PHYSICAL REVIEW B VOLUME 45, NUMBER 2 1 JANUARY 1992-II

Statics in the random quantum asymmetric Sherrington-Kirkpatrick model

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(Received 8 July 1991)

The random quantum asymmetric Sherrington-Kirkpatrick model has been studied by the extended-pair-approximation approach. The spin-glass phase is smeared by asymmetry, and the ferromagnetic phase is suppressed by the existence of the quantum fluctuations.

Recently there has been a lot of interest in the quantum version of the Sherrington-Kirkpatrick (SK) Ising spin-glass (SG) in a transverse field.[1–9] This model has been proposed to describe the frozen-proton pseudo-spin-glass (PG) phase observed in mixed hydrogen-bonded ferroelectric and antiferroelectric crystals such as $Rb_{1-x}(NH_4)_xH_2PO_4$, conventionally abbreviated as RADP. It was shown that a SG phase exists below a freezing temperature T_f, where T_f and the order parameters for the SG and ferromagnetic phase are parametrized by the transverse field Γ. For $\Gamma \geq \Gamma_c$, where $\Gamma_c = J$ and J is the distribution width of the random exchange coupling, no SG ordering is possible even for $T \rightarrow 0$.

On the other hand, the analogies between networks of formal neurons and random magnetic spin systems have to be utilized in order to apply statistical mechanics in the study of the properties of neural networks.[10] The synaptic efficacies are mapped onto exchange couplings in the spin system. The mapping assumes that the synaptic connections J_{ij} between pairs of neurons i and j are symmetric, i.e., $J_{ij}=J_{ji}$. Under this assumption, the statics and dynamics of the spin-glass-like models of neural networks can be explored on the basis of the statistical mechanical theory of spin glasses. However, the synaptic connections in biological neuron systems are usually not symmetric ($J_{ij} \neq J_{ji}$). Therefore it is of interest to understand the effect of the asymmetry on the long-time properties of the networks. Several suggestions have been made regarding the relevance of random asymmetry to the performance of associative-memory netowrks. Hertz *et al.*[11] suggested that the absence of spurious SG states improves the process of the retrieval of memories, i.e., the convergence to the retrieval states, Parisi[12] proposed that random asymmetry is important for the learning process, in that it guarantees that only the retrieval states will be enhanced by the "Hebb" learning mechanism. Crisanti *et al.*[13] have discussed the effect of the asymmetry on the long-time properties of the network by using a spherical model, in which the SG phase has been shown to be completely suppressed by the asymmetry, while the ferromagnetic phases as well as "retrieval" states are affected only slightly by weak random asymmetry.

A systematic analytic study of the random quantum asymmetric Ising spin system will be more complicated due to three reasons. (1) A transverse field Γ applied in the Ising spin system brings about a quantum effect: by causing spin flips, the requirement for noncommutativity of operators in the Hamiltonian leads to a potentially difficult technical problem.[14] (2) Because of the asymmetry the time dependence of the quantum formalism is involved; the long-time properties have to be calculated for the full dynamical problem and cannot be evaluated from statistical-mechanical averages. (3) Averaging over the quenched disorder of the bonds J_{ij} is difficult because of quantum fluctuations and asymmetry. In this paper, we will study the effects of transverse field and asymmetry on the stability of phase boundaries in the quantum randomly asymmetric fully connected Ising SK model (J_{ij} and J_{ji} being independent random variables). In order to circumvent the above difficulties we extend an approach, which combines the pair approximation for random Ising systems and the discretized path-integral representation (DPIR) for quantum spin systems,[8] to determine "the dynamics on infinite time scales" by considering time-dependent correlations. In the following, in order to calculate the free energy of the model, we use the pair approximation for random Ising systems to carry out the average over the random bonds, then we cast the problem into an equivalent random-field Ising model by introducing a suitable transformation without use of the replica trick, and we calculate time-persistent autocorrelation quantities (a long-time limit of the autocorrelation function) for the Edward-Anderson order parameter.

The model contains N spins, interacting by the Hamiltonian

$$\mathcal{H} = -\sum_{i,j} J_{ij}\sigma_i^z\sigma_j^z - \sum_i \Gamma\sigma_i^x , \tag{1}$$

where σ_i^x and σ_i^z are the Pauli matrices at the ith site of the lattice, and Γ is a transverse field. The exchange interaction matrix J_{ij} is of the form

$$J_{ij} = J_{ij}^{s} + J_{ij}^{as} , \tag{2}$$

where J^s and J^{as} are the symmetric and antisymmetric parts of J_{ij},

$$J_{ji}^{s} = J_{ij}^{s}, \quad J_{ji}^{as} = -J_{ij}^{as} . \tag{3}$$

Each of the off-diagonal elements of J^s and J^{as} is a random Gaussian variable with means J_0/N and zero, respectively. Their variance is

$$[(J_{ij}^{s}-J_0/N)^2]_J=[(J_{ij}^{as})^2]_J=J^2/2N \; . \tag{4}$$

Square brackets denote the "quenched" average with respect to the distribution of J_{ij}. The diagonal elements J_{ii}^{s} and J_{ii}^{as} are zero.

