Theory of coherent x-ray radiation by relativistic particles in a single crystal

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A quantum theory is developed for a type of coherent x-ray radiation observed recently. It is shown that the lowest-order coherent-radiation processes due to the interaction between the particle and the crystal electrons are classified into two kinds: electronic bremsstrahlung and x-ray radiation. An analytical expression for the radiation probability is derived using quantum electrodynamics. For ultrarelativistic particles, our quantum expression yields the classical formula by Ter-Mikaelian, in agreement with the experimental results.

I. INTRODUCTION

Radiation from relativistic charged particles passing through matter has been studied for a long time. Especially, radiation in a single crystal has been of special interest because it becomes very intense coherent radiation under certain conditions. For example, the periodic modulation of the trajectories of incident electrons by the crystal potential causes coherent bremsstrahlung (CB) and channeling radiation (CR). CB and CR have been intensively studied both theoretically and experimentally.[1]

Recently, a type of coherent x-ray radiation by relativistic electrons, called parametric x-ray radiation (PXR),[2] has been observed.[3-5] In contrast to CB and CR, as will be shown in Sec. II, PXR is due to the scattering of the virtual photons by the periodically distributed electrons in the crystal. Therefore, PXR is connected with x-ray diffraction. In fact, very intense x-rays have been observed around the Bragg condition of the x-rays.[3-5]

PXR was considered by Ter-Mikaelian more than two decades ago. In his book,[6] Ter-Mikaelian calculated the radiation intensity as a special kind of "resonance radiation" using classical electrodynamics. Subsequently, several theoretical studies have appeared.[7-10] However, the theories of PXR have been classical or semiclassical in that they use the Maxwell equations for obtaining the electromagnetic field in the crystal. As a result, the previous theories can only describe more or less macroscopic mechanisms of PXR. In this paper, we present a complete quantum description of PXR, giving a clear microscopic picture of it.

II. THE HAMILTONIAN AND THE DIAGRAMS

Previous studies showed that PXR is due to the interaction between the particles and the crystal electrons.[7,10] Therefore, we consider a system composed of three parts: the projectile, the crystal electrons, and the radiation field. In calculating the electromagnetic interaction in the crystal, the Coulomb gauge should be used because the binding states of the crystal electrons are taken into account.[11] The crystal electrons are assumed nonrelativistic. Thus, the total Hamiltonian for the system is given by

$$H_{\rm tot}=H_p+H_c+H_r+H' \;, \tag{1}$$

where H_p, H_c, and H_r are the nonperturbative Hamiltonians of the particle, of the crystal electrons, and of the radiation field, respectively. The unperturbed states for the particle satisfy the Dirac equation

$$H_p\psi_{\rm ps}=E(\mathbf{p})\psi_{\rm ps} \;, \quad \psi_{\rm ps}=u_{\rm ps}\exp(i\mathbf{p}\cdot\mathbf{r})/\sqrt{V} \;, \tag{2}$$

where $u_{\rm ps}$ is the spinor normalized to unity and V is the volume of the crystal. The unperturbed states for the crystal electrons as a whole satisfy the Schrödinger equation

$$H_c|n\rangle=\varepsilon_n|n\rangle \;. \tag{3}$$

The unperturbed Hamiltonian for the radiation field in the crystal has the usual form

$$H_r=\sum_{\mathbf{k},a}\hbar\omega(\mathbf{k})(a^\dagger_{\mathbf{k}a}a_{\mathbf{k}a}+\tfrac{1}{2}) \;. \tag{4}$$

However, as shown in the Appendix, the photon states are different from those in vacuum because of the interaction with the crystal electrons. They now satisfy the following dispersion relation:

$$\omega(\mathbf{k})=c^*|\mathbf{k}| \;. \tag{5}$$

The "velocity of light" in the crystal c^* is defined by

$$c^*=c(1-\tfrac{1}{2}\chi_0) \;, \tag{6}$$

where χ_0 is the electric susceptibility:

$$\chi_0=-\omega_0^2/c^2|\mathbf{k}|^2 \;, \quad \omega_0=(4\pi\rho_0e^2/m_e)^{1/2} \;. \tag{7}$$

For x rays, Eq. (6) is consistent with the classical relation[12] $c^*=c/\sqrt{\epsilon}$, where ϵ is the dielectric constant in the medium (see the Appendix).

