PHYSICAL REVIEW B VOLUME 45, NUMBER 13 1 APRIL 1992-I

Simulation of vortex motion in underdamped two-dimensional arrays of Josephson junctions

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(Received 4 November 1991)

We report numerical simulations of classical vortex motion in two-dimensional arrays of underdamped Josephson junctions. A very efficient algorithm was developed, using a piecewise linear approximation for the Josephson current. We find no indication for ballistic motion, in square arrays nor in triangular arrays. Instead, in the limit of very low damping, there appears to be an effective viscosity due to excitation of the lattice behind the moving vortex.

Theoretical research in the past few years has suggested the possibility of ballistic vortex motion in arrays of highly underdamped Josephson junctions in the quantum[1,2] or in the classical[3] regime. This possibility has been investigated experimentally and it was very recently claimed to have been confirmed.[4] In the experiment[4] vortices were created by a small magnetic field and accelerated by a current in one region of a triangular array and then launched through a channel into a field and current free region. The launching direction was parallel to one of the array axes. At very low temperatures a voltage probe opposite to the launching channel at a distance of 40 lattice cells showed the arrival of vortices. The ratio of the Josephson and charging energy was about 250 and therefore the quantum fluctuations of the phases of the superconducting order parameter of the islands are very likely negligible. Under this assumption the motion of the vortex follows from the classical equations for the island phases. This set of equations is in general hard to solve analytically without making crude approximations, but can of course be solved numerically for not-too-large arrays. Indeed, such numerical calculations were performed some time ago[5] for a square array and for one particular value of the damping (in those calculations, in contrast to ours, also an inductance was included). However, no systematic study of the dependence on the strength of the damping has yet been made. This will be the focus of the present work.

Adopting the resistively shunted junction (RSJ) model and assuming that the only important capacitance is the capacitance C between nearest neighbors, the equations of motion for the phases ϕ_j of the superconducting order parameter at the islands j ($j=1,2,\ldots,N$) of the array read

$$\sum_k G_{jk}\left[C\frac{\hbar\ddot{\phi}_k}{2e}+\frac{1}{R}\frac{\hbar\dot{\phi}_k}{2e}\right] = -\sum_k i_c\sin(\phi_j-\phi_k)+i_{jx}\,, \tag{1}$$

where the sums are over nearest-neighbor islands k and the overdots denote time derivatives. The matrix G_{jk} is the lattice Green's function ($G_{jk}=-1$ if j and k nearest neighbors, $G_{jj}=4$ for the square array, $G_{jj}=6$ for the triangular array; at the edges of the array G_{jk} has to be adjusted appropriately). R is the shunting resistance, i_c the critical current, and i_{jx} the external current fed into island j. We assume that the temperature is low enough to neglect the thermal noise generated by the resistors. The nonlinearity of the Josephson current $i_c\sin(\phi_j-\phi_k)$ prohibits the determination of the exact solution of these equations. What we will do is replace the sine function for every junction by a piecewise linear approximation and determine the exact solution for each linear branch in the N-dimensional space of the junction phases. The approximation for $\sin 2\pi x$ used in our algorithm is depicted in Fig. 1. For each linear branch the equations of motion then acquire the form

$$\sum_k G_{jk}(\ddot{\phi}_k+\beta_c^{-1/2}\dot{\phi}_k) = -\sum_k \tilde{G}_{jk}\phi_k+b_j\,, \tag{2}$$

where the matrix $\tilde{G}$ is the same as the matrix G except for the elements $\tilde{G}_{jj},\tilde{G}_{kk},\tilde{G}_{jk}=\tilde{G}_{kj}$ for which $\min_n|\phi_j-\phi_k-2\pi n|>0.4\pi$ (see Fig. 1), which depend on the particular linear branch we have to use. Time has been redefined in units of $1/\omega_p$, with ω_p the plasma frequency, $\omega_p=(8\tilde{E}_jE_c)^{1/2}/\hbar$, where $E_c=e^2/2C$ is the charging energy. We had to define an adjusted Josephson energy $\tilde{E}_j=(5/2\pi)E_j$ ($E_j=\Phi_0 i_c/2\pi$, with $\Phi_0=h/2e$ the flux quantum) because of the slightly different slope at the origin of the linear approximation (see Fig. 1). The elements b_j in Eq. (2) contain the external current contributions and contributions which depend on the specific linear branch. The strength of the damping is regulated by the

