf p'\to\mathbf p}$ all information about the boundary condition. By casting the definition of the slip length into the form $\int_0^\infty d\tilde\alpha\;[u'(\tilde\alpha)-1] = \zeta/\lambda - u(0)$, one obtains from Eq. (1) an approximation for ζ

$$\frac{\zeta}{\lambda} \approx \frac{(L_2-M_{1,2})\;(L_2-M_{2,1})\;+(L_1+M_{1,1})(\;L_3-M_{2,2}\;)}{2L_2\;(\;L_1+M_{1,1}\;)}, \tag{3}$$

where $L_n = L_n(0)$, $M_{n,m} = M_{n,m}(0,0)$. One can show that the results for ζ from (3) are very close to those obtained from a numerical evaluation of (1). For a more detailed description see Ref. 7, where the special cases of elastical specular, elastic backscattering, and diffuse scattering at the smooth wall are discussed. The results show that an agreement with the experiments[1,2] is only possible for a sufficiently high contribution of backscattering. However this is physically not justified at a smooth wall.

We now turn to the case of rough interfaces, which may be described by $z = \xi(x,y)$, $\xi_x = \partial\xi/\partial x$, $\xi_y = \partial\xi/\partial y$ or in a statistical way by the probability densities $\mathrm{w}_1(\xi)$, $\mathrm{w}_2(\xi_x,\xi_y)$ for the height and the slopes of the surface, respectively. With these densities, one can define the moments $\sigma^2 = \langle\xi^2\rangle$, $\varepsilon^2 = \frac{1}{2}\left(\langle\xi_x^2\rangle + \langle\xi_y^2\rangle\right)$. Two main complications occur due to a boundary roughness. First, there is now a dependence on all three space dimensions and thus the problem is very difficult to handle . We therefore introduce a distribution function $\bar f$ which is averaged over the surface

$$\bar f_{\mathbf p}(z) = \;f^0(\varepsilon_{\mathbf p} - p_x u(z)) + \bar g_{\mathbf p}(z) = \;\langle\,f_{\mathbf p}(\mathbf r)\,\rangle_{x,y}$$

and an effective scattering law $\bar w_{\mathbf p'\to\mathbf p}$ connecting the average particle currents related to a reference plane at $z=0$

$$\bar f^>_{\mathbf p}(z=0)\;|v_z| = \int_{v'_z<0} d^3p'\;\bar w_{\mathbf p'\to\mathbf p}\;\bar f^<_{\mathbf p'}(z=0)\;|v'_z| \quad .$$

In the following we assume that the length scales σ and σ/ε, characterizing the roughness, are smaller than the mean free path λ, which is the scale on which the distribution function varies. Then the treatment for smooth surfaces remains valid for such a roughness, provided one replaces f by $\bar{f}$ and $w_{\mathbf{p}'\to\mathbf{p}}$ by $\bar{w}_{\mathbf{p}'\to\mathbf{p}}$.

Another problem concerns the particle-surface interaction at the rough boundary. We assume that only elastic scattering, adsorption, and desorption occur and that other possible interaction effects may be neglected. The number of particles is conserved. Elastic scattering depends on the ratio of roughness (σ, σ/ε) to the wavelength of the particles ($2\pi/k_F$) on the Fermi surface. As a roughness smaller than the particle wavelength is not resolved anyway, we restrict ourselves here to the regime of geometrical optics with σ, $\sigma/\varepsilon \gg 2\pi/k_F$. Then the elastic scattering may be regarded as pure specular scattering at the surface normals $\mathbf{n}(x,y)$.

Hence we limit our considerations to a boundary roughness with $2\pi/k_F \ll \sigma$, $\sigma/\varepsilon \ll \lambda$. For normal-phase liquid ^{3}He there is $2\pi/k_F \approx 7$ Å (Ref. 8) and $\lambda \geq 1\ \mu$m at some mK and zero pressure.[2] We split the averaged scattering law $\bar{w}_{\mathbf{p}'\to\mathbf{p}}$ into two parts, one for the elastic scattering $\bar{w}^s_\varrho$ and a second part for the adsorption and desorption D_ϱ

$$\bar{w}_{\mathbf{p}'\to\mathbf{p}} = \bar{w}^s_{\varrho,\mathbf{p}'\to\mathbf{p}} + D_\varrho(\mathbf{p},\mathbf{p}'). \tag{4}$$