The interaction Hamiltonian is given by

$$\begin{aligned}H'&=H_{p-r}+H_{p-c}+H^{(1)}_{c-r}+H^{(2)}_{c-r}\\&=-e\alpha\cdot\mathbf{A}(\mathbf{r})+\sum_i\frac{e^2}{|\mathbf{r}-\mathbf{r}_i|}-\frac{e}{m_ec}\sum_i\mathbf{A}(\mathbf{r}_i)\cdot\hat{\mathbf{p}}_i\\&\quad+\frac{e^2}{2m_ec^2}\sum_i\mathbf{A}^2(\mathbf{r}_i) \;,\end{aligned} \tag{8}$$

where $\mathbf{r}$ and $\mathbf{r}_i$ are the coordinates of the particle and the ith crystal electron, respectively. the field operator $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r})=\sum_{\mathbf{k},a}[\,\mathbf{A}(\mathbf{k},a)a_{\mathbf{k}a}\exp(i\mathbf{k}\cdot\mathbf{r})+\text{H.c.}\,]\ , \tag{9a}$$

$$\mathbf{A}(\mathbf{k},a)=\left[\frac{2\pi\hbar c^{*2}}{V\omega(\mathbf{k})}\right]^{1/2}\mathbf{e}_{\mathbf{k}a}\ , \tag{9b}$$

where $a_{\mathbf{k}a}$ is the annihilation operator for the photon having the wave vector $\mathbf{k}$ and the polarization a and $\mathbf{e}_{\mathbf{k}a}$ is the polarization vector for the photon. It should be noted that the third and fourth terms in Eq. (8), which represent the interaction between the crystal electrons and the radiation field, have been neglected in the theory of CB and CR.

Now, we consider the lowest-order Feynman diagrams for the radiation processes due to the interaction between the relativistic charged particles and the crystal electrons. They are shown in Fig. 1.[13] The wavy lines, the solid lines, and the twin-solid lines represent the transverse photons, the incident particle, and the crystal electrons as a whole, respectively. The four-point contact vertices for the particles and the crystal electrons represent the static Coulomb interaction H_{p-c} and the other contact vertices in which the two photons are attached to the crystal electrons ("seagull" parts) represent the interaction due to $H^{(2)}_{c-r}$. The diagrams (c) and (f) show the well-known "electronic" bremsstrahlung.[11] However, the processes associated with the diagrams (a), (b), (d), and (e) have not been studied in detail. Indeed, these four processes as a whole correspond to PXR when the final state of the crystal electrons as a whole is the same as the initial state except for the quasimomentum transfer to the crystal electrons as a whole. It should be noted that, in contrast to processes (c) and (f), representing *radiation emitted by the particle itself* when it collides with the crystal electrons, processes (a) and (d) as well as (b) and (e) represent *radiation emitted by the crystal electrons* under the action of the field of the particle. As will be discussed later, this cause a typical difference between the two types of radiation on the dependence on the mass of the particle.

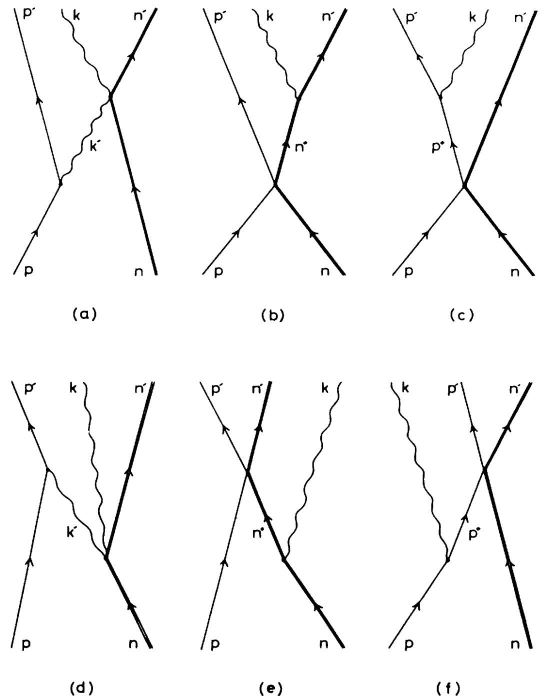

FIG. 1. The second-order radiation processes due to interactions of a relativistic charged particle with electrons in a crystal. The solid lines, the twin-solid lines, and the wavy lines represent the projectile, crystal electrons as a whole, and the photons, respectively. (a), (b), (d), and (e) show the fundamental processes of PXR while (c) and (f) represent the ordinary electronic bremsstrahlung.