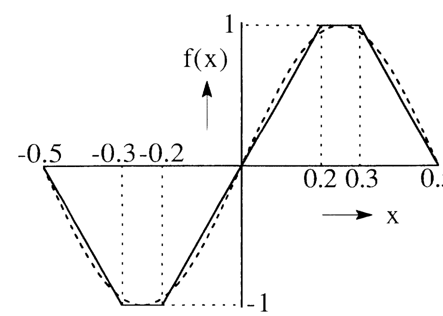

FIG. 1. The approximation (solid line) of $\sin 2\pi x$ (dashed line) used in the algorithm.

McCumber parameter β_c:

$$\beta_c = (2\pi)^2 \tilde{E}_j R^2 C / \Phi_0^2 . \tag{3}$$

The junctions are in the underdamped regime if $\beta_c > 1$.

The algorithm now works as follows. In the first stage, starting from some initial phase configuration on a specific linear branch, the exact solution of Eq. (2) is calculated by expanding the phase configuration into a static part, solving Eq. (2) with zero left-hand side, and eigenvectors ϕ_k^λ of the equation

$$\sum_k \tilde{G}_{jk} \phi_k^\lambda = \lambda \sum_k G_{jk} \phi_k^\lambda , \tag{4}$$

which have a time dependence

$$\phi_k^\lambda(t) = \phi_k^\lambda(0) \exp(-\tfrac{1}{2}[\beta_c^{-1/2} \pm (\beta_c^{-1} - 4\lambda)^{1/2}]t) .$$

In the second stage, this solution is propagated in time until we have to cross over to another linear branch. At the crossing time we match the junction phases and their time derivatives and repeat the process. What makes the algorithm so efficient is (1) the availability of an exact solution on each linear branch and (2) the fact that the matrix $\tilde{G}$ only differs from G for islands j in the immediate neighborhood of the vortex. This means that all vectors ϕ_k with zero components for these junctions trivially satisfy Eq. (4) with $\lambda = 1$ (in the terminology of Ref. 2 these junctions are treated as a "linear medium"). The determination of the other eigenvectors then reduces to an $M \times M$ instead of an $N \times N$ problem, with M the number of junctions j for which $\min_n |\phi_j - \phi_k - 2\pi n| > 0.4\pi$ (j and k nearest neighbors). If there are not too many vortices present in the array we have $M \ll N$ and an important gain is obtained. The crossing times can be calculated very efficiently by a Newton-Raphson root-finding procedure. The CPU time involved in the algorithm is effectively linear in N.

In Figs. 2 and 3 we show results for the vortex velocity v as a function of the driving current for the square and the triangular array, for several values of β_c. The distance between the centers of neighboring plaquettes is a. We feed a current i for the square array and $2i$ for the triangular array into each island on one side of the array and extract it at the opposite side. Antiperiodic boundary conditions are imposed in the direction perpendicular to the current. The vortex is introduced into the array by an initial guess of the phase configurations. Physically, this geometry corresponds to a cylindrical array with *two* vortices of equal sign opposite to one another. For both the square and the triangular array the current was injected perpendicular to one of the axes of the array. The depinning current was found to be $0.095 i_c$ for the square array and $0.025 i_c$ for the triangular array, in good agreement with the theoretical values.[6]