Let us assume that the effective Hamiltonian for the ith spin is of the form

$$\mathcal{H}_i=-\Gamma\sigma_i^x-h_i(t)\sigma_i^z \; , \tag{5}$$

where the local field at the site i is given by

$$h_i(t)=h_i^0(t)+\sum_j J_{ij}\sigma_j^z \; . \tag{6}$$

Here $h_i^0(t)$ is an external field. It was shown that in the randomly asymmetric fully connected Ising system, the local field $h_i(t)$ can be replaced by a time-dependent Gaussian random field. We have[15]

$$h_i(t)=h_i^0(t)+J_0 m+\phi(t) \; , \tag{7}$$

where $\phi(t)$ is a Gaussian variable with zero mean and variance

$$\langle \phi(t)\phi(t_0)\rangle_\phi=J^2C(t-t_0) \; , \tag{8}$$

which reflects the effect of the random interactions J_{ij} on the dynamics of a single spin. In the present case, owing to the asymmetry, the field induced by J_{ij} is not a static one (even at equilibrium) as in the symmetric case. The average magnetization m and autocorrelation $C(t)$ are determined self-consistently by

$$m=\langle\langle \sigma_i\rangle\rangle_\phi=\int d\phi(2\pi)^{-1/2}\exp(-\phi^2/2)\langle\sigma_i\rangle \; , \tag{9}$$

$$C(t)=\lim_{\tau\to\infty}\langle\langle \sigma_i(t+\tau)\sigma_i(t)\rangle\rangle_\phi \; , \tag{10}$$

where $\langle\sigma_i\rangle$ is the thermal average of the spin fluctuations, and $\langle\ \rangle_\phi$ denotes the average over the Gaussian field $\phi(t)$.

The corresponding one-body partition function becomes

$$Z_i=\mathrm{Tr}\exp(-\beta\mathcal{H}_i)=2\cosh\{\beta[h_i^2(t)+\Gamma^2]^{1/2}\} \; . \tag{11}$$

The average magnetization m and the local static transverse susceptibility χ induced by the local field h_i (with $h^0=0$) are given by

$$m=\int d\phi(2\pi)^{-1/2}\exp(-\phi^2/2)\frac{J_0m+\phi}{[(J_0m+\phi)^2+\Gamma^2]^{1/2}}\tanh\{\beta[(J_0m+\phi)^2+\Gamma^2]^{1/2}\} \; , \tag{12}$$

$$\chi=\int d\phi(2\pi)^{-1/2}\exp(-\phi^2/2)\left[\frac{\Gamma^2}{[(J_0m+\phi)^2+\Gamma^2]^{3/2}}\tanh\{\beta[(J_0m+\phi)^2+\Gamma^2]^{1/2}\}\right.$$
$$\left.+\beta\frac{(J_0m+\phi)^2}{[(J_0m+\phi)^2+\Gamma^2]}\mathrm{sech}^2\{\beta[(J_0m+\phi)^2+\Gamma^2]^{1/2}\}\right] \; . \tag{13}$$

Within the context of the dynamical theory, the spin fluctuations are viewed as local, thermally averaged, time-dependent magnetic moments. These moments are induced by excess (i.e., nonthermal) internal noise which is time dependent in the presence of asymmetry. The static Edwards-Anderson order parameter q has to be determined by time-persistent quantities. We have

$$q=\lim_{t\to\infty}q(t)=\lim_{t\to\infty}C(t)$$
$$=\int dx\,(2\pi)^{-1/2}\exp(-x^2/2)\left[\int dy\,(2\pi)^{-1/2}\exp(-y^2/2)\frac{(J_0m+\sqrt{1-J^2q}\,y+\sqrt{J^2q}\,x)}{[(J_0m+\sqrt{1-J^2q}\,y+\sqrt{J^2q}\,x)^2+\Gamma^2]^{1/2}}\right.$$
$$\left.\times\tanh\{\beta[(J_0m+\sqrt{1-J^2q}\,y+\sqrt{J^2q}\,x)^2+\Gamma^2]^{1/2}\}\right]^2 \; . \tag{14}$$

The pair Hamiltonian in the pair approximation is given by

$$\mathcal{H}_{ij}=-J_{ij}\sigma_i^z\sigma_j^z-h_i^*(t)\sigma_i^z-h_j^*(t)\sigma_j^z-\Gamma(\sigma_i^x+\sigma_j^x) \; ,$$
$$h_i^*(t)=h_i^0(t)+\sum_{l\neq i,j}J_{il}\sigma_l \; , \tag{15}$$
$$h_j^*(t)=h_j^0(t)+\sum_{l\neq j,i}J_{jl}\sigma_l \; ,$$

where $h_i^*(t)$ is the local field on site i coming from other spins except from site j, and equals the one-body local field h_i in the limit of infinite range interactions, where every spin couples equally with every other spin. The corresponding pair partition function becomes

$$Z_{ij}=\mathrm{Tr}\exp(-\beta\mathcal{H}_{ij}) \; . \tag{16}$$

In order to obtain the pair partition function, we will reformulate the Hamiltonian in DPIR. The idea in DPIR is to convert the quantal two-state spin on each lattice site into a P-component vector, and eventually let