III. THE PROBABILITY AMPLITUDES

Once the diagram for a fundamental process is given, the calculation of the probability amplitude becomes straightforward. For example, the compound matrix element for process (a) in Fig. 1 is given by

$$K^{(a)}_{FI}=\sum_{M}\frac{\langle\psi_{\mathbf{p}'s'},\mathbf{k}_a,n'|H^{(2)}_{c-r}|\psi_{\mathbf{p}'s'},\mathbf{k}'_{a'},n\rangle}{E_I-E_M}\times\langle\psi_{\mathbf{p}'s'},\mathbf{k}'_{a'},n|H_{p-r}|\psi_{\mathbf{p}s},0,n\rangle\ , \tag{10}$$

Where I, M, and F are the quantum numbers of the initial, the intermediate, and the final states of the system, respectively: $F=(\mathbf{p}',s',\mathbf{k},a,n')$, etc.

As mentioned in Sec. II, the radiation processes are coherent when the initial state of the crystal electrons is the same as the final state, except for the quasimomentum transfer $\hbar\mathbf{h}$ to the crystal electrons as a whole. In the coherent case, the initial and the final energy of the crystal electrons are degenerate; $\varepsilon_{n'}=\varepsilon_n$. Thus, substituting Eqs. (8) and (9) into Eq. (10), we obtain

$$K^{(a)}_{FI}=-\left[\frac{e^3}{m_e c^2}\right]\sum_{\substack{\mathbf{k}',a'\\(\mathbf{k}\neq\mathbf{k}')}}\mathbf{A}(\mathbf{k}',a')\cdot(u^\dagger_{\mathbf{p}'s'}\boldsymbol{\alpha}u_{\mathbf{p}s})\mathbf{A}(\mathbf{k},a)\cdot\mathbf{A}(\mathbf{k}',a')\mathcal{F}_{nn}(\mathbf{k}'-\mathbf{k})\delta(\mathbf{p}'+\hbar\mathbf{k}'|\mathbf{p})\frac{1}{\hbar[\omega(\mathbf{k})-\omega(\mathbf{k}')]}\ , \tag{11}$$

where $\delta(\mathbf{Q}|\mathbf{Q}')$ is the Kronecker delta and $\mathcal{F}_{nn}(\mathbf{Q})$ is the form factor defined by

$$\mathcal{F}_{nn}(\mathbf{Q})=\left\langle n\left|\sum_i\exp(-i\mathbf{Q}\cdot\mathbf{r}_i)\right|n\right\rangle\ . \tag{12}$$

Assuming that the crystal electrons are initially in their ground states, $n=n_0$, the corresponding form factor $\mathcal{F}_{00}(\mathbf{Q})$ is related to the electron density $\rho(\mathbf{r})$. Since $\rho(\mathbf{r})$ is a periodic function, we obtain

$$\mathcal{F}_{00}(\mathbf{Q})=V\sum_{\mathbf{h}}\rho_{\mathbf{h}}\delta(\mathbf{h}|\mathbf{Q}) , \tag{13}$$

where $\mathbf{h}$ are the reciprocal lattice vectors and $\rho_{\mathbf{h}}$ are the Fourier components of $\rho(\mathbf{r})$.

The matrix element for the process (d) in Fig. 1 can be calculated in the same manner. The total matrix element for the processes (a) and (d) become

$$K_{FI}^{(I)}=K_{FI}^{(a)}+K_{FI}^{(d)}=\sum_{\mathbf{h}(\neq 0)}\sum_{a'}\left[\frac{e}{\epsilon}\right]\left[\frac{2\pi\hbar c^{*2}}{V\omega(\mathbf{k})}\right]^{1/2}\chi_{-\mathbf{h}}(u^{\dagger}_{\mathbf{p}'s'}\mathbf{e}_{\mathbf{h}a'}\cdot\boldsymbol{\alpha}u_{\mathbf{p}s})(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{e}_{\mathbf{h}a'})\frac{\omega(\mathbf{k})^2}{\omega(\mathbf{k})^2-\omega(\mathbf{k}_{\mathbf{h}})^2}\delta(\mathbf{p}'+\hbar\mathbf{k}_{\mathbf{h}}|\mathbf{p}) , \tag{14}$$

where $\mathbf{e}_{\mathbf{h}a'}$ is the polarization vectors for the photon $\mathbf{k}_{\mathbf{h}}(=\mathbf{k}+\mathbf{h})$ and we have defined that

$$\chi_{\mathbf{h}}=-\omega_{\mathbf{h}}^2/|c\mathbf{k}|^2 , \quad \omega_{\mathbf{h}}=(4\pi\rho_{\mathbf{h}}e^2/m_e)^{1/2} . \tag{15}$$