The most striking feature in Figs. 2 and 3 is the saturation of the vortex velocity in the limit of low damping, meaning that the vortex experiences a *nonvanishing* viscosity in this limit. This is in contrast to the theoretical predictions of *ballistic* vortex motion. Theoretically[7] it was predicted that the vortex viscosity should be proportional to $1/R$ and hence should vanish in the underdamped limit ($R \to \infty$). By inspecting the phase configuration we learn that in the wake of the vortex the island phases are oscillating at the plasma frequency ω_p. In previous experiments on vortex motion in underdamped arrays[8] this was already suggested as a possible source of the unexpectedly high viscosity. Most of the energy appears to be transferred to the junction just behind the vortex. An estimate of the viscosity can be made by assuming that every time the vortex moves one cell an amount of energy of the order of a few times E_j is transferred to the lattice. This leads to a viscosity η of the order of a few times $E_j/(a^2\omega_p)$. Equating the force $i\Phi_0/a$ due to the current to the drag force ηv we find a velocity of v the order of $a\omega_p i/i_c$, in agreement with the simulations. We also checked in our simulations whether the vortex keeps on moving for some distance if the current is stepped from a certain value to zero, because of a vortex "mass." What we observed, however, is that a vortex moving with its maximal velocity $v \sim a\omega_p$ (see below) retraps after crossing at most one junction. This is consistent with the above picture since with a vortex mass $M_v \sim \Phi_0^2 C/a^2$ (Refs. 1-3) the maximal kinetic energy is of the order of a few times E_j and is lost in crossing just one junction. On the other hand, in the square array we find a retrapping

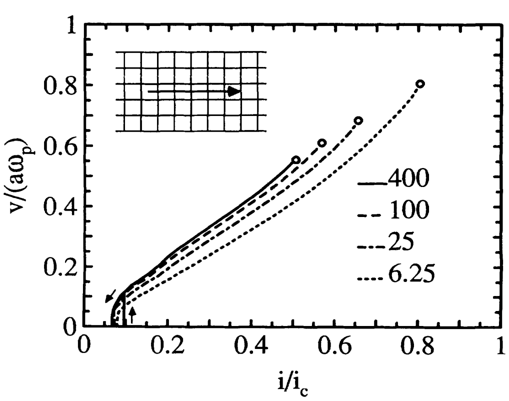

FIG. 2. Vortex velocity vs current in a 100×10 square array for different values of β_c. The arrows indicate whether the current was increased or decreased.

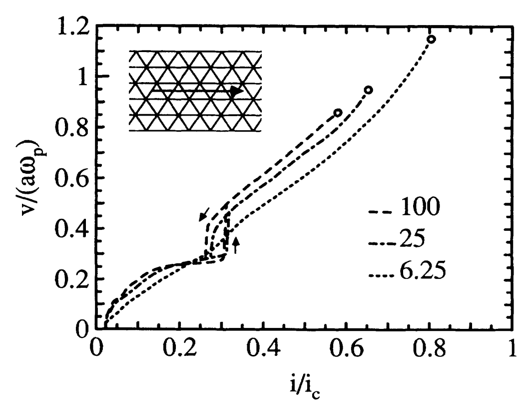

FIG. 3. Vortex velocity vs current in a 10 junction wide triangular array for different values of β_c. The length of the array was 20 junctions for $\beta_c = 6.25$ and 25, and 40 junctions for $\beta_c = 100$.

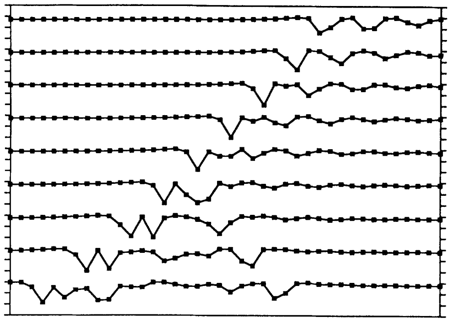

FIG. 4. Snapshots of the supercurrent in part of the central row of junctions in the 100×10 square array for $\beta_c=25$. The current has just been stepped from $i/i_c=0.625$ to $i/i_c=0.65$. The time interval is $4/\omega_p$. The division on the vertical axis is i_c and the curves are shifted by successive increments of $3i_c$ along the current axis. Time increases from top to bottom.