P go to infinity. Each component is taken to be a classical two-state variable, and the net effect is to represent the quantum uncertainty by creating many copies, or replicas, of the original variable. By means of the DPIR, the pair Hamiltonian can be broken up into a reference part involving only the single-site terms, and an interaction part. The corresponding free energy can be expressed in terms of the free energy of the reference part and a cumulant expansion. By taking the first cumulant, we obtain the expression[8]

$$\begin{aligned}\ln Z_{ij} =& \ln\{2\cosh[\beta(h_i^{*2}+\Gamma^2)^{1/2}]\}\\ &+\ln\{2\cosh[\beta(h_j^{*2}+\Gamma^2)^{1/2}]\}\\ &+\beta J_0\frac{\tanh[\beta(h_i^{*2}+\Gamma^2)^{1/2}]}{(h_i^{*2}+\Gamma^2)^{1/2}}h_i^*\\ &\times\frac{\tanh[\beta(h_j^{*2}+\Gamma^2)^{1/2}]}{(h_j^{*2}+\Gamma^2)^{1/2}}h_j^* . \end{aligned} \tag{17}$$

The free energy of the full system in the pair approximation is given by the expression[8,16]

$$-\beta F=\int d\phi(2\pi)^{-1/2}\exp(-\phi^2/2) \times\left[\sum_i \ln Z_i+\sum_{i,j}(\ln Z_{ij}-\ln Z_i-\ln Z_j)\right] . \tag{18}$$

We first consider the possibility of a SG phase, i.e., a phase with $m=0$ and $q\neq 0$. It is straightforward to see from Eq. (14) that $q=0$ is the only solution for all $T>0$, leading to the conclusion that the system does not have a SG phase. Unlike the static noise in the symmetric SK case, a time-dependent random Gaussian field which plays the role of a dynamic noise in the asymmetric SK model destroys the SG phase which exists in the symmetric model. The classical result, in agreement with the conclusions of Parisi,[12] is recovered for $\Gamma=0$.

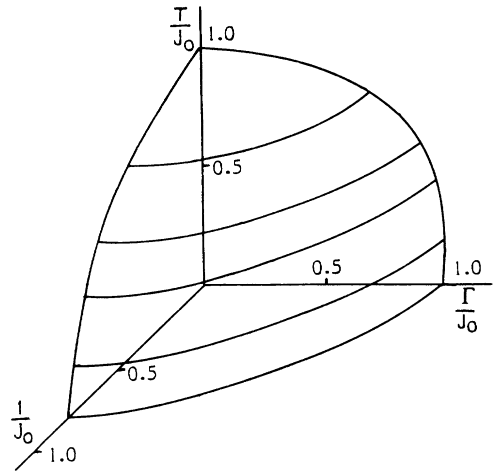

FIG. 1. The paramagnetic-ferromagnetic (P-F) phase boundaries of the random quantum full asymmetric SK model (in units of J_0).

In order to investigate the stability of the phase boundary between paramagnetic and ferromagnetic order, we expand the free energy in terms of the magnetization m. We take the second derivative of the free energy and let it go to zero. The second order instability lines are given by

$$\int d\phi(2\pi)^{-1/2}\exp(-\phi^2/2)\left[\frac{\Gamma^2}{(\Gamma^2+\phi^2)^{3/2}}\tanh[\beta(\Gamma^2+\phi^2)^{1/2}]+\beta\frac{\phi^2}{\Gamma^2+\phi^2}\operatorname{sech}^2[\beta(\Gamma^2+\phi^2)^{1/2}]\right]=\frac{1}{J_0} . \tag{19}$$

The critical surface separating the paramagnetic and the ferromagnetic phases is determined by the relation $J_0\chi=1$. The resulting phase diagram in $(T/J_0,\Gamma/J_0,1/J_0)$ space is shown in Fig. 1. From this figure, we see that the regions of the ferromagnetic phase as well as "retrieval" states in neural networks decrease with the increase of the strength of transverse field Γ, and when $\Gamma=J_0$ the ferromagnetic phase disappears. In the present case, the main effects of the transverse field make the ferromagnetic ordering unstable at all temperatures.

This is a study of the quantum randomly asymmetric fully connected Ising SK model by using the extended pair-approximation approach. The main effect of asymmetry is the smearing of the SG phase. Quantum fluctuations have the effect of destroying the ordered phase; the ferromagnetic phase disappears for $\Gamma\geq J_0$. Further investigations of the general random asymmetric SK model with a transverse field are planned to follow.

The project was supported by the National Natural Science Foundation of China.

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