Next, we calculate the matrix elements for the processes (b) and (e) in Fig. 1. Similar calculations to the above yield the following matrix elements:

$$\begin{aligned}K_{FI}^{(II)}&=K_{FI}^{(b)}+K_{FI}^{(e)}\\&=-\left[\frac{e^3}{m_ec^2}\right]\sum_{\mathbf{q}}\left[\frac{4\pi c}{V\mathbf{q}^2}\right](u^{\dagger}_{\mathbf{p}'s'}u_{\mathbf{p}s})\delta(\mathbf{p}'+\hbar\mathbf{q}|\mathbf{p})\\&\quad\times\sum_m\mathbf{A}(\mathbf{k},a)\cdot\left[\frac{\left\langle n\left|\sum_i\exp(-i\mathbf{k}\cdot\mathbf{r}_i)\hat{\mathbf{p}}_i\right|m\right\rangle\left\langle m\left|\sum_j\exp(i\mathbf{q}\cdot\mathbf{r}_j)\right|n\right\rangle}{\hbar\omega(\mathbf{k})+\varepsilon_{nm}}\right.\\&\qquad\left.-\frac{\left\langle n\left|\sum_i\exp(iq\cdot\mathbf{r}_i)\right|m\right\rangle\left\langle m\left|\sum_j\exp(-i\mathbf{k}\cdot\mathbf{r}_j)\hat{\mathbf{p}}_j\right|n\right\rangle}{\hbar\omega(\mathbf{k})-\varepsilon_{nm}}\right] ,\end{aligned} \tag{16}$$

where $\varepsilon_{nm}=\varepsilon_n-\varepsilon_m$. It is very difficult to compute the summations over all the eigenstates of the crystal electrons. Instead, for obtaining an analytical expression, we introduce the approximation that the excitation energies $|\varepsilon_{nm}|$ are much smaller than the energies of the emitted photons:

$$|\varepsilon_{nm}|\ll\hbar\omega(\mathbf{k}) . \tag{17}$$

For x rays, $\hbar\omega\gtrsim 10$ KeV is satisfied whereas the average value of ε_{nm} is of the order of the mean excitation energy, $I\lesssim 1$ KeV. Therefore, Eq. (17) is reasonable. By using (17), Eq. (16) reduces to the much simpler form

$$K_{FI}^{(II)}=e\sum_h\chi_{-\mathbf{h}}\left[\frac{2\pi\hbar c^{*2}}{V\omega(\mathbf{k})}\right]^{1/2}(u^{\dagger}_{\mathbf{p}'s'}u_{\mathbf{p}s})(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{k}_{\mathbf{h}})\frac{\omega(\mathbf{k})}{c\mathbf{k}_{\mathbf{h}}^2}\delta(\mathbf{p}'+\hbar\mathbf{k}_{\mathbf{h}}|\mathbf{p}) , \tag{18}$$

where we have used the completeness relation $\sum_m|m\rangle\langle m|=1$ and the identity

$$\hat{\mathbf{p}}_i\sum_j\exp(i\mathbf{q}\cdot\mathbf{r}_j)=\hbar\mathbf{q}\exp(i\mathbf{q}\cdot\mathbf{r}_i)+\sum_j\exp(i\mathbf{q}\cdot\mathbf{r}_j)\hat{\mathbf{p}}_i . \tag{19}$$

In calculating the polarization of the photons, we fix the directions of the two independent polarization vectors, such that the one is parallel to the $\mathbf{k}-\mathbf{h}$ plane and the other is normal to the plane. We write them as $\mathbf{e}_{\parallel}$ and $\mathbf{e}_{\perp}$, respectively. Thus, we have

$$\mathbf{e}_{\mathbf{h}\perp}=\mathbf{e}_{\mathbf{k}\perp} , \quad \mathbf{e}_{\mathbf{h}\parallel}=\frac{|\mathbf{k}|}{|\mathbf{k}_{\mathbf{h}}|(\mathbf{k}\cdot\mathbf{k}_{\mathbf{h}})}[\mathbf{k}_{\mathbf{h}}^2\mathbf{e}_{\mathbf{k}\parallel}-(\mathbf{e}_{\mathbf{k}\parallel}\cdot\mathbf{h})\mathbf{k}_{\mathbf{h}}] . \tag{20}$$