The elastic part $\bar{w}^s_{\varrho,\mathbf{p}'\to\mathbf{p}}$ represents the probability for a particle reaching the surface with momentum $\mathbf{p}'$ and to leave it with $\mathbf{p}$ after one or more elastic scattering events, averaged over all surface normals $\mathbf{n}(\xi_x,\xi_y) = (-\xi_x,-\xi_y,1)/\sqrt{1+\xi_x^2+\xi_y^2}$ and over all possible multiple reflections. In each encounter of a particle with the surface, elastic scattering occurs with a probability ϱ. Hence we can write $\bar{w}^s_\varrho$ as

$$\bar{w}^s_{\varrho,\mathbf{p}'\to\mathbf{p}} = \varrho \int d\xi_x \int d\xi_y\, P(\hat{\mathbf{p}}',\xi_x,\xi_y)\Big(\Theta(\hat{p}_z)\, Q(\hat{\mathbf{p}}',\hat{\mathbf{p}})\, R(\xi_x,\xi_y,\mathbf{p}'\to\mathbf{p}) + \int d^3\tilde{p}[1-\Theta(\hat{\tilde{p}}_z)\, Q(\hat{\mathbf{p}}',\hat{\tilde{\mathbf{p}}})\,]\, R(\xi_x,\xi_y,\mathbf{p}'\to\tilde{\mathbf{p}})\, \bar{w}^s_{\varrho,\tilde{\mathbf{p}}\to\mathbf{p}}\Big). \tag{5}$$

Here

$$P(\hat{\mathbf{p}},\xi_x,\xi_y) \equiv \mathrm{w}_2(\xi_x,\xi_y)\, \frac{\Theta(-\mathbf{n}\cdot\hat{\mathbf{p}})\,(-\mathbf{n}\cdot\hat{\mathbf{p}})}{\int d\tilde{\xi}_x \int d\tilde{\xi}_y\, \mathrm{w}_2(\tilde{\xi}_x,\tilde{\xi}_y)\,\Theta(-\tilde{\mathbf{n}}\cdot\hat{\mathbf{p}})\,(-\tilde{\mathbf{n}}\cdot\hat{\mathbf{p}})}\,,$$

is the probability density that a reflection occurs at the normal $\mathbf{n}(\xi_x,\xi_y)$, $\Theta(\hat{p}_z)\,Q(\hat{\mathbf{p}}',\hat{\mathbf{p}})$ is the probability density that no further scattering occurs and

$$R(\xi_x,\xi_y,\mathbf{p}'\to\mathbf{p}) \equiv \delta(\mathbf{p} - (\mathbf{p}' - 2\mathbf{n}\,[\mathbf{n}\cdot\mathbf{p}']))$$

is the specular scattering transition probability. The features of roughness are contained in P and Q. Under the quite general assumption that the adsorption and desorption process transfers all momentum p_x of the incoming particles to the surface on the average, D_ϱ contributes nothing to the kernel $M_{n,m}$ (2). Hence the second term involving Eq. (4) for $\bar{w}$ can be dropped in the calculations of the velocity and the slip length.

For a given rough structure one has first to determine the function Q. Then the slip length is obtained by inserting (5) in (2) and $M_{n,m}$ in (3). We consider here explicitly a Gaussian roughness shape in height and slopes

$$\mathrm{w}(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\xi^2}{2\sigma^2}\right),$$

$$\mathrm{w}_2(\xi_x,\xi_y) = \frac{1}{2\pi\varepsilon^2} \exp\left(-\frac{\xi_x^2+\xi_y^2}{2\varepsilon^2}\right),$$

for which the probability Q has been found by general statistical considerations about the intersection of a straight line with the rough surface.[9] It turns out that Q depends only on the mean slope ε. Hence also the $M_{n,m}$ and the slip length ζ are only functions of ε. Figure 2 shows our numerically calculated results for the slip length ζ/λ as a function of the mean slope ε for different contributions of elastic scattering running from pure elastic scattering ($\varrho = 1$) to pure adsorption and desorption ($\varrho = 0$). The slip length does not depend on the roughness in the latter limit, but specular reflection leads to a

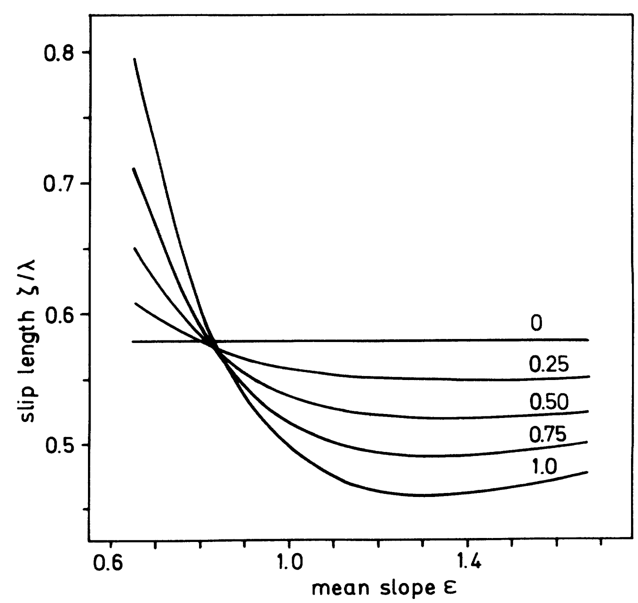

FIG. 2. Slip length ζ as function of the mean slope ε for a Gaussian-shaped roughness and for different portions $\varrho = 0, 0.25, 0.5, 0.75, 1.0$ of elastic scattering.

strong dependence of the slip length on ε.