current slightly below the depinning current, like in experiments.[8,9] We do not find a clear signature of this kind of hysteresis in the triagonal array. However, for the triagonal array we find at higher currents a small hysteretic loop for large β_c ($\beta_c=25,100$). At the velocity where the upward jump occurs, $v\approx 0.3a\omega_p$, the vortex moves roughly two cells per plasma oscillation period. Behind the vortex a pattern of oscillations commensurate with the lattice is built up, leading to a phase locking of the vortex velocity to the plasma frequency. This phase locking is suddenly broken if the current exceeds the value $i/i_c\approx 0.3$. For smaller β_c ($\beta_c=6.25$) a less pronounced structure remains.

One could argue that the nonvanishing viscosity is somehow caused by the nonanalyticity of the piecewise linear approximation of the Josephson current (see Fig. 1). However, calculations using the correct Josephson current and a Runge-Kutta integration method[10] give, for $\beta_c=25$ and an 8×8 lattice, results for the current-velocity characteristic which differ only a few percent from results obtained with the present method. Because of the too-small lattice size it is hard to draw conclusions from those calculations pertaining to an infinite lattice size, but the validity of our approach is certainly supported by them.

Another interesting feature in the simulations is the creation of vortex-antivortex pairs behind the moving vortex above a critical value of the current. This phenomenon was also observed in the simulations of Ref. 5. It can qualitatively be understood as follows. The oscillations behind the moving vortex can be interpreted as a sequence of bound vortex-antivortex pairs. When the vortex moves faster, the distance between the bound vortices increases and at some point the unbinding force $i\Phi_0/a$ due to the current exceeds the binding force $2\pi E_j/r$. Estimating $r\sim 2\pi v/\omega_p$ and $v\sim a\omega_p i/i_c$ we find the right order of magnitude $v\sim a\omega_p$ for this critical velocity. In Fig. 4 we show a time evolution for $\beta_c=25$ of the Josephson currents in part of the central row of junctions of the square array just after the current has been raised above the value where the vortex-antivortex creation starts. Clearly observable are the oscillations behind the vortex which develop into an antivortex moving into the opposite direction and a vortex of the same sign chasing the initial vortex. A cascade of such processes will occur and eventually switching of the whole row into the normal state.

In the simulations it was important to have sufficiently long arrays, so that the oscillations could die out before the periodic image of the vortex reached them. This was particularly important for large β_c. For the square array we could exploit the symmetry about the central row of junctions. For the triangular array this symmetry is absent and the computations took a considerably longer amount of time (other factors also increased the computation time). We therefore adjusted the length of the array to β_c. The width of the array was 10 junctions both for the square and the triangular array, which turned out to be sufficient for our purposes. The computations were done on SUN4c systems. The calculation of each current-velocity curve took a few hours to a few days of CPU time.

The important conclusion we draw from our simulations is that there is no ballistic vortex motion in the classical regime. In the quantum regime, however, ballistic motion of vortices could still be a possibility. Because of the single frequency ω_p involved an amount of energy $\hbar\omega_p$ is required to excite the lattice. If the kinetic energy of the vortex is below this value excitation of the lattice is impossible and the vortex moves without damping. It seems unlikely, considering the large ratio between Josephson and charging energy, that this would be the explanation for the experimental observations.[4] Another explanation could be that the measurements do not reflect a single vortex property but some collective behavior, i.e., the interaction between the vortices plays an important role. Further experiments should decide this.

The author is grateful to Harvard University for hospitality, financial support (NSF Grant No. DMR-89-12927), and the use of computer facilities. He appreciates invaluable discussions with M. Rzchowski, T. Tighe, E. Granato, M. Tinkham, H. van der Zant, A. van Otterlo, and G. Schön. The author's research has been made possible by partial financial support from the Royal Dutch Academy of Arts and Sciences.

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