In this geometry, the summation over the polarizations in Eq. (14) becomes obvious. From Eq. (14) and Eq. (18), we obtain the total matrix element for PXR

$$K(\mathbf{k},\mathbf{a})=\sum_{\mathbf{h}(\neq 0)}\frac{e\chi_{-\mathbf{h}}\omega(\mathbf{k})^2}{\epsilon}\left[\frac{2\pi\hbar c^{*2}}{V\omega(\mathbf{k})}\right]^{1/2}\delta(\mathbf{p}|\mathbf{p}'+\hbar\mathbf{k}_{\mathbf{h}})Y(\mathbf{k},\mathbf{a}) , \tag{21}$$

$$Y(\mathbf{k},a)=\left[\frac{(u^{\dagger}_{\mathbf{p}'s'}\mathbf{e}_{\mathbf{h}a}\cdot\boldsymbol{\alpha}u_{\mathbf{p}s})(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{e}_{\mathbf{h}a})}{\omega(\mathbf{k})^2-\omega(\mathbf{k}_{\mathbf{h}})^2}+\frac{c(u^{\dagger}_{\mathbf{p}'s'}u_{\mathbf{p}s})(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{h})}{\omega(\mathbf{k})\omega(\mathbf{k}_{\mathbf{h}})^2}\right] , \tag{22}$$

where $a=\parallel,\perp$ and $\mathbf{e}_{\mathbf{k}\perp}\cdot\mathbf{h}=0$ has been used.

Next, we calculate the spinors. In this paper, we do not consider polarized particles. Therefore, we should calculate

$$|\bar{Y}(\mathbf{k},a)|^2=\frac{1}{2}\sum_{s,s'=1,2}\sum |Y(\mathbf{k},a)|^2 . \tag{23}$$

After some calculations, we obtain

$$|\bar{Y}(\mathbf{k},a)|^2=|P(\mathbf{k},a)|^2\left[(\mathbf{e}_{\mathbf{h}a}\cdot\boldsymbol{\beta})^2-(\mathbf{e}_{\mathbf{h}a}\cdot\boldsymbol{\beta})(\mathbf{e}_{\mathbf{h}a}\cdot\delta\boldsymbol{\beta})+\frac{1}{2}\left[\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}+\frac{1}{\gamma^2}-\frac{1}{\gamma\gamma'}\right]\right]$$
$$+2P(\mathbf{k},a)Q(\mathbf{k},a)[(\mathbf{e}_{\mathbf{h}a}\cdot\boldsymbol{\beta})-\tfrac{1}{2}(\mathbf{e}_{\mathbf{h}a}\cdot\delta\boldsymbol{\beta})]+|Q(\mathbf{k},a)|^2\left[1-\tfrac{1}{2}\left[\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta}+\frac{1}{\gamma^2}-\frac{1}{\gamma\gamma'}\right]\right] , \tag{24}$$

where

$$P(\mathbf{k},a)=\frac{\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{e}_{\mathbf{h}a}}{\omega(\mathbf{k})^2-\omega(\mathbf{k}_\mathbf{h})^2} , \quad Q(\mathbf{k},a)=\frac{c(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{h})}{\omega(\mathbf{k})\omega(\mathbf{k}_\mathbf{h})^2} , \tag{25}$$

$$\boldsymbol{\beta}=c\mathbf{p}/E(\mathbf{p}) , \quad \boldsymbol{\beta}'=c\mathbf{p}'/E(\mathbf{p}'), \quad \delta\boldsymbol{\beta}=\boldsymbol{\beta}-\boldsymbol{\beta}' , \quad \gamma=\sqrt{1-\beta^2} , \quad \gamma'=\sqrt{1-\beta'^2} . \tag{26}$$

IV. RADIATION PROBABILITY

The probability of PXR per unit time is now obtained by the golden rule:

$$w(\mathbf{k},a)=\frac{2\pi}{\hbar}|\bar{K}(\mathbf{k},a)|^2\rho_F . \tag{27}$$

The density of final states ρ_F is given by

$$\rho_F=\left[\frac{\partial\omega(\mathbf{k})}{\partial E}\right]\rho_\omega$$
$$=\left[\frac{E(\mathbf{p}')}{E(\mathbf{p})}\right]\left[1-\boldsymbol{\beta}^*\cdot\mathbf{n}-(1-\epsilon)\left[\frac{\hbar\omega(\mathbf{k})}{E(\mathbf{p})}\right]\right]^{-1}$$
$$\times\frac{V\omega(\mathbf{k})^2 d\Omega}{(2\pi)^3\hbar c^{*3}} , \tag{28}$$