The experimentally determined slip length[1,2,14] is $\zeta/\lambda = 0.34, \ldots, 0.52$, the theoretical investigations[5,7,10] assuming diffuse scattering on a plane wall yield $\zeta/\lambda = 0.58$, as we obtain for $\varrho = 0$. As an important result we find that the slip length may, for a certain range of roughness, lie well below the universal result for diffuse scattering off a plane wall and that this explains these experiments, if the roughness is rather large, $\varepsilon > 1$. Unfortunately, nothing is known about the size or the shape of the roughness in these experiments and in two recent experiments[3,11] which measured the roughness, the conditions are outside the range of our considerations. In Ref. 3 the roughness height is estimated to be of order 1 μm, while the mean free path is much smaller than 5 μm and in Ref. 11 the roughness (~ 20 Å) is of the same order as the particle wavelength. However, it should be noted that we obtain finite slip lengths for pure specular scattering due to the roughness without the assumption of a high amount of diffuse scattering of an unclarified physical origin. Hence one message of this paper is to stimulate experiments examining specifically the influence of rough boundaries on degenerate quantum liquid properties. These will give the required data to check and improve our concept and support further theoretical investigations. This will also help to clarify some related questions as the role of a ^{4}He boundary layer[3,11–13] in context with an existing roughness and the pressure dependence of ζ/λ in normal fluid ^{3}He as observed by Einzel and Parpia.[14] In the case of a ^{4}He boundary layer the experiments suggest an enhancement of specular scattering and an increase in slip length.[11] This is assumed to come from eliminating an existing microscopic roughness (atomic scale) and reducing the adsorption of ^{3}He by the superfluid ^{4}He films. In the case of a much larger roughness, as treated by us, the ^{4}He layer may change the local boundary condition to higher specularity ($\varrho \to 1$) resulting in a smaller slip length for certain roughness parameters (see Fig. 2). This should be proved experimentally.

As in other approaches,[4,5,7] our derivation yields so far no pressure dependence of ζ/λ . This changes, if the disregarded Fermi-liquid interaction term $(\partial f^0/\partial\varepsilon)\delta\varepsilon(0)$ mentioned above is included. Neglecting angular-momentum components with $\ell \geq 3$, the change of energy at the boundary $z = 0$ in stationary mass flow is[7]

$$\delta\varepsilon_{\mathbf{p}}(0) = \frac{m}{m*}\left(\frac{F_1^s}{3}p_x u(0) \;+\; \frac{F_2^s}{1+F_2^s/5}\frac{p_x v_z}{nmv_F^2}\Pi_{xz}(0)\right)$$

with the bare particle mass m and the $\ell = 1$ and $\ell = 2$ spin-symmetric Fermi-liquid interaction parameters F_1^s, F_2^s. Taking into account this energy change $\delta\varepsilon(0)$ one has to replace $u(\alpha)$ in the integral equation (1) by

$$u(\alpha) - \frac{F_2^s}{5+F_2^s}\alpha - \frac{F_1^s}{1+F_1^s/3}u(0).$$

This yields a relation between the pressure-independent slip length calculated above — now called $(\zeta/\lambda)_{\rm ind}$ — and the new pressure-dependent slip length $(\zeta/\lambda)_{\rm dep}$ given by

$$\left(\frac{\zeta}{\lambda}\right)_{\rm dep} = \left(1 - \frac{F_2^s}{5+F_2^s}\right)\left[\left(\frac{\zeta}{\lambda}\right)_{\rm ind} + \frac{F_1^s}{3}u_{\rm ind}(0)\right], \quad (6)$$

where $u_{\rm ind}(0)$ represents the velocity at $\alpha = 0$ when disregarding $\delta\varepsilon(0)$. Since we have not solved the integral equation (1) explicitly, we do not know the exact value of $u_{\rm ind}(0)$. Taking the interaction parameters F_1^s, F_2^s from Refs. 8 and 15 and noting that $u_{\rm ind}(0)$ is larger than zero and may lie close to $(\zeta/\lambda)_{\rm ind}$, we can make the following statements by inspecting relation (6). On the one hand, there is an enhancement of the slip length, $(\zeta/\lambda)_{\rm dep} > (\zeta/\lambda)_{\rm ind}$ and on the other hand $(\zeta/\lambda)_{\rm dep}$ increases with increasing pressure, qualitatively similar to the experimental result for the slip coefficient $a_s = 6\,\zeta/\lambda$ of Ref. 14.

In summary we have shown a method to account for rough boundaries in the theoretical description of fluid flow in ^{3}He. Our approach can be extended to other flows, to other roughness shapes, and further parameters like the site of the Knudsen minimum[1] can also be calculated. Other systems like solutions of ^{3}He in ^{4}He or the normal component of suprafluid ^{3}He can also be included. Additionally, we found a source of pressure dependence of the slip length due to Fermi-liquid interactions. This should be a matter of a critical review of the existing theoretical description and further theoretical and experimental investigations. A more detailed account of this concept and the results will be published elsewhere.

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