where we have defined $\boldsymbol{\beta}^*=\boldsymbol{\beta}c/c^*$ and $\mathbf{n}=\mathbf{k}/|\mathbf{k}|$. Substituting Eqs. (24) and (28) into Eq. (27), we obtain an analytical expression for the radiation probability of PXR without any approximations except for Eq. (17). However, for ultrarelativistic particles satisfying $\gamma \gg 1$, we can obtain a much simpler formula. Since the energies of the emitted photons are in the x-ray region, in the relativistic limit we can neglect the recoil energy due to the photon emission:

$$\hbar\omega(\mathbf{k}) \ll E(\mathbf{p}),E(\mathbf{p}') . \tag{29}$$

Using (29), Eq. (24) reduces to

$$|\bar{Y}(\mathbf{k},a)|^2\approx|P(\mathbf{k},a)(\mathbf{e}_{\mathbf{h}a}\cdot\boldsymbol{\beta})+Q(\mathbf{k},a)|^2 . \tag{30}$$

It is worthwhile to point out that Eq. (30) can be obtained directly by the calculation neglecting the particle spin. By using $|\chi_0|=|1-\epsilon| \ll 1$, the density of final states becomes

$$\rho_F=\left[1-\boldsymbol{\beta}^*\cdot\mathbf{n}\right]^{-1}\frac{V\omega(\mathbf{k})^2 d\Omega}{(2\pi)^3\hbar c^{*3}} . \tag{31}$$

Using Eqs. (30) and (31) and taking into account Eq. (20), we obtain from Eq. (27)

$$w(\mathbf{k},a)=\left[\frac{e^2}{\hbar c}\right]\frac{\omega(\mathbf{k})d\Omega}{2\pi(1-\boldsymbol{\beta}^*\cdot\mathbf{n})\epsilon^{1/2}}\sum_{\mathbf{h}\,(\neq 0)}|\chi_{-\mathbf{h}}|^2\left|\left[\frac{\omega(\mathbf{k})}{c^*}\right]\frac{\{[\omega(\mathbf{k})/c]\boldsymbol{\beta}-\mathbf{h}\}\cdot\mathbf{e}_{\mathbf{k}a}}{\mathbf{k}_\mathbf{h}^2-[\omega(\mathbf{k})/c^*]^2}\right|^2 , \tag{32}$$

where we have neglected the terms equal to or smaller than the order of $|\chi_0|(\lesssim 10^{-5})$. In the above, we have also used the following dispersion relation for PXR:

$$\omega(\mathbf{k})\cong\mathbf{k}_\mathbf{h}\cdot\mathbf{v} . \tag{33}$$

Equation (33) has been easily obtained from energy and momentum conservation laws. Except for the dependence on ϵ, Eq. (32) coincides with our previous result derived by semiclassical considerations.[10] Using Eq. (33), the denominator in Eq. (32) can be rewritten as

$$\mathbf{k}_\mathbf{h}^2-[\omega(\mathbf{k})/c^*]^2\approx\mathbf{k}_{\mathbf{h}\perp}^2+[\omega(\mathbf{k})/c]^2/|\beta\gamma|^2 , \tag{34}$$

where $\mathbf{k}_{\mathbf{h}\perp}$ is the component of $\mathbf{k}_\mathbf{h}$ transverse to the velocity of the incident particle. If we sum up the polarization of emitted x rays, we obtain the classical results[4,6] by using the relation

$$\sum_a |\{[\omega(\mathbf{k})/c]\boldsymbol{\beta}-\mathbf{h}\}\cdot\mathbf{e}_{\mathbf{k}a}|^2=|\{[\omega(\mathbf{k})/c]\boldsymbol{\beta}-\mathbf{h}\}\times\mathbf{n}|^2 . \tag{35}$$

The reduction to the classical formula is quite reasonable because calculations neglecting the quantum recoil and the spin of the particle generally yield the classical results.[14] This is the reason why the classical formula explains the experimental result well.[4] Obviously, the quantum result is needed when we consider the polarized particles. Also, for relatively low-energy particles ($\gamma\sim 1$), the quantum recoil should be taken into account. In the latter case, however, the radiation probability becomes small as seen in Eq. (32) and Eq. (34).

V. INTERFERENCE WITH ELECTRONIC COHERENT BREMSSTRAHLUNG

As one can see in Fig. 1, the amplitude of PXR may interfere with that of "electronic" coherent bremsstrahlung (ECB) represented by diagrams (c) and (f). In fact, the dispersion relation for ECB is the same as Eq. (33). However, the interference effect should be small because Eq. (32) in itself explains the experimental results. Now, we confirm this explicitly.

The amplitude of ECB is easily obtained by neglecting the quantum recoil and the particle spin. The result is

$$K^{(\mathrm{ECB})}(\mathbf{k},a)=\sum_{\mathbf{h}} e\left[\frac{2\pi\hbar c^{*2}}{V\omega(\mathbf{k})}\right]^{1/2}\chi_{\mathbf{h}}\frac{\omega(\mathbf{k})(\mathbf{e}_{\mathbf{k}a}\cdot\mathbf{h})}{c\mathbf{h}^2(1-\boldsymbol{\beta}^*\cdot\mathbf{n})\gamma}\times\delta(\mathbf{p}'+\hbar\mathbf{k}_{\mathbf{h}}|\mathbf{p}) . \tag{36}$$

Equation (36) shows that ECB is intense only in the forward direction satisfying

$$(1-\boldsymbol{\beta}^*\cdot\mathbf{n})^{-1}\approx 2\gamma^2 .$$

On the other hand, PXR is intense in the direction that the denominator of Eq. (32), i.e., Eq. (34), becomes very small:

$$\mathbf{k}_\perp\approx-\mathbf{h}_\perp . \tag{37}$$

Equation (33) suggests that PXR has sharp peaks in the vicinity of the Bragg condition for the "virtual" photon $\mathbf{k}_v=\omega\boldsymbol{\beta}/c$. Since the directions satisfying Eq. (37) are far from the forward direction, we may neglect interference effects.

VI. CONCLUDING REMARKS

In this paper, we have developed the quantum theory of PXR. We have derived an analytical expression including quantum recoil and spin. Neglecting these quantum features, our expression yields the classical formula, in agreement with the experimental result.

In certain experimental geometries, the absorption of the emitted photons becomes important.[3] We can easily take into account the effect of absorption, by including the imaginary part of the wave vector of the photon along the emitted direction.

As mentioned in Sec. II, an interesting feature of PXR is that the photons are emitted by the crystal electrons. This is similar to Čerenkov radiation (CR) and transition radiation (TR). Therefore, PXR production occurs not only for relativistic electrons and positrons but also for any other relativistic charged particles. It means that PXR might be applicable to particle detectors, like CR and TR.

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APPENDIX

It is well known that the phase velocity of light changes in matter. In classical electrodynamics, it can be written as[12]

$$c^*=c/\sqrt{\epsilon} \tag{A1}$$

where ϵ is the dielectric constant. From a quantum-electrodynamical point of view, Eq. (A1) is far from obvious. We show that Eq. (A1) holds for x rays in matter to within the first-order approximation.

The unperturbed Hamiltonian for the radiation field in vacuum is given by

$$H_r^0=\sum_{\mathbf{k}}\hbar\omega^0(\mathbf{k})(a_{\mathbf{k}a}^\dagger a_{\mathbf{k}a}+\tfrac{1}{2}) , \tag{A2}$$

where $\omega^0(\mathbf{k})=c|\mathbf{k}|$ is the frequency of a photon in vacuum. In matter, additional diagonal terms appear due to the interaction with the electrons. The simplest term can be derived by considering

$$\langle n_0|H_{c-r}^{0(2)}|n_0\rangle=\frac{e^2}{2m_ec^2}\left\langle n_0\left|\sum_i \mathbf{A}^0(\mathbf{r}_i)^2\right|n_0\right\rangle , \tag{A3}$$

where $\mathbf{A}^0(\mathbf{r})$ is the field operator in vacuum:

$$\mathbf{A}^0(\mathbf{r})=\sum_{\mathbf{k}a}\mathbf{A}^0(\mathbf{k},a)[a_{\mathbf{k}a}\exp(i\mathbf{k}\cdot\mathbf{r})+\mathrm{H.c.}] ,$$
$$\mathbf{A}^0(\mathbf{k},a)=\left[\frac{2\pi\hbar c^2}{V\omega^0(\mathbf{k})}\right]^{1/2}\mathbf{e}_{\mathbf{k}a} . \tag{A4}$$

From Eqs. (A2) and (A3), we obtain

$$\langle n_0|H_{c-r}^{0\ (2)}|n_0\rangle=\frac{e^2}{2m_ec^2}\sum_{qb}|\mathbf{A}^0(\mathbf{q},b)|^2[2F_{00}(0)(a_{\mathbf{q}b}^\dagger a_{\mathbf{q}b}+\tfrac{1}{2})+F_{00}(2\mathbf{q})a_{\mathbf{q}b}a_{\mathbf{q}b}+F_{00}(-2\mathbf{q})a_{\mathbf{q}b}^\dagger a_{\mathbf{q}b}^\dagger]$$
$$+\frac{e^2}{2m_ec^2}\sum_{\substack{\mathbf{q},\mathbf{q}'\ (\mathbf{q}\neq\mathbf{q}')\\ b,b'\ (b\neq b')}}\sum\mathbf{A}^0(\mathbf{q},b)\cdot\mathbf{A}^0(\mathbf{q}',b',)[F_{00}(\mathbf{q}+\mathbf{q}')a_{\mathbf{q}b}a_{\mathbf{q}'b'}+F_{00}(-\mathbf{q}+\mathbf{q}')a_{\mathbf{q}b}^\dagger a_{\mathbf{q}'b'}$$
$$+F_{00}(\mathbf{q}-\mathbf{q}')a_{\mathbf{q}b}a_{\mathbf{q}'b'}^\dagger+F_{00}(-\mathbf{q}-\mathbf{q}')a_{\mathbf{q}b}^\dagger a_{\mathbf{q}'b'}^\dagger] . \tag{A5}$$

Using $F_{00}(0)/V=\rho_0$ and $\chi_0(\mathbf{q})$ defined by Eq. (7), the first term in Eq. (A5) can be rewritten as

$$-\sum_{\mathbf{q},b}\frac{1}{2}\chi_0(\mathbf{q})\hbar\omega^0(\mathbf{q})\left[a^\dagger_{\mathbf{q}b}a_{\mathbf{q}b}+\frac{1}{2}\right]\equiv H_r' \ . \tag{A6}$$

Taking into account Eq. (A6), the unperturbed Hamiltonian in matter is modified to the form

$$\begin{aligned} H_r &= H_r^0+H_r' \\ &= \sum_{\mathbf{k}a}\hbar\omega^0(\mathbf{k})[1-\tfrac{1}{2}\chi_0(\mathbf{k})](a^\dagger_{\mathbf{k}a}a_{\mathbf{k}a}+\tfrac{1}{2}) \ . \end{aligned} \tag{A7}$$

Now, defining the photon frequency in matter by

$$\omega(\mathbf{k})=\omega^0(\mathbf{k})[1-\tfrac{1}{2}\chi_0(\mathbf{k})] \ , \tag{A8}$$

we obtain Eq. (4).

It should be noted that Eq. (A8) is consistent with Eqs. (5) and (6). Moreover, if we define

$$\epsilon=1+\chi_0 \ , \tag{A9}$$

then we obtain Eq. (A1) from Eq. (6) because $|\chi_0|\ll 1$ for x rays.

Equation (A7) must be consistent with the classical Hamiltonian for a dielectric, i.e.,

$$H_r^{(cl)}=\frac{1}{8\pi}\int d\mathbf{r}[\epsilon|\mathbf{E}|^2+|\mathbf{B}|^2] \ . \tag{A10}$$

If we define the field operator in matter by Eq. (9), we obtain Eq. (A10) from Eq. (A7).

We have considered only the dispersion due to the interaction caused by $H^{(2)}_{c-r}$. If we calculate the second-order perturbation, we will find that the interaction by $H^{(1)}_{c-r}$ also contributes to the dispersion.[15] It must be emphasized that our derivation of the dispersion relation is based on the perturbation theory, i.e., because of the fact that x rays satisfy $|\chi_0|\lesssim 10^{-5}$. It would be much more difficult to give a consistent description of QED in the matter for arbitrary frequencies. In fact, even in classical electrodynamics, the dispersion effects in general have not been clearly understood.[12] We will not discuss this problem further in this work.

Finally, we point out that, in accordance with Eq. (A7), H_r' should be subtracted from the interaction Hamiltonian $H^{(2)}_{c-r}$. This has been done in our calculations.

[1] *Coherent radiation sources,* edited by A. W. Sáenz and H. Überall (Springer-Verlag, Berlin, 1985).

[2] This radiation could also be called quasi-Čerenkov radiation, resonance radiation, parametric Čerenkov radiation, etc. Though we are not sure whether "parametric x-ray radiation" is the most suitable term, we use it henceforth for the sake of convenience.

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[13] There are other processes of order e^3. They can be calculated by third-order perturbation theory by considering the interaction due to H_{p-r} and $H^{(1)}_{c-r}$. However, for x-ray emissions these contributions are quite small, just as the Kramers-Heisenberg term is much smaller than the Waller term in the coherent scattering cross section of photons by atomic electrons (Ref. 11).

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